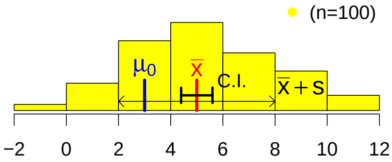
Surrounding the one-sample *t*-test



For comments please email: M. Messer, michael.messer@tuwien.ac.at

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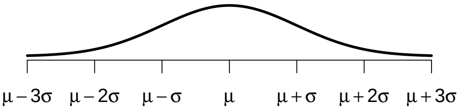

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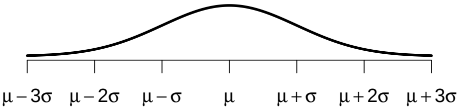

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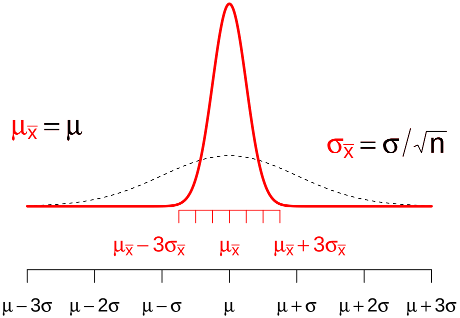

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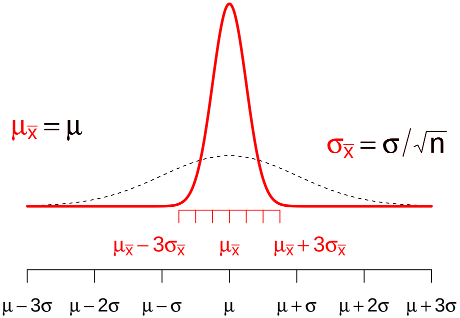

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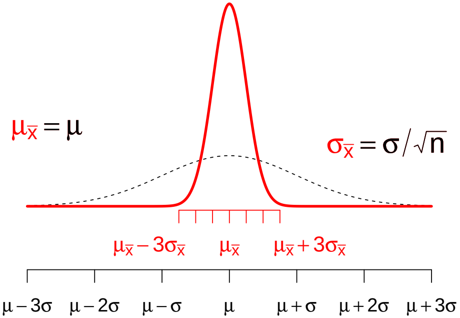
    Interpretation: the typical deviation of the mean from its expectation is $\sigma/\sqrt{n}$



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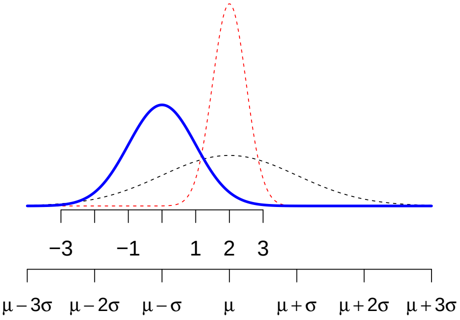
- Standardization: $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$



From the z-test to the t-test

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- Plug in estimator S

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$$\boxed{\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1)}$$

t-distributed with $n-1$ degrees of freedom.

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t-distribution

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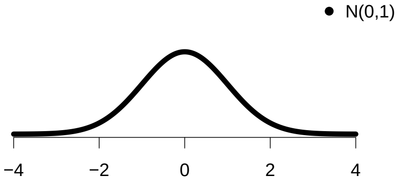

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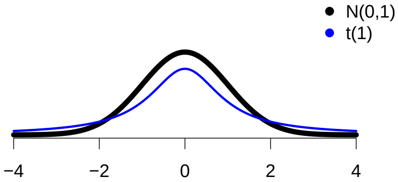

The $t(n)$-distribution ($n \in \{1, 2, \dots\}$)

- Definition: $X \sim t(n) :\Leftrightarrow X$ has density $f(x) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi} \cdot \Gamma(n/2)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}, x \in \mathbb{R}$

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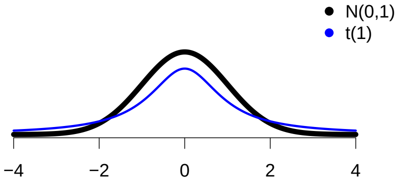

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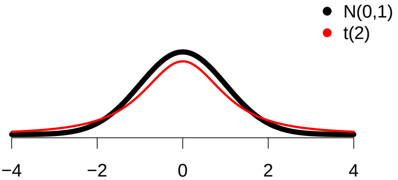

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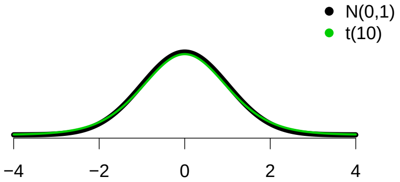

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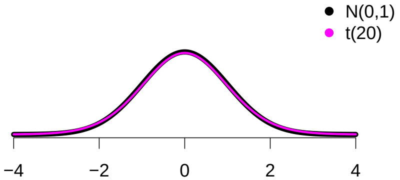

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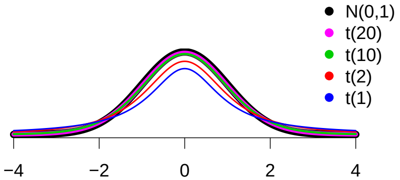

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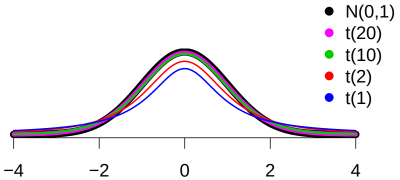

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- $t(n)$ has a one parameter: degrees of freedom n,
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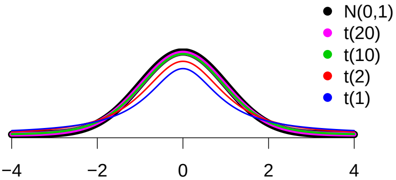

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- $t(n)$ symmetric around 0
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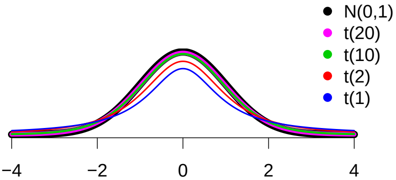

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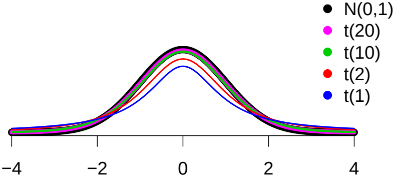

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- $t(n)$ has heavier tails (polynomial) than the $N(0,1)$-distribution (exponential)
  - Intuition: estimation of σ via S increases the variability of the rescaled mean
- the density of $t(n)$ converges pointwise to the density of $N(0,1)$

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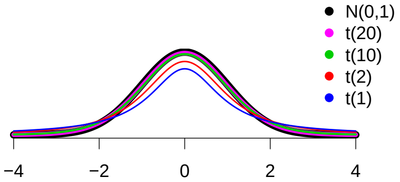

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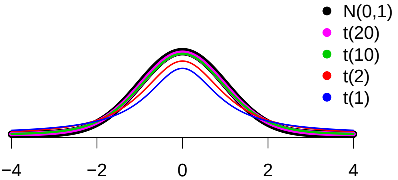

The $t(n)$-distribution ($n \in \{1, 2, \ldots\}$)

- Definition: $X \sim t(n) :\Leftrightarrow X$ has density $f(x) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\cdot\Gamma(n/2)}\left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}, x \in \mathbb{R}$
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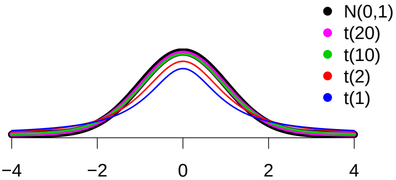

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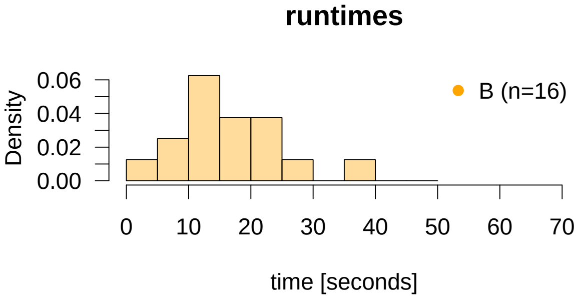
From what is said, it is (hopefully) plausible, but not rigorously proven, that $\frac{\bar{X} - \mu}{S / \sqrt{n}} \sim t(n-1)$.

For details see e.g., Messer, M. and Schneider, G. *Statistik: Theorie und Praxis im Dialog*, Springer Berlin

Example from last lecture

Reminder:

- Runtimes of an algorithm implemented by $n = 16$ students that took a certain programming class.



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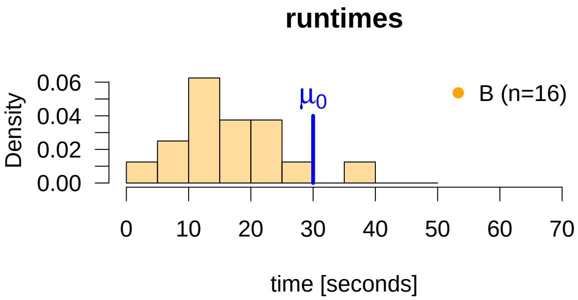

- Delicate assertion from a colleague of the lecturer: " The course was held by the lecturer a couple of times before. If all participants that have ever taken the course had implemented this algorithm, then the mean runtime would have been $\mu_0 = 30$

From z-test to the t-test

- Assertion about a huge ′population′

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- Note that T is a proper *statistic* in the sense that it is a bare function of the random variables / the data. Particularly, it does not depend on unknown parameters

The (one-sample) t-Test according to 'Student' (google: W.S. Gosset)

- *Set significance level*: Choose (e.g.,) $\alpha = 5\%$
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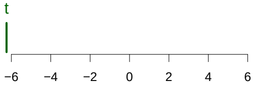

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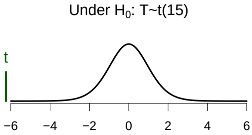

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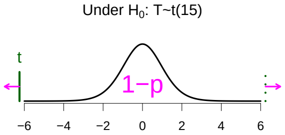

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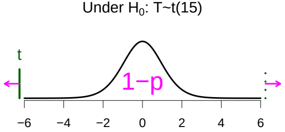

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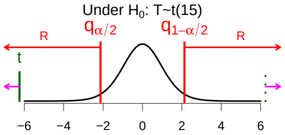

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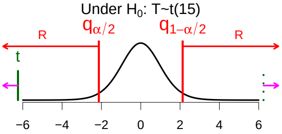

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- *Interpretation*: If H_0 holds true, then something very unlikely was observed. In that sense, the data are hardly compatible with H_0.

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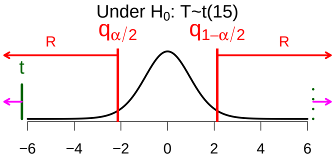

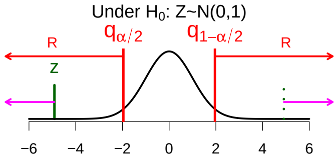

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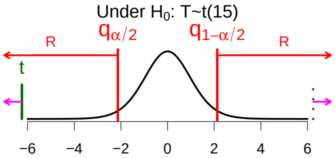

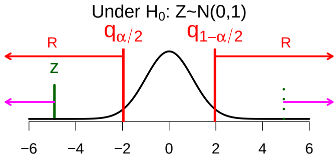

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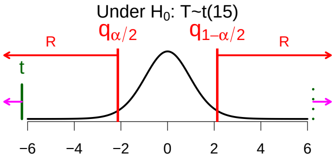

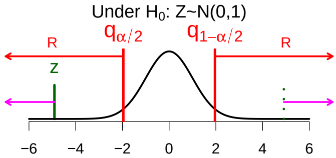

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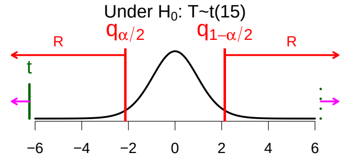

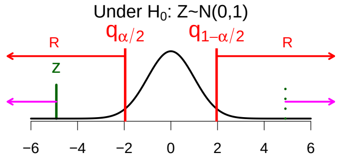

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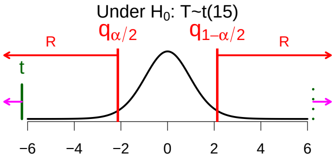

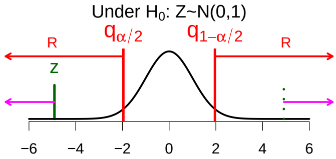

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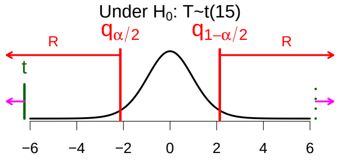

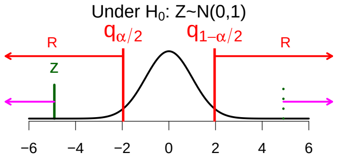

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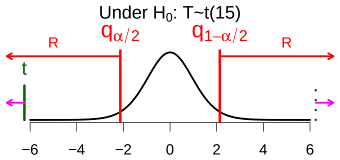
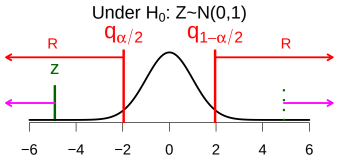
  - However: p-value in t-test ($\approx 1.5 \cdot 10^{-5}$) larger than p in z-test ($\approx 9 \cdot 10^{-7}$)
  - No contradiction, because the tails of the t-distribution are heavier than those of $N(0, 1)$ (visually not observable in the graphics)
  - For the same reason: in the t-test slightly smaller rejection area R: for $\alpha = 5\%$ it holds first regarding $t(15)$ that $q_{\alpha/2} = -2.13$, and second regarding $N(0, 1)$ that $q_{\alpha/2} = -1.96$ (while $q_{\alpha/2}$ denotes the $\alpha/2$-quantiles)





$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \approx \frac{16.5 - 30}{8.7/4} \approx -6.2 \qquad z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \approx \frac{16.5 - 30}{11/4} \approx -4.9$$

The standard error of the mean

- Let $X_1, \ldots, X_n$ be i.i.d. RVs and $\sigma^2 = \mathbb{V}ar(X_1) \in (0, \infty)$

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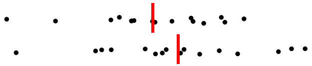

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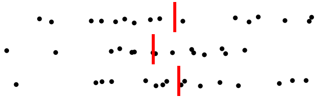

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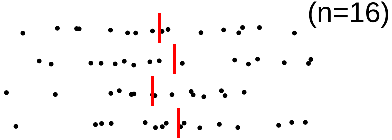

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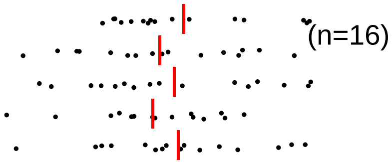

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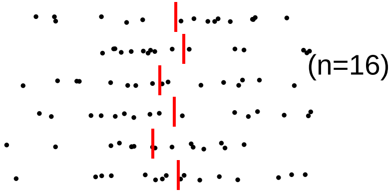

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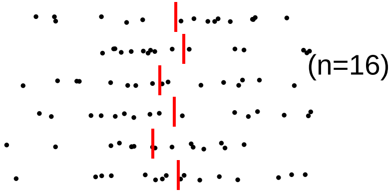

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$$\boxed{\underline{\text{Definition:}} \qquad SEM := \frac{S}{\sqrt{n}}}$$

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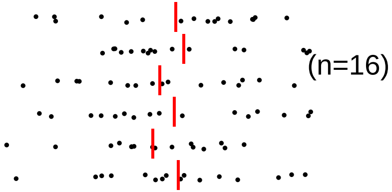

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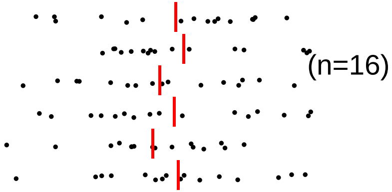
Standard Error of the Mean

- Meaning: the estimated variability of the mean



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Standard Error of the Mean

- Meaning: the estimated variability of the mean
- The t-statistic measures the discrepancy $\bar{X} - \mu_0$ in the units SEM

$$\boxed{T = \frac{\bar{X} - \mu_0}{SEM}}$$

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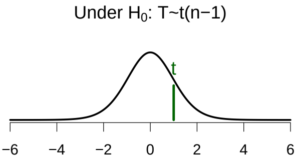

- Intuition: If $H_0 : \mu = \mu_0$ holds true
  - $|t| = 1 \Leftrightarrow |\bar{x} - \mu_0| = 1 \cdot sem$
    it is not unlikely to observe a discrepancy of <u>one</u> *sem*

The standard error of the mean

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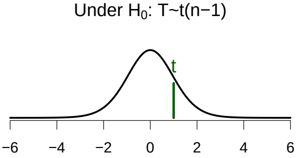
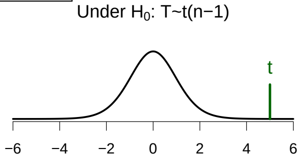


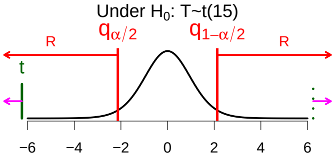


- Intuition: If $H_0 : \mu = \mu_0$ holds true
  - $|t| = 1 \Leftrightarrow |\bar{x} - \mu_0| = 1 \cdot sem$
    it is not unlikely to observe a discrepancy of <u>one</u> *sem*
  - $|t| = 5 \Leftrightarrow |\bar{x} - \mu_0| = 5 \cdot sem$
    it is very unlikely to observe a discrepancy of <u>five</u> *sem*
    ’chance has a hard time realizing this’

From the standard error to the confidence interval

- Let $X_1, \ldots, X_n$ be i.i.d. RVs and $X_1 \sim N(\mu, \sigma^2)$

$$\boxed{T = \frac{\bar{X} - \mu_0}{SEM}} \qquad SEM = \frac{S}{\sqrt{n}}$$



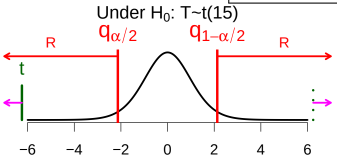
- In our example: $t = \frac{\bar{x} - \mu_0}{sem} \approx \frac{16.5 - 30}{8.7/4} \approx -6.2$

  it is very unlikely to observe a discrepancy $|\bar{x} - \mu_0|$ of $6.2 \cdot sem$ under H_0.

From the standard error to the confidence interval

- Let $X_1, \ldots, X_n$ be i.i.d. RVs and $X_1 \sim N(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$ and $\sigma^2 > 0$

$$\boxed{T = \frac{\bar{X} - \mu_0}{SEM}} \qquad\qquad SEM = \frac{S}{\sqrt{n}}$$



$$(*) \qquad\qquad \alpha = \mathbb{P}_{H_0}(T \in R)$$

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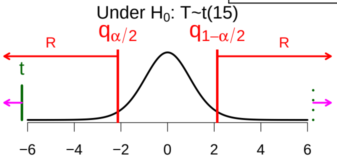

$$(*) \qquad \begin{aligned} \alpha &= \mathbb{P}_{H_0}(T \in R) \\ &= \mathbb{P}_{H_0}(|\bar{X} - \mu_0| \geqslant q_{1-\alpha/2} \cdot SEM) \end{aligned}$$

($q_{1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of $t(n-1)$)

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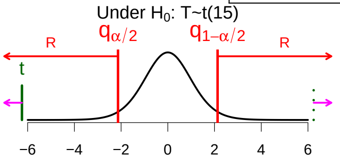

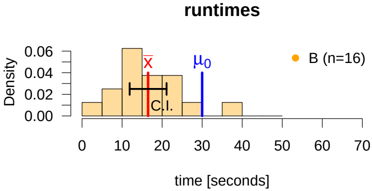

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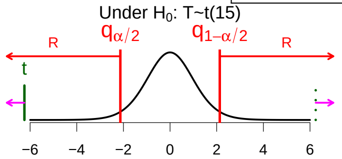

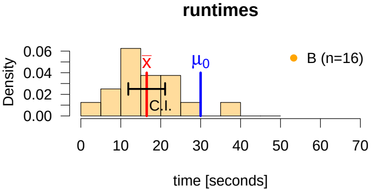

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($q_{1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of $t(n-1)$)

- Meaning: Under H_0, the interval

$$I := (\bar{X} - q_{1-\alpha/2} \cdot SEM, \bar{X} + q_{1-\alpha/2} \cdot SEM)$$

overlaps the parameter μ_0 with probability $1 - \alpha$

From the standard error to the confidence interval

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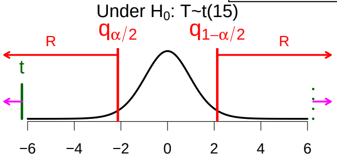

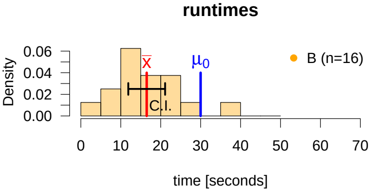

$$
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(*) \qquad \alpha &= \mathbb{P}_{H_0}(T \in R) \\
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- I is called a $(1-\alpha)$-*confidence interval* for μ (abbreviate: C.I.)

Confidence interval

- Let $X_1, \ldots, X_n$ be i.i.d. RVs and $X_1 \sim N(\mu, \sigma^2)$,with $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, and let $q_{1-\alpha/2}$ be the $(1-\alpha/2)$-quantile of $t(n-1)$

Under $H_0 : \mu = \mu_0$, the confidence interval

$$I := \left(\bar{X} - q_{1-\alpha/2} \cdot SEM,\ \bar{X} + q_{1-\alpha/2} \cdot SEM\right)$$

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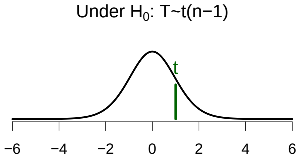

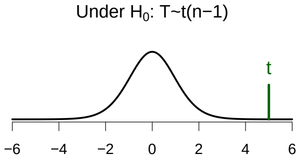

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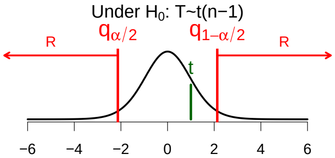

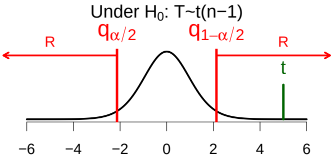

- $t \in R$
  Reject H_0

Confidence interval

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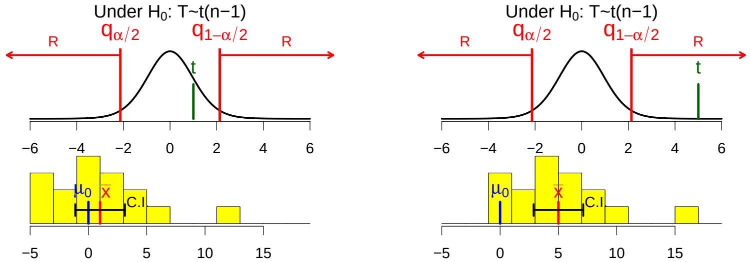

- $t \in R \Leftrightarrow i \not\ni \mu_0$ (see $(*)$ in previous slide)
  Reject H_0 if and only if μ_0 is not overlapped by the confidence interval i

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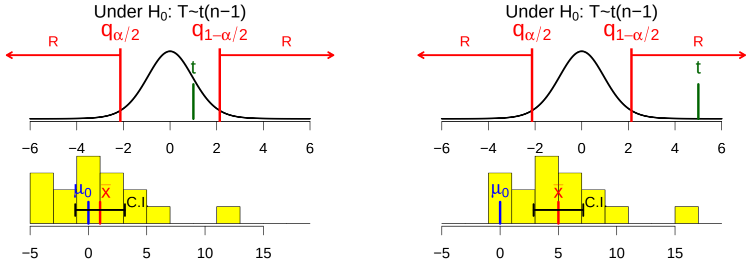

- $t \in R \Leftrightarrow i \not\ni \mu_0$ (see $(*)$ in previous slide)
  Reject H_0 if and only if μ_0 is not overlapped by the confidence interval i
- Meaning: Equivalence of test and confidence interval ($\rightarrow$ 'Student's C.I.')

Confidence interval

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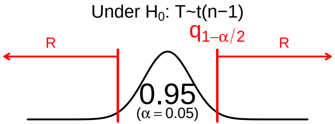

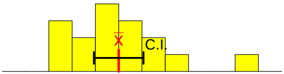

- Decrease of α
  $\leftrightarrow$ increase of $q_{1-\alpha/2}$

Confidence interval

- Let $X_1, \ldots, X_n$ be i.i.d. RVs and $X_1 \sim N(\mu, \sigma^2)$,with $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, and let $q_{1-\alpha/2}$ be the $(1-\alpha/2)$-quantile of $t(n-1)$

Under $H_0 : \mu = \mu_0$, the confidence interval

$$I := \left(\bar{X} - q_{1-\alpha/2} \cdot SEM,\ \bar{X} + q_{1-\alpha/2} \cdot SEM\right)$$

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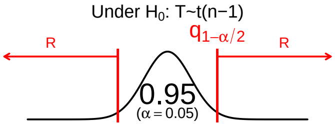

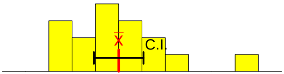

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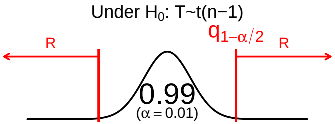

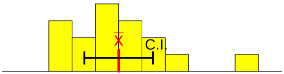

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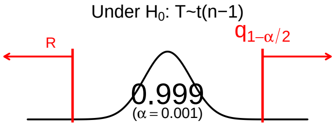

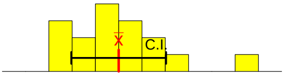

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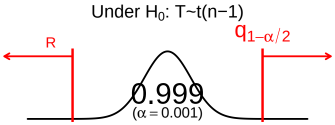

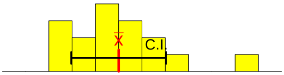

- Decrease of α
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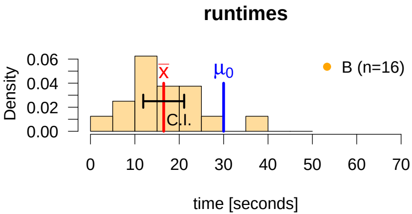
  Plausible: 'If μ_0 shall be overlapped with large probability, then I must be large'

Interpretation

- Delicate assertion from a colleague of the lecturer: " The course was held by the lecturer a couple of times before. If all participants that have ever taken the course had implemented this algorithm, then the mean runtime would have been $\mu_0 = 30'$



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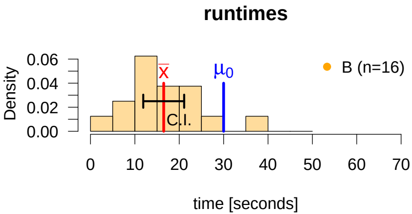

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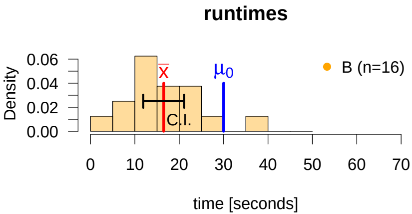

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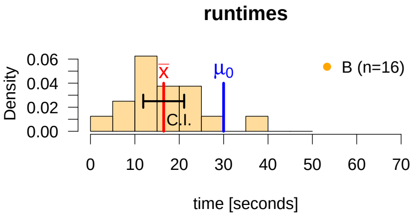

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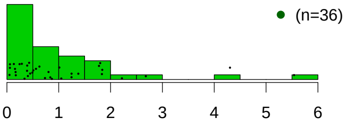
- In that sense the data are barely compatible with the assertion
- Intuitively: $q_{1-\alpha/2} \approx 2.13 \rightarrow$ the c.i. has a diameter of about $4sem \rightarrow$ the distance $|\bar{x} - \mu_0|$ is about $6sem$. That is a large distance if we take in account that the typical deviation of $\bar{X}$ from μ_0 is about $1SEM$

Data not bell-shaped



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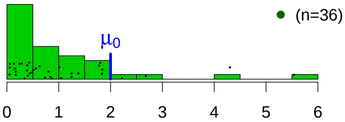

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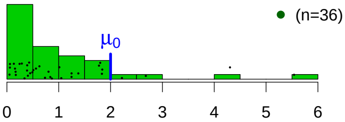

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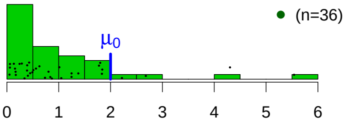

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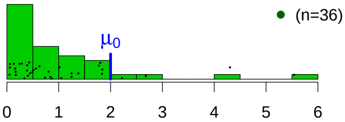

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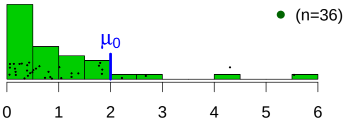

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- Way out: asymptotic normality of the mean

Reminder

- How is the mean distributed under normal distribution?
- Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables and $X_1 \sim N(\mu, \sigma^2)$
- For the mean it holds $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N(\mu, \sigma^2/n)$
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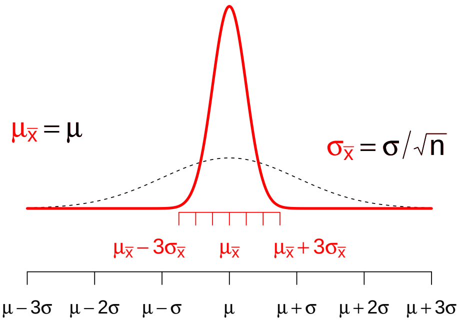
    Interpretation: the typical deviation of the mean from its expectation is $\sigma/\sqrt{n}$



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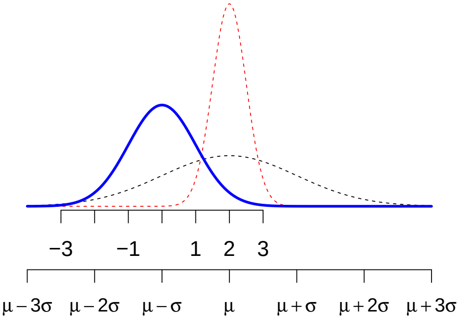

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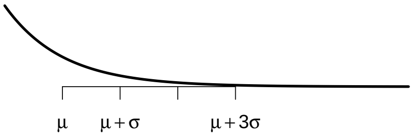

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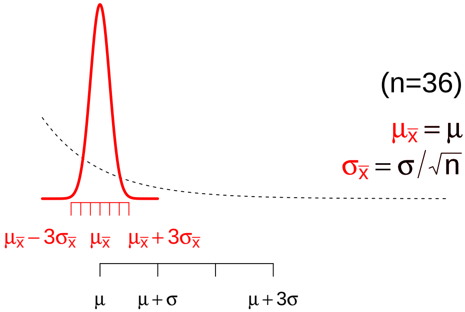

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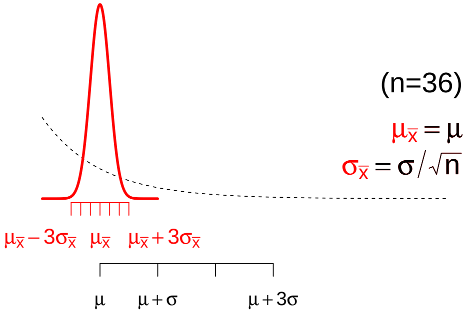

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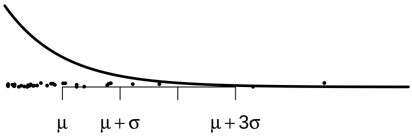

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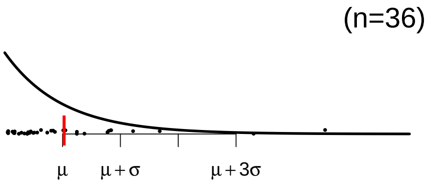

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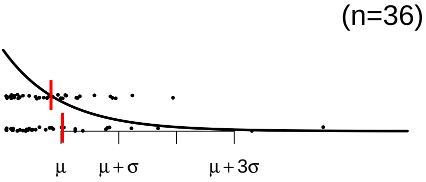

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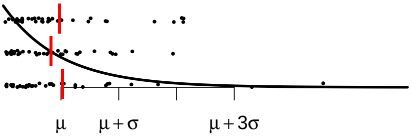

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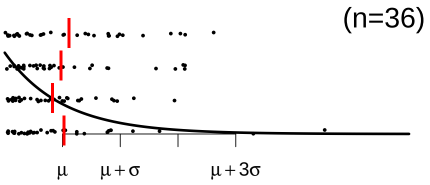

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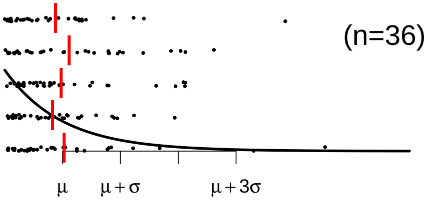

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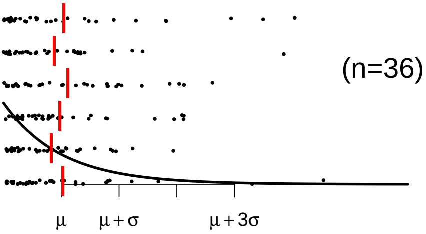

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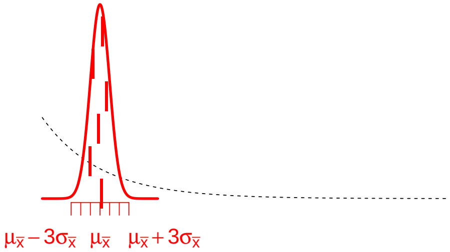

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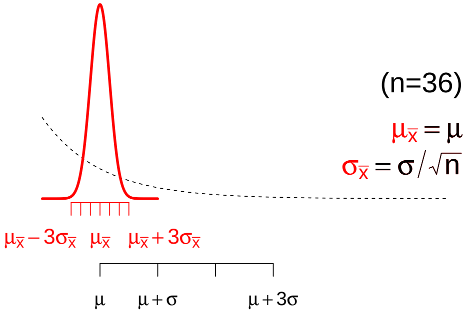
  - $\bar{X}$ has expectation $\mu_{\bar{X}} = \mu$
  - $\bar{X}$ has standard deviation $\sigma_{\bar{X}} = \sigma/\sqrt{n}$
- Standardization: $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$ for $n \to \infty$ (central limit theorem)



Asymptotic normality of the mean

- How is the mean distributed for a large sample size n?
- Let $X_1, \ldots, X_n$ i.i.d. random variables and $\mathbb{V}ar(X_1) \in (0, \infty)$ (not necessarily normally distributed!)
- Then $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \overset{d}{\approx} N(\mu, \sigma^2/n)$ approximately
  - $\bar{X}$ is approximately normally distributed
  - $\bar{X}$ has expectation $\mu_{\bar{X}} = \mu$
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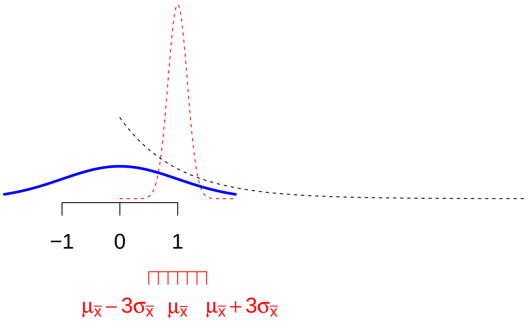


Asymptotic one-sample test and confidence interval

- Let $X_1, \dots, X_n$ be i.i.d. RVs with $\mu \in \mathbb{R}$ and $\mathbb{V}ar(X_1) \in (0, \infty)$, and let $q_{1-\alpha/2}$ be the $(1-\alpha/2)$-quantile of $N(0,1)$

Under $H_0 : \mu = \mu_0$ it approximatively holds for large n

$$T = \frac{\bar{X} - \mu_0}{SEM} \overset{d}{\approx} N(0,1)$$

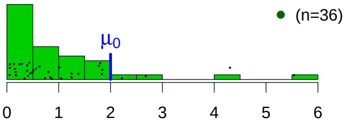
and equivalently: the confidence interval

$$I := \left(\bar{X} - q_{1-\alpha/2} \cdot SEM,\ \bar{X} + q_{1-\alpha/2} \cdot SEM\right)$$

overlaps the parameter μ_0 with probability about $1 - \alpha$

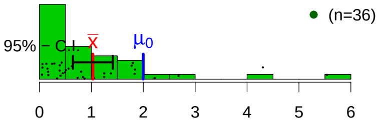
- i.e.,: test and confidence interval are constructed according to 'Student'. Only difference: Use the quantiles of $N(0,1)$, instead of $t(n-1)$

Approximate procedure



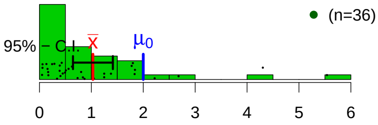
- Here: $n = 36$ data
- $H_0 : \mu = \mu_0$

Approximate procedure



- Here: $n = 36$ data
- $H_0 : \mu = \mu_0$
- Approximate 95%-confidence interval does not overlap μ_0

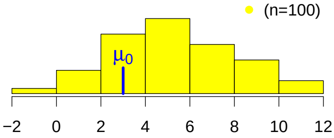
Approximate procedure



- Here: $n = 36$ data
- $H_0 : \mu = \mu_0$
- Approximate 95%-confidence interval does not overlap μ_0
  $\rightarrow$ reject H_0 on the 5% level

Question

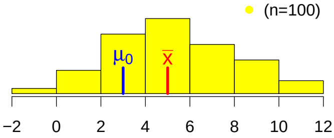
Can $H_0 : \mu = \mu_0 = 3$ be rejected on the 5%-level?



Here *t*-test naively:

Question

Can $H_0 : \mu = \mu_0 = 3$ be rejected on the 5%-level?



Here t-test naively:

- $\bar{x} \approx 5$ (balance in equilibrium)

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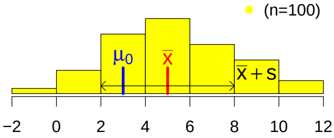

Here t-test naively:

- $\bar{x} \approx 5$ (balance in equilibrium)
- $s \approx 3$ (bell-shaped distribution, 2/3 of the data captured)

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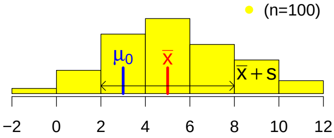

Here *t*-test naively:

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- *sem* $\approx 3/10$ (100 data points)

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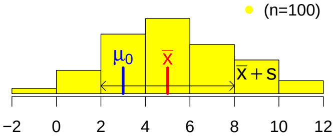

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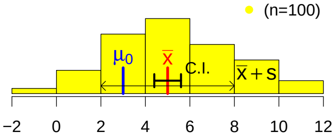

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- Together: C.I. $i \approx (4.4, 5.6)$ ($\bar{x} \pm q \cdot sem$)

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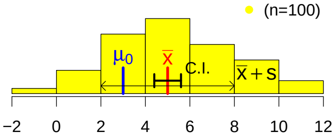

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- Together: C.I. $i \approx (4.4, 5.6)$ ($\bar{x} \pm q \cdot sem$)
- Hence: Reject H_0 (C.I. does not overlap μ_0)

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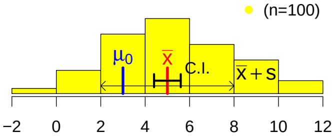

Here t-test naively:

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- Together: C.I. $i \approx (4.4, 5.6)$ ($\bar{x} \pm q \cdot sem$)
- Hence: Reject H_0 (C.I. does not overlap μ_0)

Of course, this is not 'precise', but the important message is that $\bar{x}$ is further than $6sem$ away from μ_0. This is extremely far! More precise estimates would have yielded the same massage!

t-test in R

```
# Enter data
x <- c(...)
# perform t-test
t.test(x,mu=3,...)
# Output

        One Sample t-test

data:  x
t = 8.3512, df = 99, p-value = 4.22e-13
alternative hypothesis: true mean is not equal to 3
95 percent confidence interval:
 4.745578 5.833572
sample estimates:
mean of x
5.289575
```

- Our naive estimates were plausible
  - Mean and C.I. fit well
  - $\bar{x}$ is even further than 8*sem* apart from μ_0

Thank you!