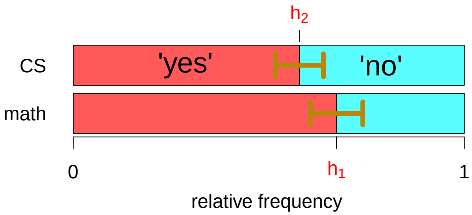
Proportions



For comments please email: M. Messer, michael.messer@tuwien.ac.at

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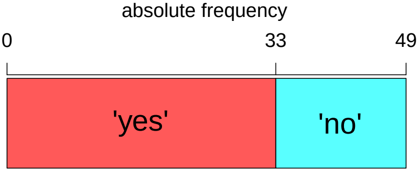
- Thus, graphical representation, e.g., in a *barplot* (in R: `barplot()`)



Relative frequencies

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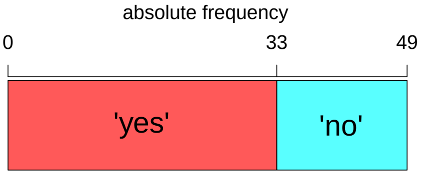

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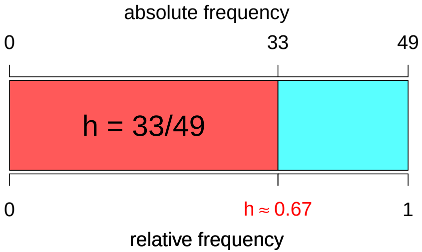

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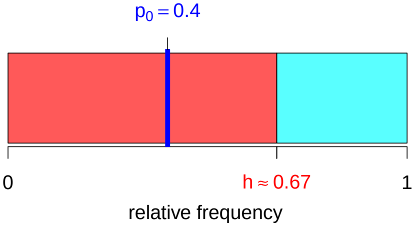

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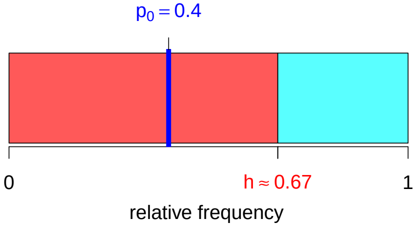

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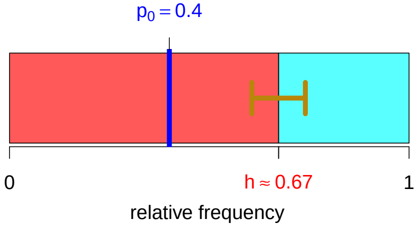

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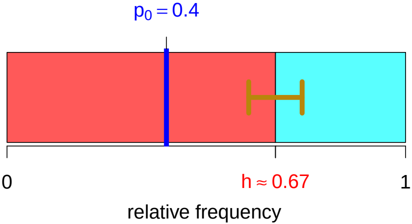
- n data in two categories (here: 'yes' and 'no')



Note that we deal here with categorical (or nominal) data which is a completely different data type as the metric data considered in the previous lectures. However, we mention here that the proportion h is nothing but a *mean*. Thus, despite the different data type, we will methodologically proceed analogously to the t-tests, comparing means. In that sense, nothing new is going to happen in the following...

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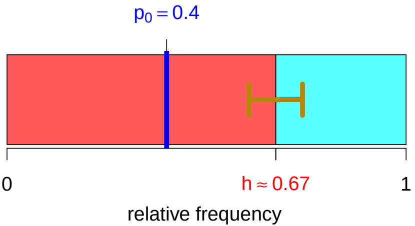

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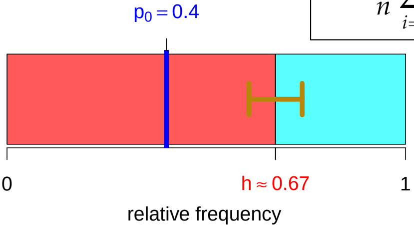

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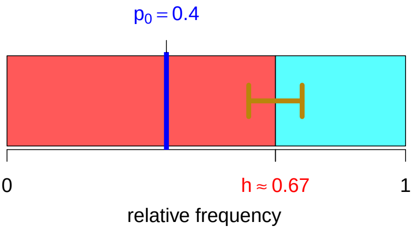

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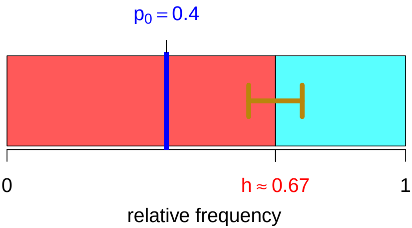

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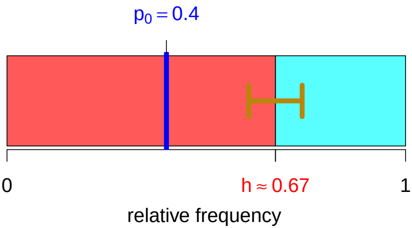

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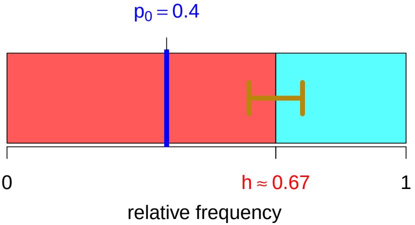

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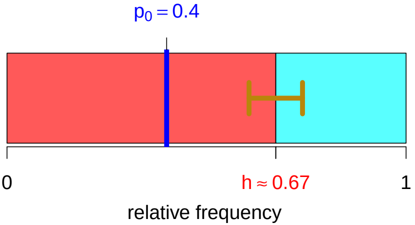

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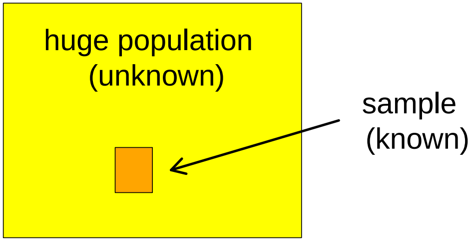

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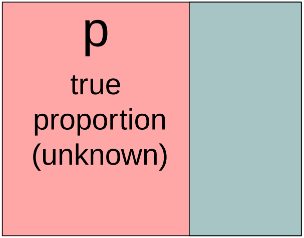

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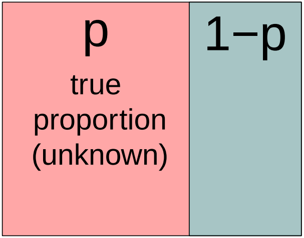

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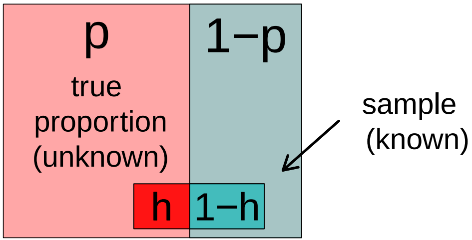

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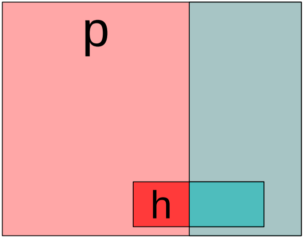

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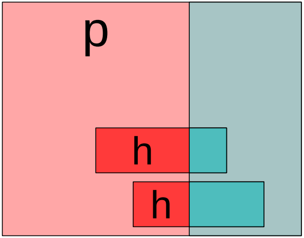

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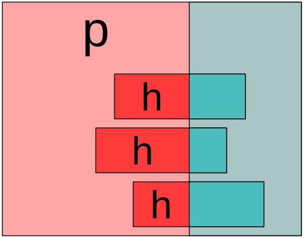

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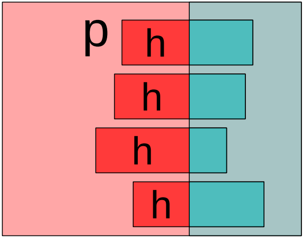

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- and for $n \to \infty$

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'central limit theorem'

Note: in the t-test we found the rescaled mean to be exactly(!) t-distributed, $(\bar{X} - \mu)/\sqrt{S^2/n} \sim t(n-1)$, when assuming normally distributed RVs. Here, we do not make the normal assumption on the RVs and thus will make use of the normal approximation of H

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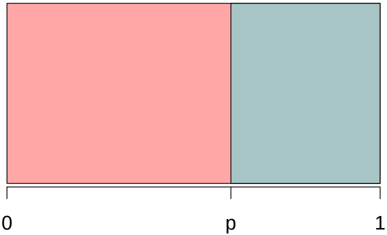

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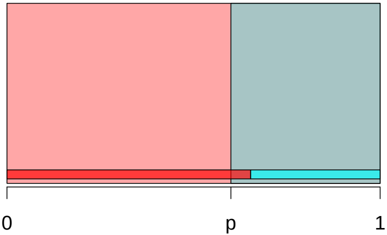

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- Let $Y_1, \dots, Y_n$ be i.i.d. RVs with $Y_1 \sim ber(p)$. Let

note that $\sum^n Y_i \sim b(n,p)$

$$H = \frac{1}{n} \sum_{i=1}^{n} Y_i$$

- It holds for the expectation and the variance

'linearity of the expectation and independence'

$$\mathbb{E}[H] = p \qquad \text{and} \qquad \mathbb{V}ar(H) = \frac{p(1-p)}{n}$$

- and for $n \to \infty$

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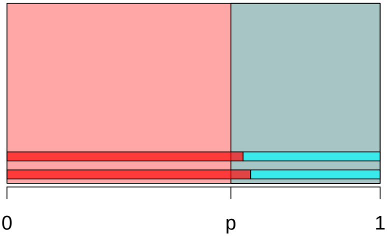

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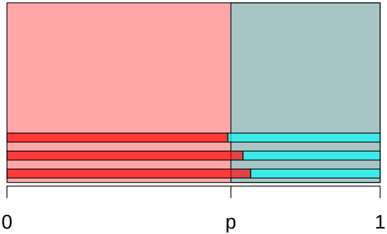

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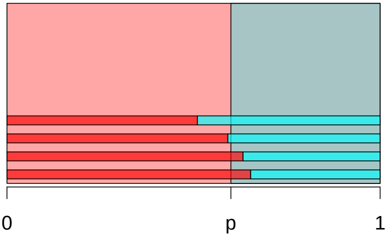

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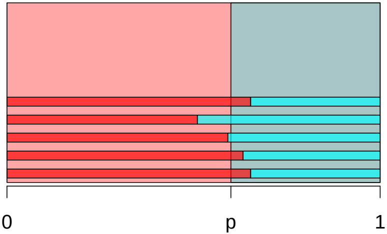

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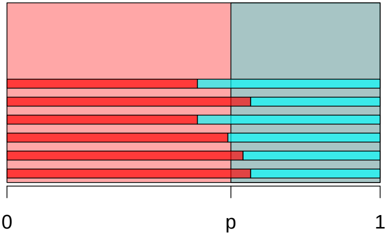

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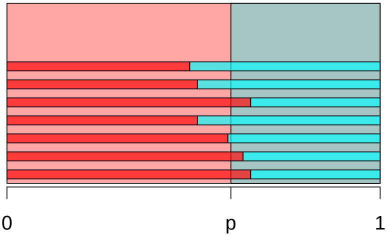

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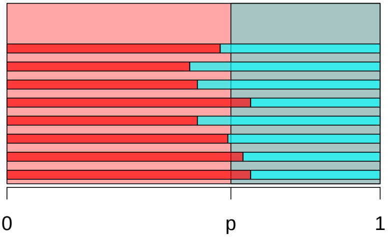

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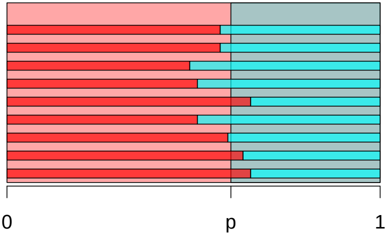

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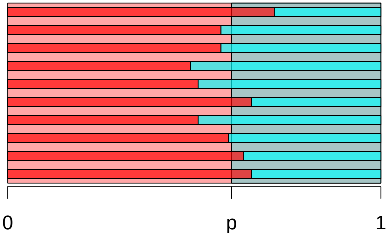

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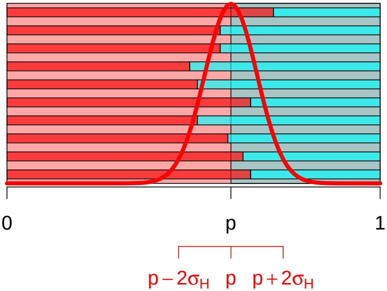

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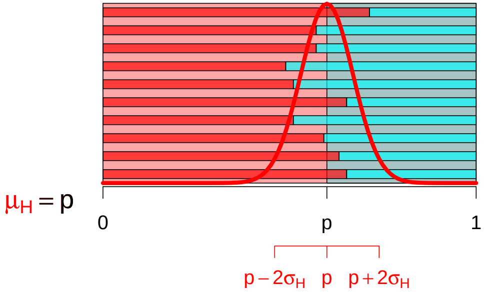

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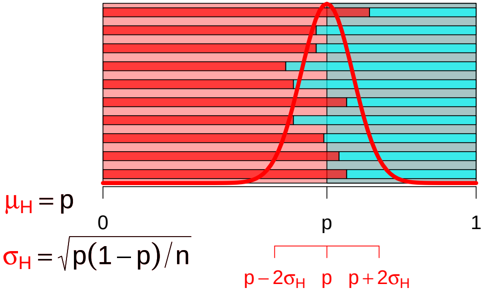

The standard error of the relative frequency

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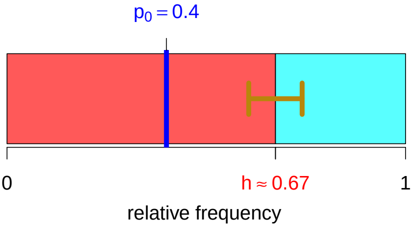
  - deviations from the expectation of order $1 \cdot SE_H$ are not unlikely
  - but deviations of 'many' SE_H are unlikely

The standard error of the relative frequency

- the estimation based on the data yields

$$se_h = \sqrt{\frac{h(1-h)}{n}} \approx 0.07$$



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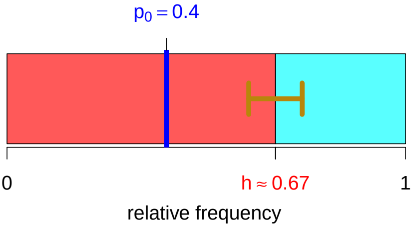

- the discrepancy of the observed frequency $h = 0.67$ and the claimed proportion $p_0 = 0.4$ is

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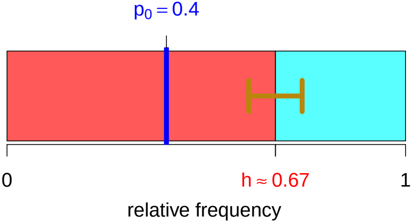

- the discrepancy of the observed frequency $h = 0.67$ and the claimed proportion $p_0 = 0.4$ is

$$|h - p_0| \approx 4.1 \cdot se_h$$

- this is extremely far, given the typical deviation of H to be about $1 \cdot SE_H$

Asymptotic one-sample test for frequencies

- Let $Y_1, \ldots, Y_n$ be i.i.d. RVs with $Y_1 \sim ber(p)$ and $p \in (0,1)$
- and let $q_{1-\alpha/2}$ be the $(1-\alpha/2)$-quantile of the $N(0,1)$-distribution

Under $H_0 : p = p_0$ it holds approximately for large n that

$$Z := \frac{H - p_0}{SE_H} \overset{d}{\approx} N(0,1)$$

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and equivalently: the confidence interval

$$I := \left(H - q_{1-\alpha/2} \cdot SE_H,\ H + q_{1-\alpha/2} \cdot SE_H\right)$$

overlaps the parameter p_0 with probability about $1 - \alpha$

- 'Structure' as in the t-test: $Z = (\spadesuit - \spadesuit)/\spadesuit$ and $I = (\spadesuit - q \cdot \spadesuit,\ \spadesuit + q \cdot \spadesuit)$

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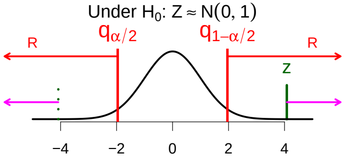
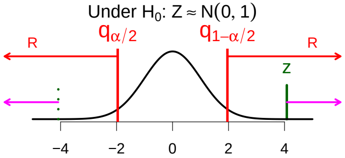
- Thus again: Equivalence of test and confidence interval
  $\alpha = \mathbb{P}_{H_0}(Z \in R) = \cdots = \mathbb{P}_{H_0}(I \not\ni p_0)$ (while R denotes the rejection area of the two-sided test)

Evaluation of the data



- For the significance level $\alpha = 5\%$ it is $q_{1-\alpha/2} \approx 1.96$

Evaluation of the data



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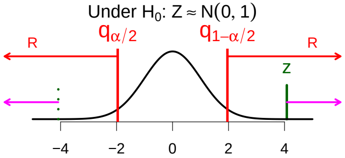

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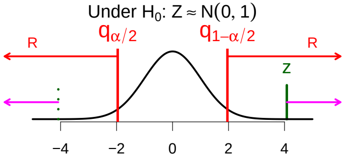

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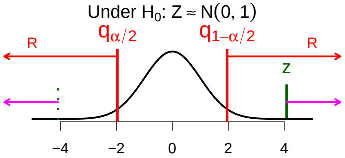

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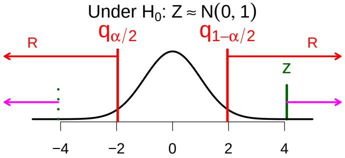

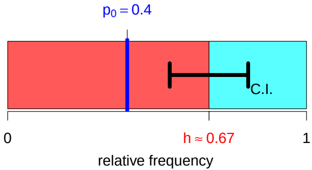

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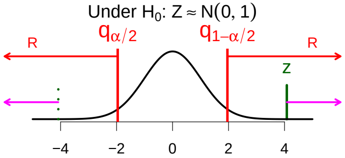

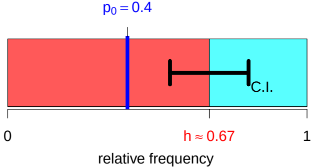

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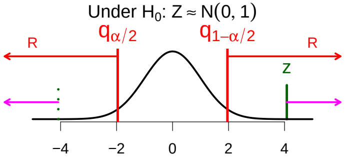

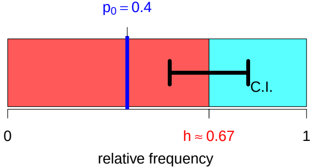

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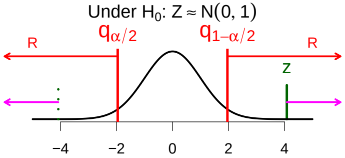

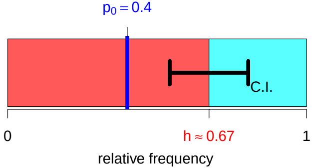

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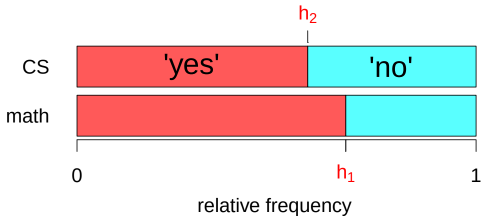

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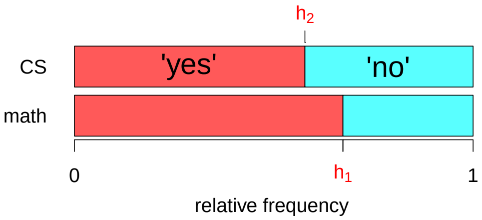

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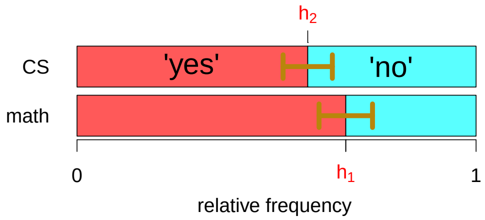

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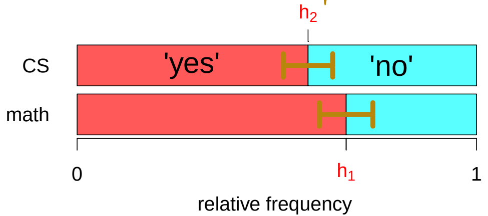

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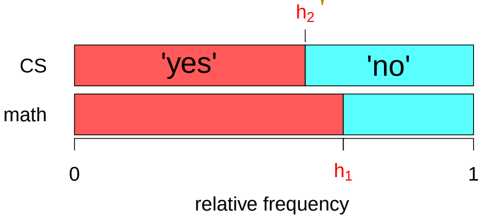

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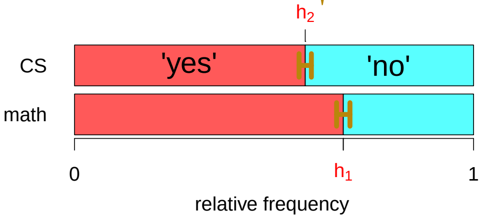

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- direct generalization for the difference $p_2 - p_1 = d_0 \neq 0$

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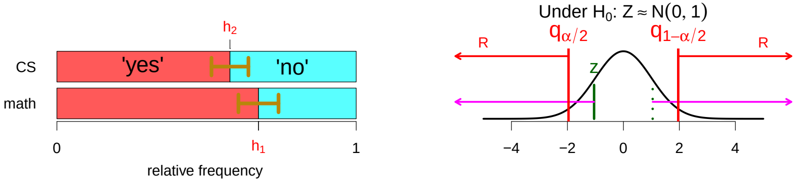

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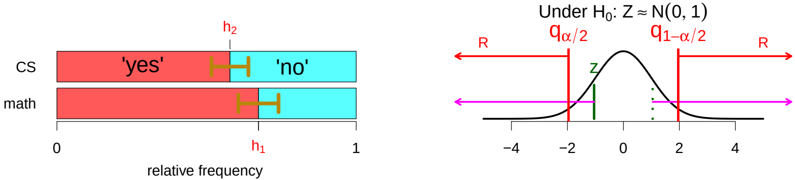

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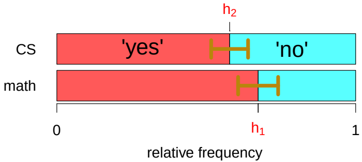

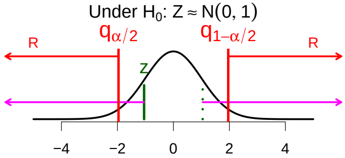

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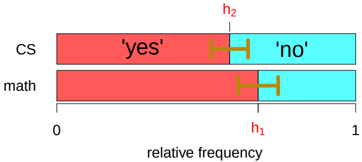

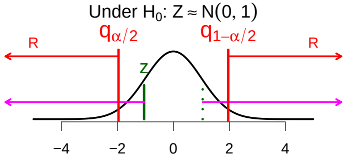

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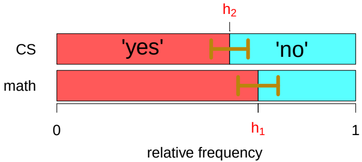

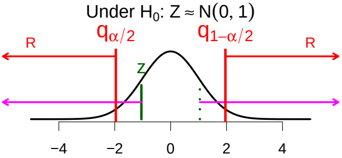

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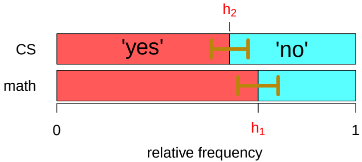

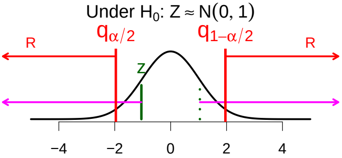

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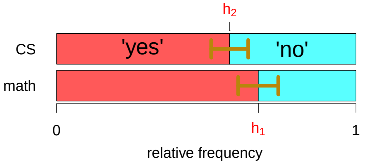

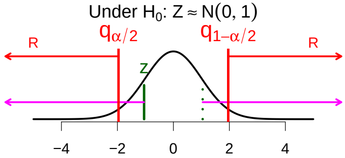

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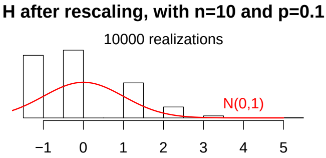

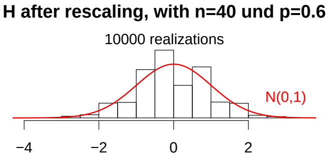

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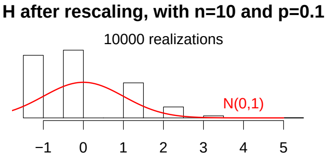

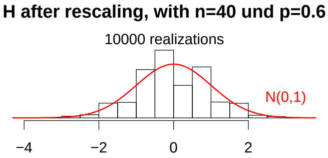

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- left: approximation unreasonable :-(right: quite plausible :-)

Thank you!