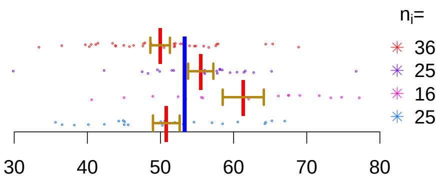
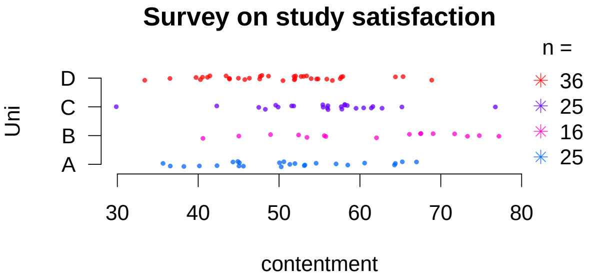
Analysis of variance and multiple testing



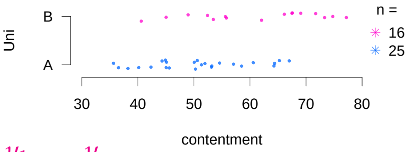
For comments please email: M. Messer, michael.messer@tuwien.ac.at

Reminder

At four universities students of a certain study program were interviewed regarding the level of their satisfaction with the study situation. An extensive survey had to be filled out. Subsequently, for every respondent a global value of 'contentment' was evaluated.



Reminder: two-sample situation



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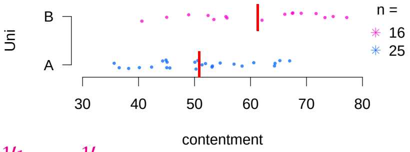

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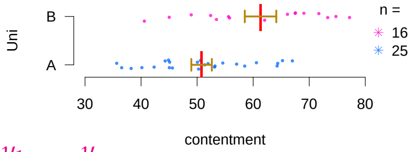

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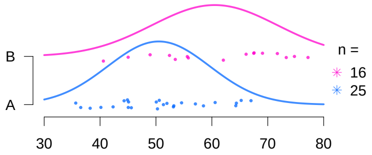

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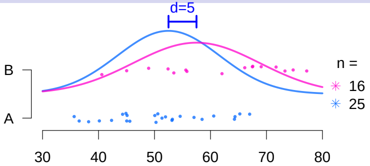

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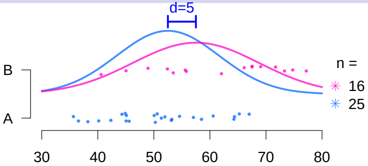

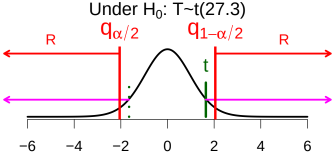

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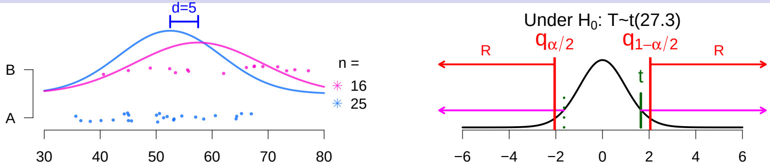

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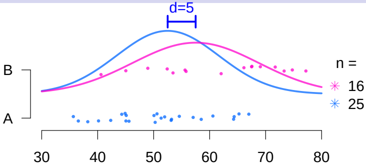

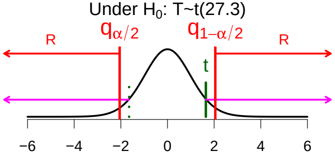

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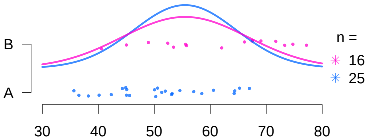

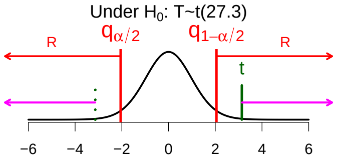

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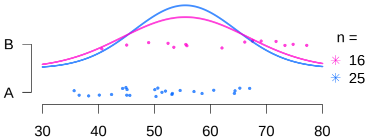

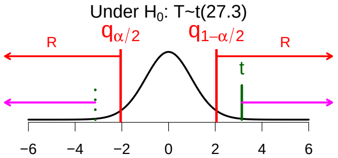

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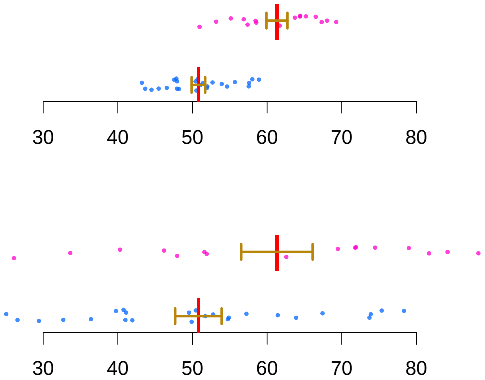
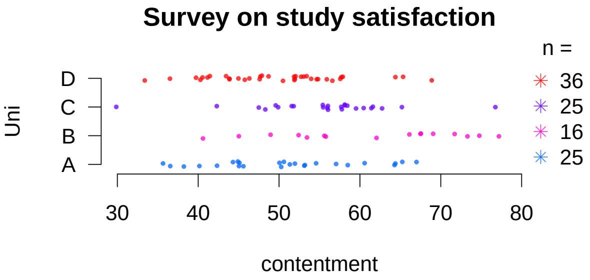
This is the Welch-test. Remember that there is also Student's t-test, were equal variances $\sigma_1^2 = \sigma_2^2$ are assumed, for which we found under H_0 that $T \sim t(n_1 + n_2 - 2)$ while a different scaling was used in T ('pooled variance')

Message



The *t*-statistic says:
the discrepancy between the groups is large,
if the variability within the groups is small

Today: Comparison of $k \geqslant 2$ groups



Here $k = 4$ groups

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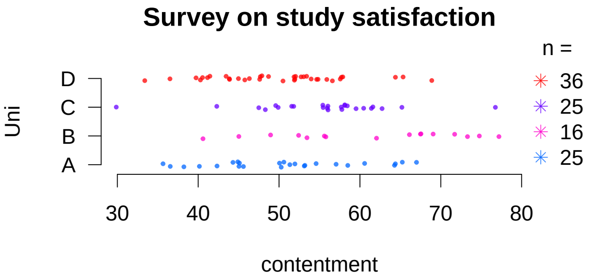

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Two possibilities to deal with that:

- Multiple pairwise t-tests

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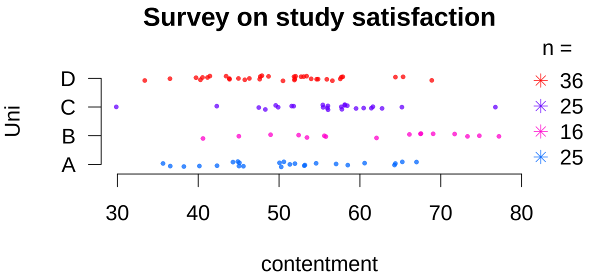

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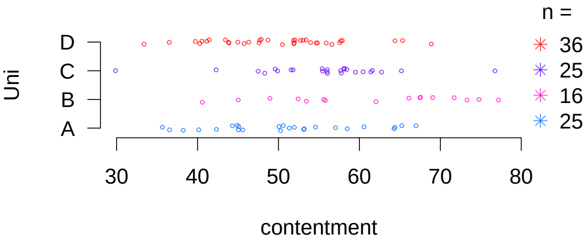
Two possibilities to deal with that:

- Multiple pairwise t-tests
- Analysis of variance (ANOVA)

Multiple testing

Multiple pairwise t-tests: for every pair a two-sample t-test



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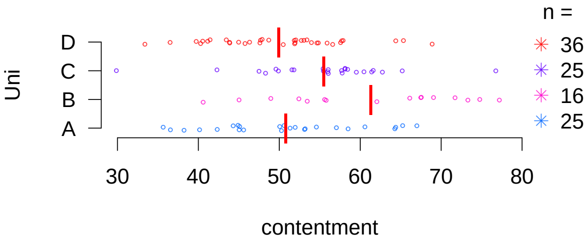

- means

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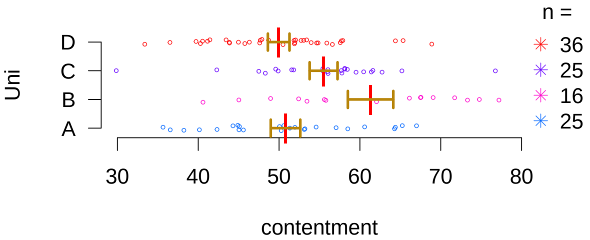

- means ± standard errors

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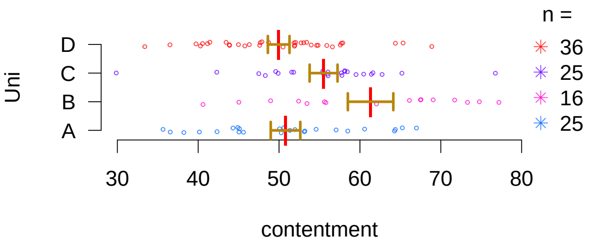

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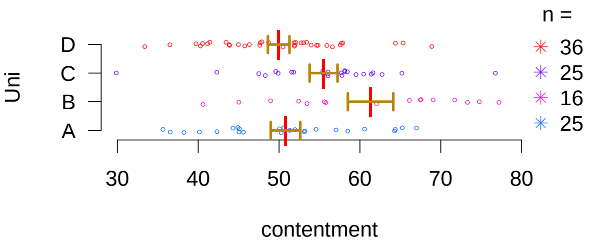

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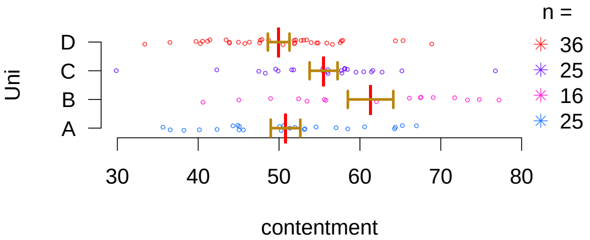

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groups	A/B	A/C	A/D	B/C	B/D	C/D
p-value	red	green	green	green	red	red

red: $p < 5\%$, green: $p \geqslant 5\%$

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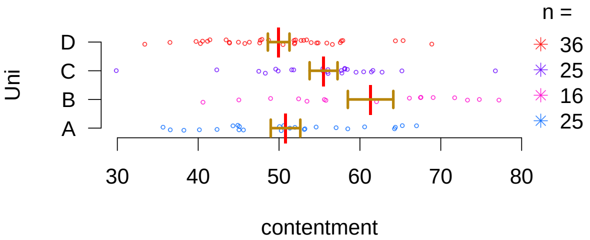

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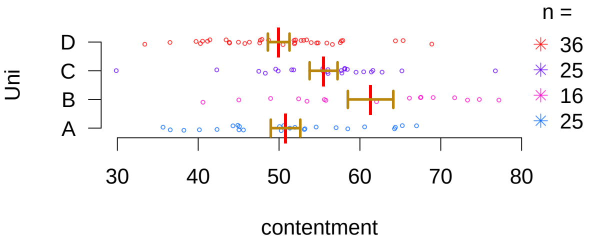

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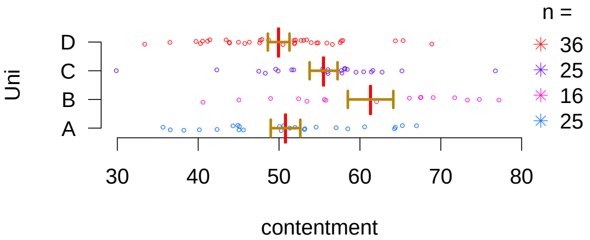

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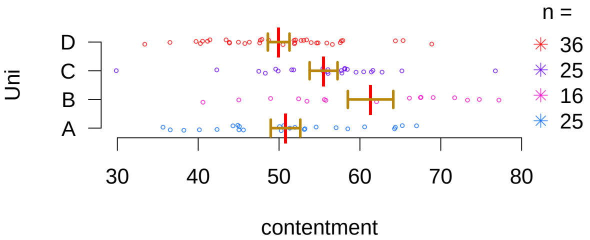

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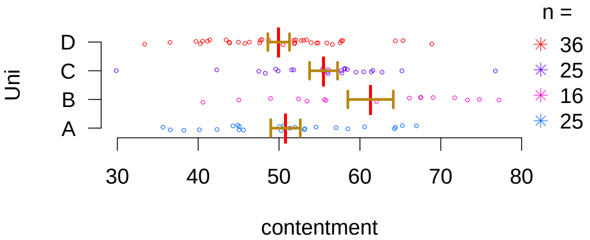

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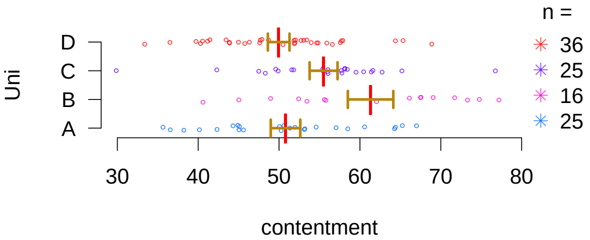

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- … and the application of <u>many statistical tests</u> brings a <u>conceptional problem</u> …

The multiple testing problem

gedankenexperiment

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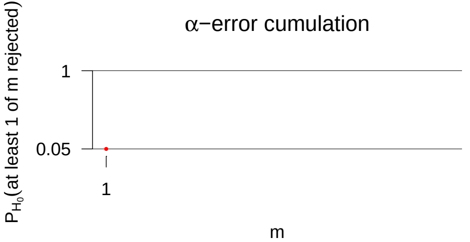

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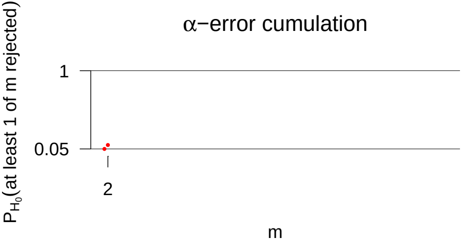

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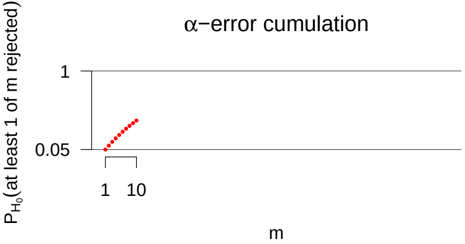

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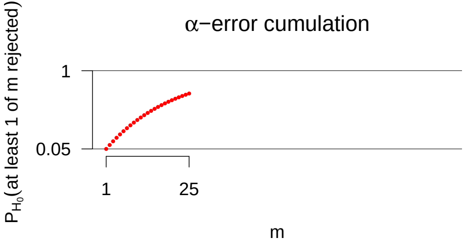

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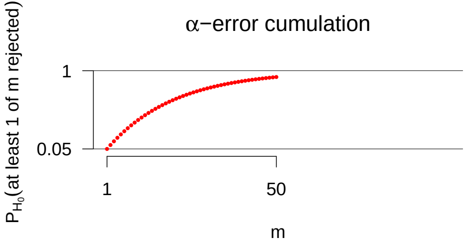

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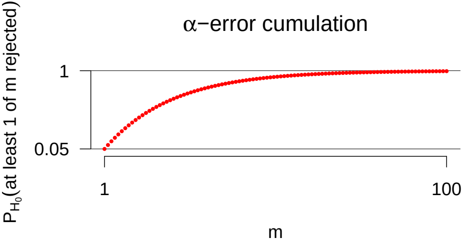

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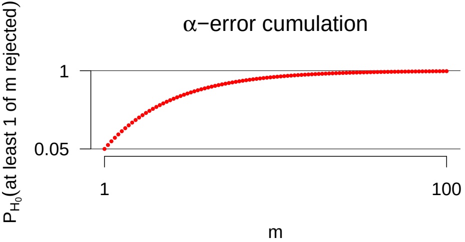

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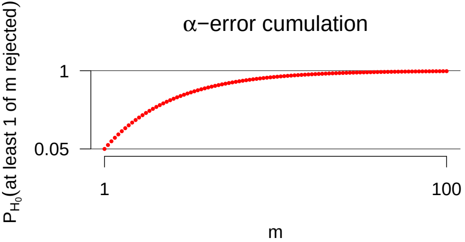

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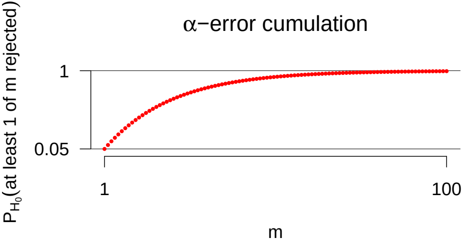

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- Meaning for practice: do not perform unnecessary tests!

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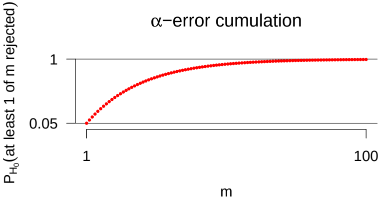

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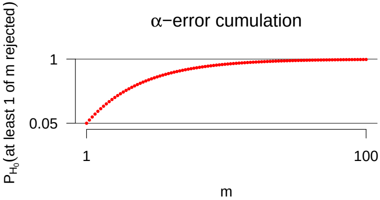

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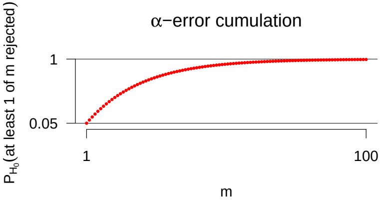

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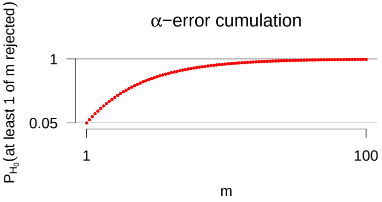

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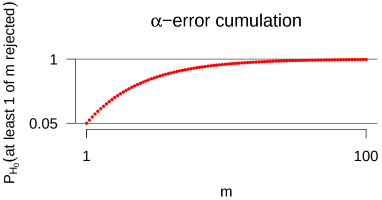

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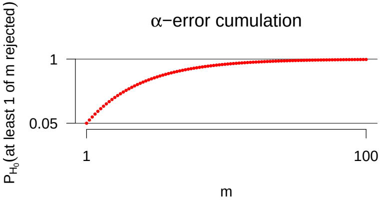

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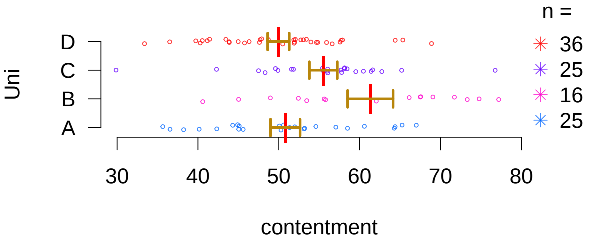
- Equivalent, as $p \leqslant \alpha^* \Leftrightarrow p^* \leqslant \alpha$

Multiple testing and Bonferroni-correction



groups	A/B	A/C	A/D	B/C	B/D	C/D
p-value p (prior Bonferroni)	0.004	0.067	0.710	0.090	0.001	0.014
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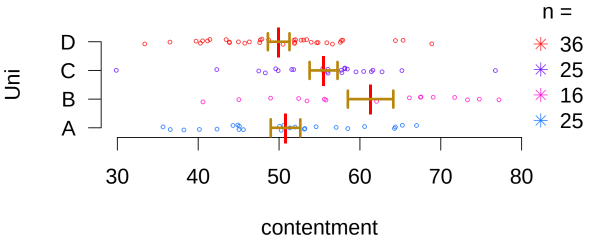

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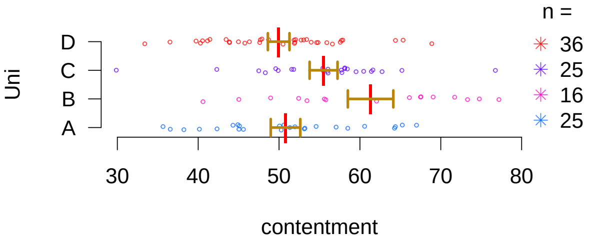

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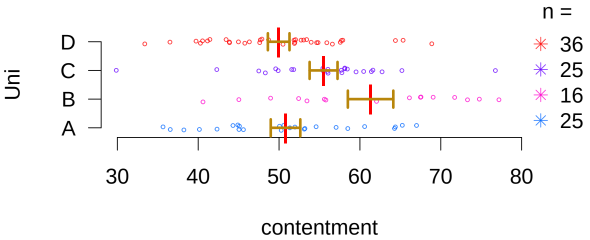

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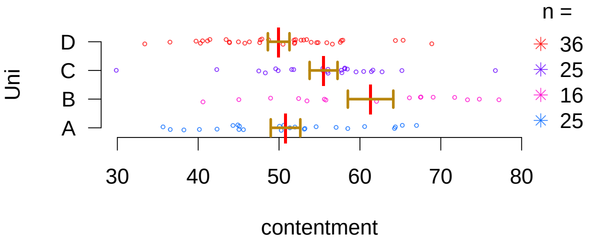

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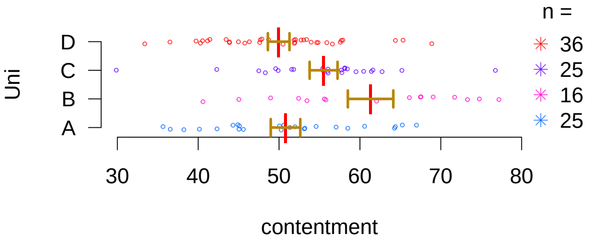

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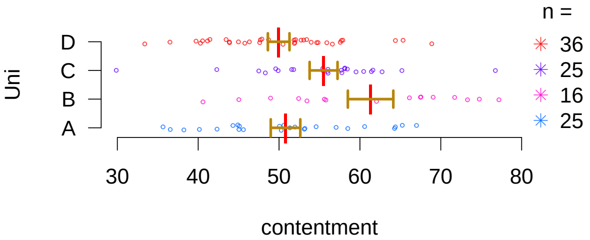

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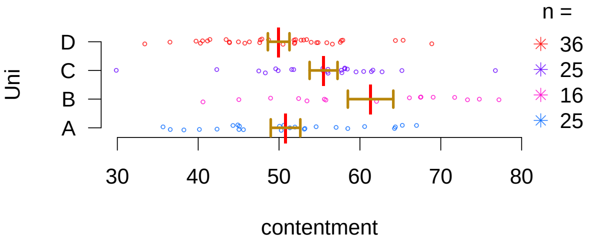

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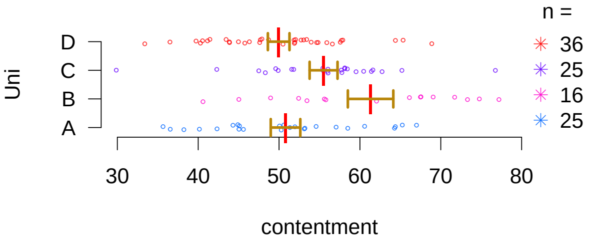

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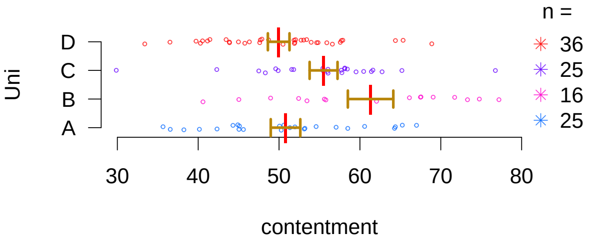
Analysis of variance



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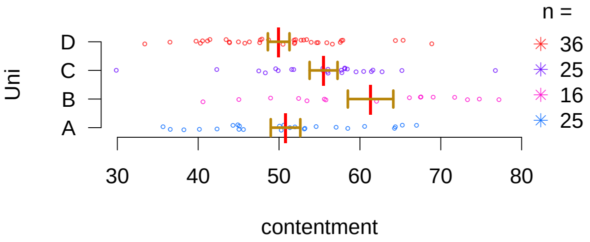

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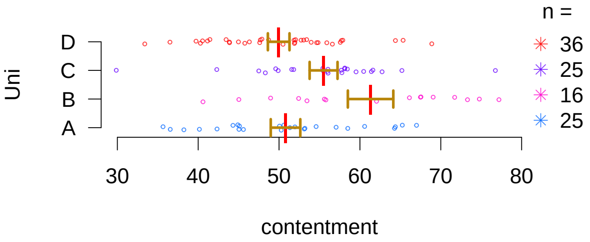

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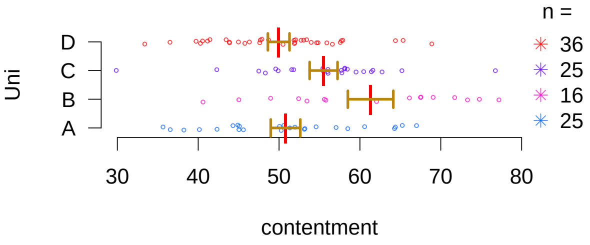

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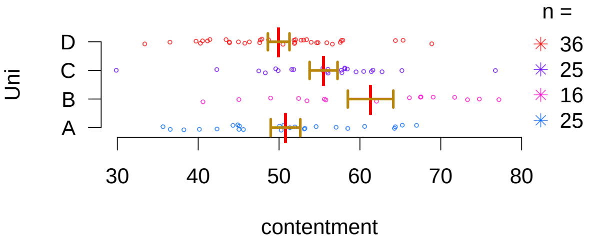

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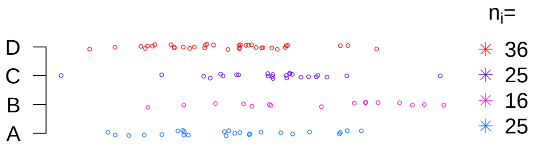
- Intuitively in the graphic: this is not very plausible, because the group means are partly far away (multiple sem) from each other

ANOVA - Model

Model:

Let $X_{1,1}, \ldots, X_{1,n_1}, \quad X_{2,1}, \ldots, X_{2,n_2}, \quad \ldots, \quad X_{k,1}, \ldots, X_{k,n_k}$ be independent RVs and for $i = 1, \ldots, k$ let $X_{i,j} \sim N(\mu_i, \sigma^2)$ for $j = 1, \ldots, n_i$, with $(\mu_1, \ldots, \mu_k, \sigma^2) \in \mathbb{R}^k \times \mathbb{R}^+$



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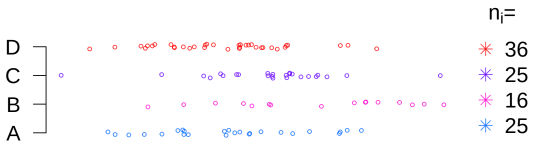
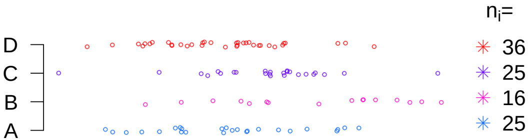


- Notation
  - k groups

ANOVA - Model

Model:

Let $X_{1,1},\ldots,X_{1,n_1},\quad X_{2,1},\ldots,X_{2,n_2},\quad \ldots,\quad X_{k,1},\ldots,X_{k,n_k}$ be independent RVs and for $i=1,\ldots,k$ let $X_{i,j}\sim N(\mu_i,\sigma^2)$ for $j=1,\ldots,n_i$, with $(\mu_1,\ldots,\mu_k,\sigma^2)\in\mathbb{R}^k\times\mathbb{R}^+$

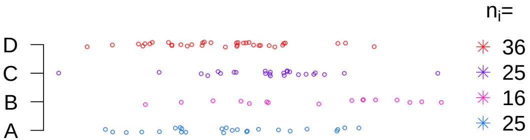


- Notation
  - k groups
  - n_i observations in the i-th group

ANOVA - Model

Model:

Let $X_{1,1}, \ldots, X_{1,n_1}, \quad X_{2,1}, \ldots, X_{2,n_2}, \quad \ldots, \quad X_{k,1}, \ldots, X_{k,n_k}$ be independent RVs and for $i = 1, \ldots, k$ let $X_{i,j} \sim N(\mu_i, \sigma^2)$ for $j = 1, \ldots, n_i$, with $(\mu_1, \ldots, \mu_k, \sigma^2) \in \mathbb{R}^k \times \mathbb{R}^+$



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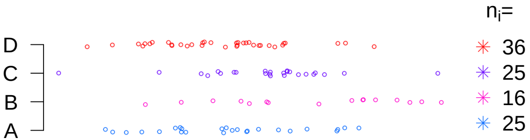

- Notation
  - k groups
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- Assumptions
  - independence

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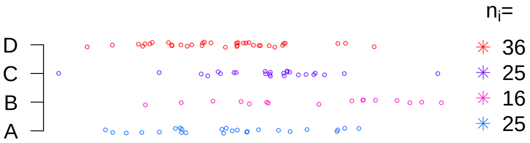

- Notation
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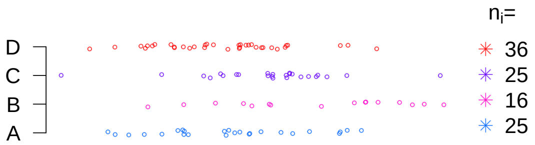

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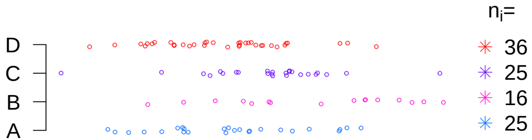

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  - the variance σ^2 is constant

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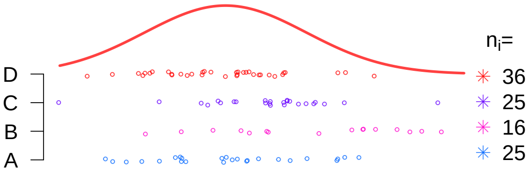

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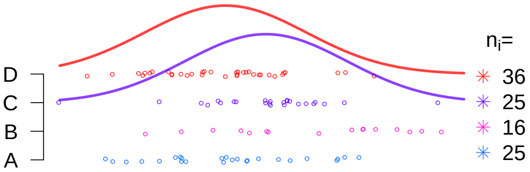

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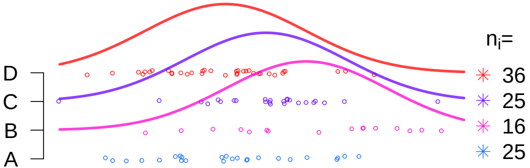

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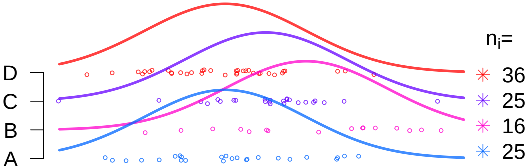

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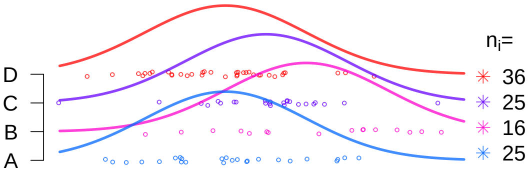

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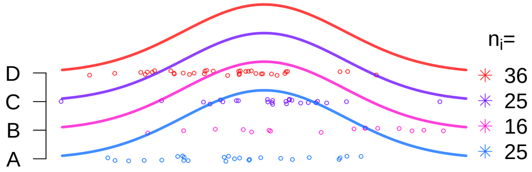
Note that for $k = 2$ groups, this model equals the model in Student's two-sample t-test.

ANOVA - Null hypothesis

Let $X_{1,1},\dots,X_{1,n_1},\;\; X_{2,1},\dots,X_{2,n_2},\;\; \dots,\;\; X_{k,1},\dots,X_{k,n_k}$ be independent RVs and for $i=1,\dots,k$ let $X_{i,j}\sim N(\mu_i,\sigma^2)$ for $j=1,\dots,n_i$, with $(\mu_1,\dots,\mu_k,\sigma^2)\in\mathbb{R}^k\times\mathbb{R}^+$



- Null hypothesis: $H_0:\mu_1=\mu_2=\cdots=\mu_k$
  the distributions are equal, in particular no shift

ANOVA - Null hypothesis

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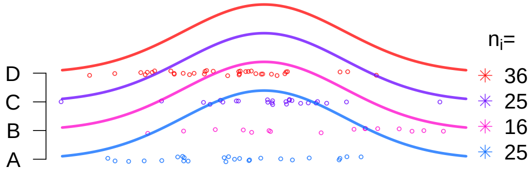

- Null hypothesis: $H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$
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- Alternative: at least two expectations are not equal, $\mu_i \neq \mu_\ell$

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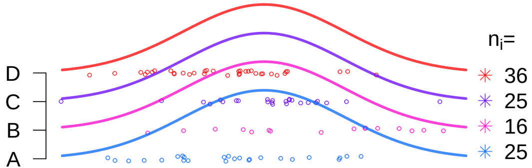
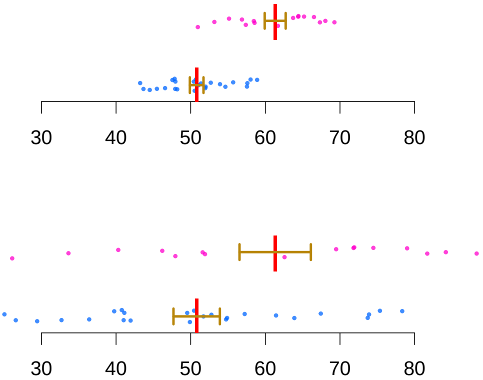


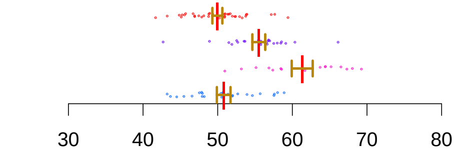
- Null hypothesis: $H_0 : \mu_1 = \mu_2 = \dots = \mu_k$
  the distributions are equal, in particular no shift
- Alternative: at least two expectations are not equal, $\mu_i \neq \mu_\ell$
- How can we test the null hypothesis?

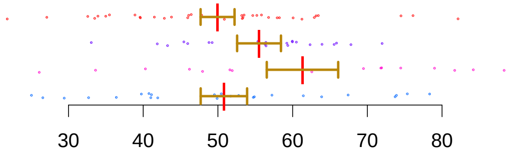
Reminder: *t*-Statistic



The *t*-statistic says:
the discrepancy <u>between</u> the groups is large,
if the variability <u>within</u> the groups is small

Idea of the ANOVA analog



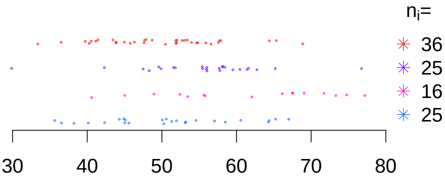


The f-statistic will tell us:
the variability <u>between</u> the groups is large,
if the variability <u>within</u> the groups is small

F-Statistic

Naive:

$$f \stackrel{!}{=} \frac{\text{'variability between the groups'}}{\text{'variability within the groups'}}$$



F-Statistic

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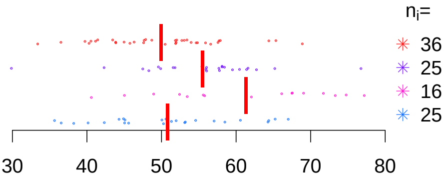

- Need 1.: group means, for $i = 1, \ldots, k$

$$\bar{X}_{i,\cdot} := \frac{1}{n_i} \sum_{j=1}^{n_i} X_{i,j}$$

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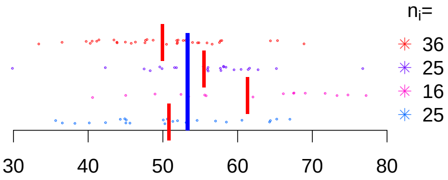


- Need 1.: group means, for $i = 1, \dots, k$

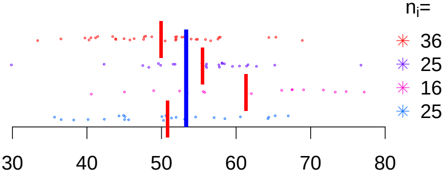
$$\bar{X}_{i,\cdot} := \frac{1}{n_i} \sum_{j=1}^{n_i} X_{i,j}$$

- Need 2.: global mean over all groups

$$\bar{X} := \frac{1}{n} \sum_{i,j} X_{i,j}$$

with $n = n_1 + \dots + n_k$ total number of observations

F-Statistic (google: R. A. Fisher)



- $\bar{X}_{i,\cdot} = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{i,j}$ and $\bar{X} = \frac{1}{n} \sum_{i,j} X_{i,j}$ with $n = \sum_{i=1}^{k} n_i$

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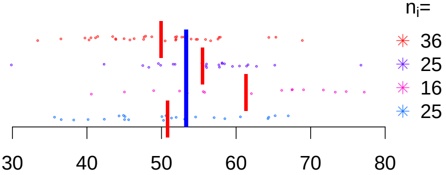

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- The F-Statistic is

$$\boxed{F := \frac{\frac{1}{k-1} \sum_{i=1}^{k} n_i (\bar{X}_{i,\cdot} - \bar{X})^2}{\frac{1}{n-k} \sum_{i,j} (X_{i,j} - \bar{X}_{i,\cdot})^2}}$$

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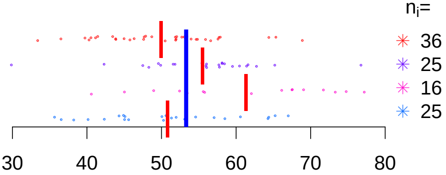

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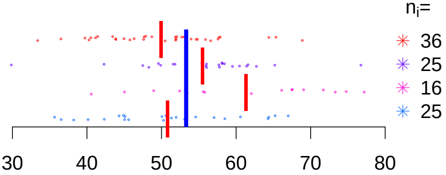

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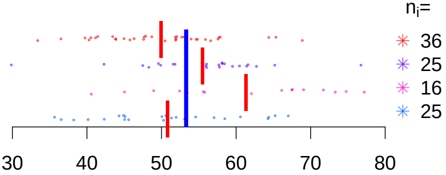
- $F \geqslant 0$, due to squaring

F-Statistic (google: R. A. Fisher)



- $\textcolor{red}{\bar{X}_{i,\cdot}} = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{i,j}$ and $\textcolor{blue}{\bar{X}} = \frac{1}{n} \sum_{i,j} X_{i,j}$ with $n = \sum_{i=1}^{k} n_i$
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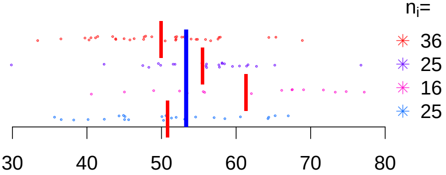
- $F \geqslant 0$, due to squaring
- If the evaluation of the data f is very large, this speaks against H_0

F-Statistic (google: R. A. Fisher)



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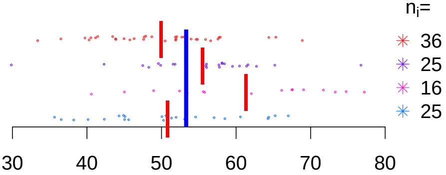
- $F \geqslant 0$, due to squaring
- If the evaluation of the data f is very large, this speaks against H_0
  - Because then the numerator is large compared to the denominator

F-Statistic (google: R. A. Fisher)



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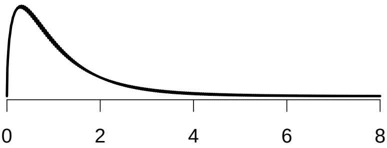
F-Statistic (google: R. A. Fisher)



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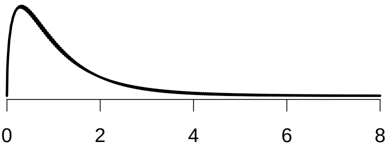

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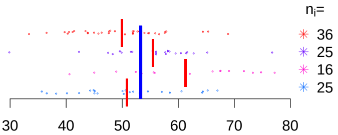
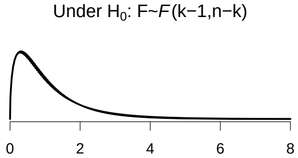
For more details on the $\mathcal{F}(k-1, n-k)$-distribution and the derivation of the distribution of F see e.g., Messer, M. and Schneider, G. *Statistik: Theorie und Praxis im Dialog*, Springer Berlin

F-test





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$$\boxed{F := \frac{\frac{1}{k-1}\sum_{i=1}^{k} n_i(\color{red}{\bar{X}_{i,\cdot}} - \color{blue}{\bar{X}})^2}{\frac{1}{n-k}\sum_{i,j}(X_{i,j} - \color{red}{\bar{X}_{i,\cdot}})^2}}$$

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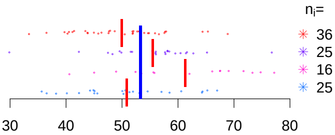
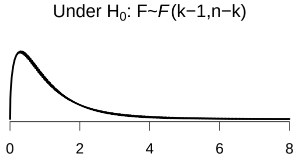
F-test





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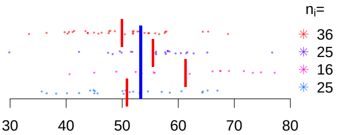

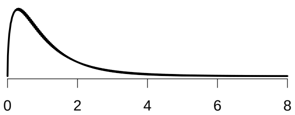

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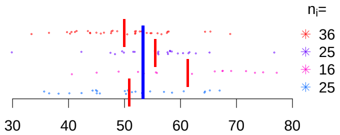

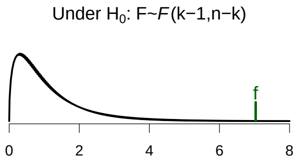

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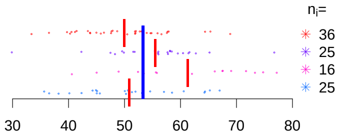

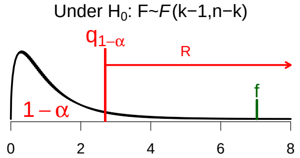

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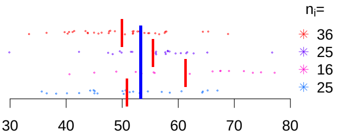

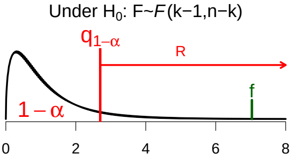

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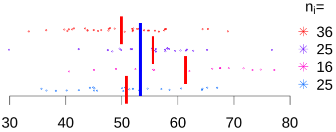

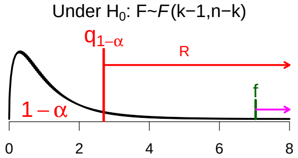

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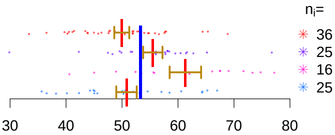

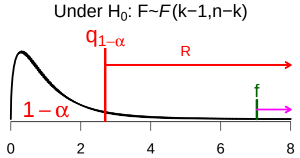

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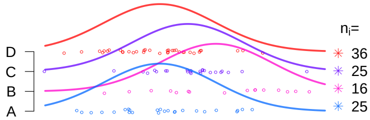
- Result plausible, given the variability (sem) of the means

ANOVA - Summary

Model:

- Let $X_{1,1},\ldots,X_{1,n_1},\ \ X_{2,1},\ldots,X_{2,n_2},\ \ \ldots,\ \ X_{k,1},\ldots,X_{k,n_k}$ be independent RVs and for $i=1,\ldots,k$ let $X_{i,j}\sim N(\mu_i,\sigma^2)$ for $j=1,\ldots,n_i$, with $(\mu_1,\ldots,\mu_k,\sigma^2)\in\mathbb{R}^k\times\mathbb{R}^+$



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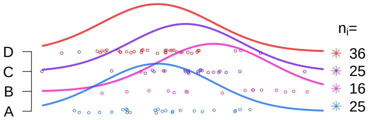
- Thus, k groups and $n = \sum_{i=1}^k n_i$ observations in total



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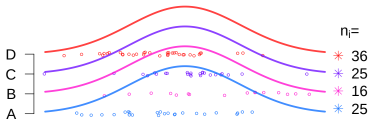
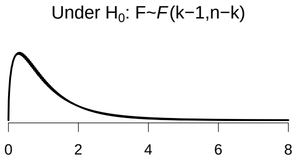
Under $H_0:\mu_1=\cdots=\mu_k$ it holds

$$F := \frac{\frac{1}{k-1}\sum_{i=1}^k n_i(\bar{X}_{i,\cdot}-\bar{X})^2}{\frac{1}{n-k}\sum_{i,j}(X_{i,j}-\bar{X}_{i,\cdot})^2}\sim\mathscr{F}(k-1,n-k)$$





ANOVA in R

```
# Sizes of groups as vector (k groups)
ni       <- c(n1,n2,...,nk)
# Data as vector, all from group 1, then all from group 2 etc..
x        <- c(...)
# Groups as factor variable
gr       <- factor(rep(1:k,ni))
# Perform ANOVA (F-test)
anova(aov(x~gr))
# Output
Analysis of Variance Table

Response: x
          Df Sum Sq Mean Sq F value   Pr(>F)
gr         3 1709.8  569.92  7.0332 0.000248 ***
Residuals 98 7941.2   81.03
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '' 1
```

- 1. column: degrees of freedom (first row numerator, second row denominator)
- 3. column: numerator and denominator of the f-statistic
- 4. column: f-statistic
- 5. column: p-value   (2.column $=$ 1.column $\cdot$ 3.column)

In practice

Procedure in practice often:

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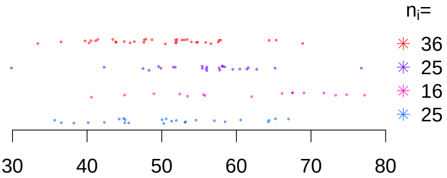
Rethink of whether the data comply with the model assumptions of the ANOVA:

- Independence and own expectation per group
  $\rightarrow$ makes sense due to the practical setup
- normal distribution and constant variance (between the groups)
  $\rightarrow$ plausible when looking at the data



Thank you!