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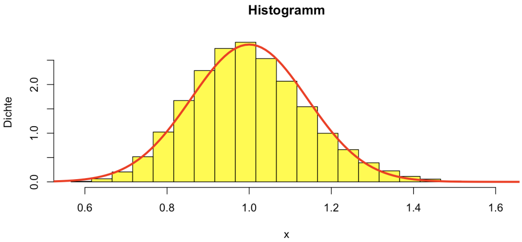
Central limit theorem and Law of large numbers



Please send you comments to: T. Levajković, tijana.levajkovic@tuwien.ac.at

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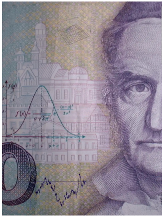

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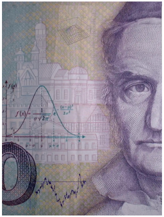

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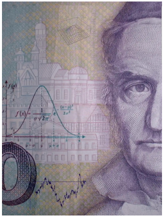

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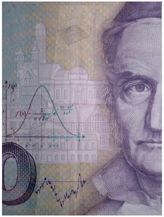

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$Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

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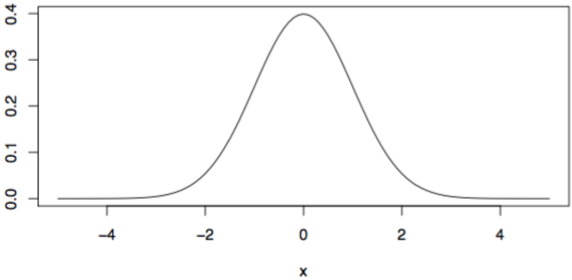

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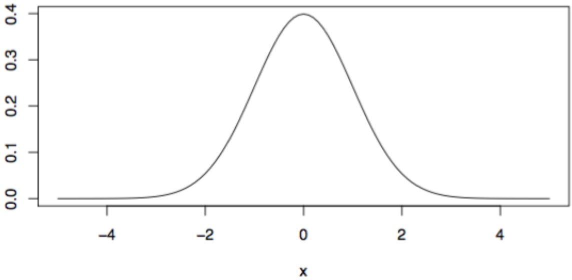

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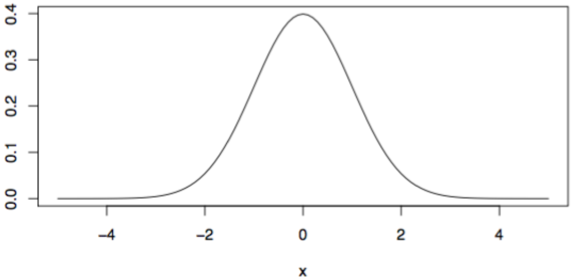

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- The following hold

$$P(-1 \leqslant Z \leqslant 1) \approx 0.68$$
$$P(-2 \leqslant Z \leqslant 2) \approx 0.95$$
$$P(-3 \leqslant Z \leqslant 3) \approx 0.997$$

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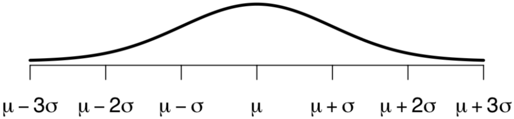

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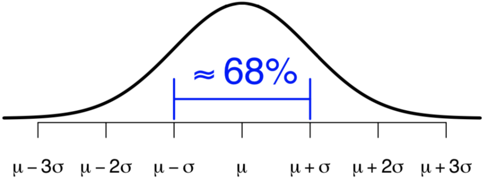

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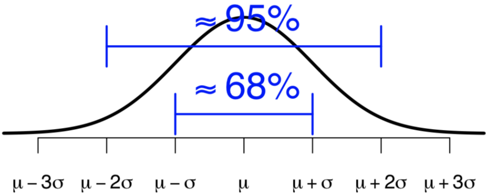

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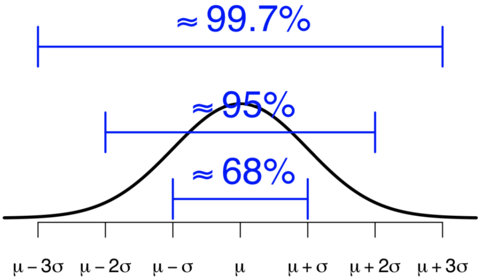

Sample mean

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- $$E(\bar{X}) = E\Big(\frac{1}{n} \sum_{i=1}^{n} X_i\Big) \overset{\text{linearity}}{=} \frac{1}{n} \sum_{i=1}^{n} E(X_i) \overset{\text{ident. distr.}}{=} \frac{1}{n} \sum_{i=1}^{n} E(X_1)$$
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- $Var(\bar{X}) = Var\Big(\frac{1}{n}\sum_{i=1}^{n} X_i\Big) = \frac{1}{n^2} Var\Big(\sum_{i=1}^{n} X_i\Big) \overset{\text{independ.}}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$
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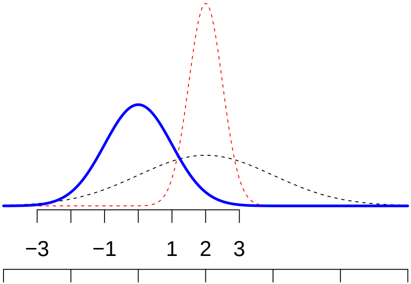
$$E(\bar{X}) = \mu \qquad \text{and} \qquad Var(\bar{X}) = \frac{\sigma^2}{n}$$

  - Especially if $X_1, \ldots X_n$ are normally distributed, the sample mean is also normally distributed $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$



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... We state the theorems for

independent and identically distributed random variables

Theorems

- Let $X_1, \dots, X_n$ be independent and identically distributed (i.i.d.) random variables with $E(X_1) = \mu$ and finite $Var(X_1) = \sigma^2 < \infty$.

... n i.i.d. random variables $X_1, \dots, X_n$ are called a random sample of size n

  - Note that $X_1, \dots, X_n$ are not necessarily normally distributed.

- We consider the sample mean

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

  - The Law of large numbers and the Central limit theorem provide information about the value and the distribution of $\bar{X}_n$.

    - LLN: As n grows, the probability of $\bar{X}_n$ being in the neighborhood of μ tends to 1.

    - CLT: For large n, the distribution of $\bar{X}_n$ is approximately $\mathcal{N}(\mu, \frac{\sigma^2}{n})$.

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  Suppose $X_1, X_2, \ldots, X_n \ldots$ are i.i.d. random variables with expectation μ and finite variance σ^2. For each n, let $\bar{X}_n$ be the mean of the first n variables. Then for any $a > 0$, we have

$$\lim_{n \to \infty} P(|\bar{X}_n - \mu| < a) = 1$$

Example

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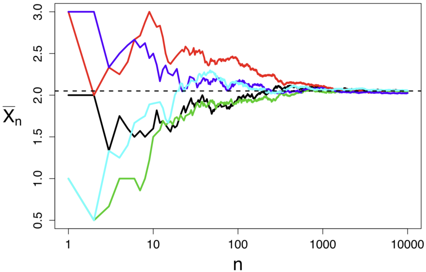

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HW Repeat the calculations and determine the probability

$$P(|\bar{X}_n - \mu| < 0.01).$$

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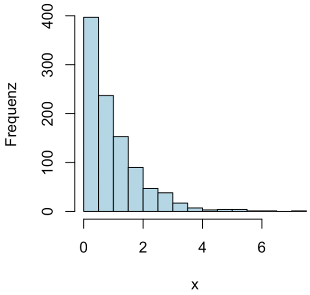

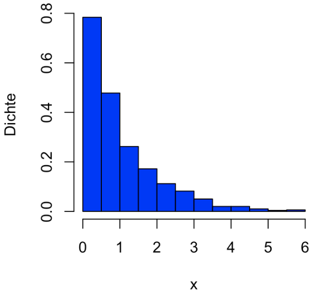

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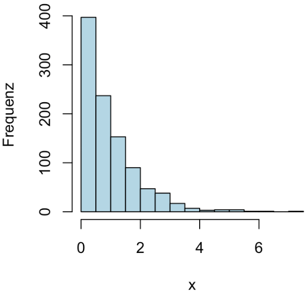
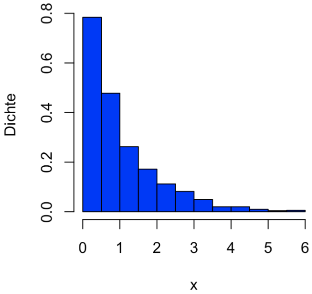
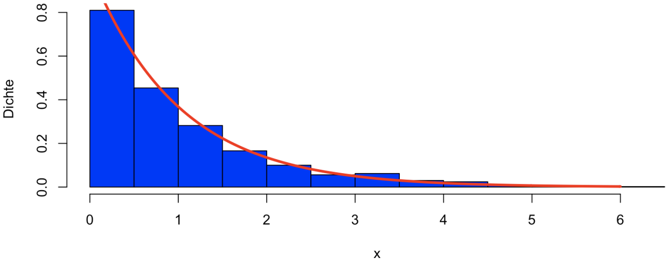




Law of Large Numbers and histograms

- The Law of Large Numbers implies that density histogram converges to probability density function.



The histogram with bin width 0.5 showing 1000 draws from an exponential $\exp(1)$ distribution. The pdf of $\exp(1)$ is given in red.

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\begin{aligned}
S_n &\approx \quad \mathcal{N}(n\mu, n\sigma^2) \\
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    - Thus, the standardized $\bar{X}_n$ and S_n have approximately $\mathcal{N}(0,1)$, i.e.

$$\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \approx \mathcal{N}(0,1) \quad \text{and} \quad \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \approx \mathcal{N}(0,1)$$

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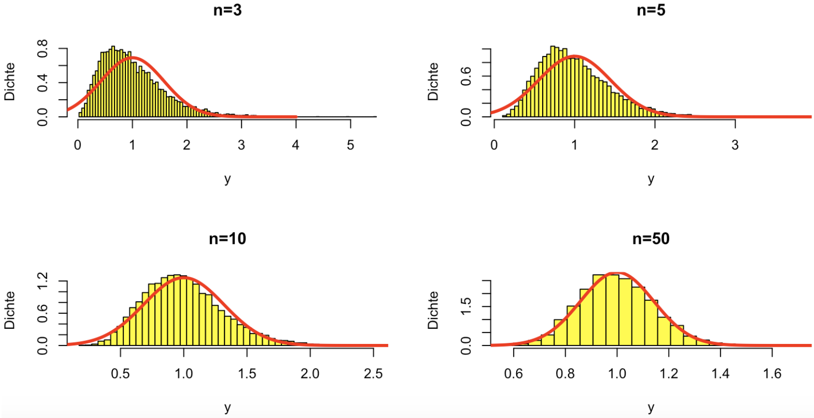
    - The standardized means Y_n

$$Y_n = \frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{Var(\bar{X}_n)}} = \frac{\bar{X}_n - 1}{\frac{1}{\sqrt{n}}} \approx \mathcal{N}(0, 1)$$

Example

- The following plots show the results for respectively $n = 3, 5, 10, 50$ based on 10000 simulated values for X_n. The density of the $\mathcal{N}(1, \frac{1}{n})$ distribution is drawn in red.



Normal approximation of $B(n,p)$ distribution

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$$P(a \leqslant S_n \leqslant b) \approx \Phi\left(\frac{b + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{a - \frac{1}{2} - np}{\sqrt{np(1-p)}}\right)$$

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  - Let T be a continuous random variable such that $E(T) = E(S_n) = np$ and $Var\,(T) = Var\,(S_n) = np(1-p)$. Then,

$$\begin{aligned}
P(a \leqslant S_n \leqslant b) &= P\left(a - \frac{1}{2} \leqslant S_n < b + \frac{1}{2}\right) \approx P\left(a - \frac{1}{2} \leqslant T < b + \frac{1}{2}\right) \\
&= P\left(\frac{a - \frac{1}{2} - E(T)}{\sqrt{Var\,(T)}} \leqslant \frac{T - E(T)}{\sqrt{Var\,(T)}} < \frac{b + \frac{1}{2} - E(T)}{\sqrt{Var\,(T)}}\right) \\
&\overset{\text{CLT}}{\approx} P\left(\frac{a - \frac{1}{2} - np}{\sqrt{np(1-p)}} \leqslant Z < \frac{b + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) \\
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- Rule of thumb: The approximation is considered reasonable, when $\min\{np, np(1-p)\} \geqslant 10$

Normal approximation of $\mathcal{P}(\lambda)$ distribution

- Let $X \sim \mathcal{P}(\lambda)$. Then, by providing the continuity correction for $a \leqslant b$, with $a, b \in \mathbb{N}_0$

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- An accountant rounds to the nearest euro. We assume the error in rounding follows uniform distribution on $(-0.5, 0.5)$. Estimate the probability that the total error in 300 entries is more than 5 euro.

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  - Then,

$$P(|S| > 5) = 1 - P(|S| \leqslant 5) \approx 1 - P(|Z| \leqslant 1) \approx 0.32$$

Examples

HW Let $X_1, X_2, \ldots, X_{25}$ be independent and identically distributed (i.i.d.) random variables and $X_1 \sim \mathcal{N}(1, 4)$. Find the probability $P(X_1 + X_2 + \ldots\ldots X_{25} \geqslant 26)$.

HW Transportation officials tell us that 60% of the population wear their seatbelts while driving. A random sample of 1000 drivers has been taken. What is the probability that between 580 and 630 of the drivers were wearing their seatbelts?

HW Let $X_1, X_2, \ldots X_{121}$ be i.i.d. with the expectation $\mu = 35$ and variance $\sigma^2 = 25$. Denote by

$$\bar{X}_{121} = \frac{1}{121}(X_1 + \cdots + X_{121})$$

the sample mean. Approximate the probability $P(\bar{X}_{121} > 35.2)$ using the Central limit theorem.

HW We toss a fair coin 100 times.
What is the probability of obtaining 60 or more heads?

Questions

A few multiple-choice questions

(1) Let $X_1, X_2, \ldots X_{81}$ be i.i.d. sample from a population with population mean $\mu = 5$ and population variance $\sigma^2 = 4$ and let $S = X_1 + X_2 + \ldots X_{81}$. Approximate the probability $P(S \notin [387, 423])$ using the Central limit theorem.

a. 68%
b. 78%
c. 45%
d. 32%

(2) Assume that X is a binomial random variable with $n = 100$ and $p = 0.1$. Use the normal probability distribution to compute $P(X \leqslant 15)$.

a. 0.5336
b. 0.9664
c. 0.0336
d. 0.4664

A few multiple-choice questions

(3) Transportation officials tell us that 60% of the population wear their seatbelts while driving. A random sample of 1000 drivers has been taken. What is the probability that between 220 and 550 of the drivers were wearing their seatbelts?

a. 1.0
b. 0.4015
c. 0.9066
d. 0

Thank you for your attention!