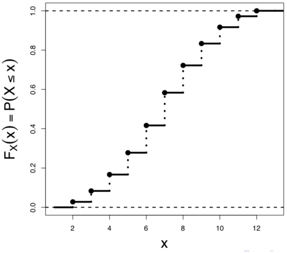
Random variables and distributions



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    - Probability table

Elementary events ω_i	ω_1	ω_2	$\dots$	ω_n	$\dots$	
Probability p_i	p_1	p_2	$\dots$	p_n	$\dots$	$\sum_i p_i = 1$

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- We now introduce random variables.

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  - Game: We win \$500 if the sum is 7, otherwise we lose \$100.
    - We denote this payoff function by

$$X(i,j) = \begin{cases} 500 & \text{if } i+j=7 \\ -100 & \text{if } i+j \neq 7. \end{cases}$$

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A random variable assigns a number to each outcome in a sample space.

Random variables

- A (measurable) function
$$X : \Omega \to \mathbb{R}$$
which assigns a real number $X(\omega) = x$ to every $\omega \in \Omega$
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    - The number $x \in \mathbb{R}$ is called the realization of X.
    - The set of values of X is the image space (or feature space)
    $$\{ x \mid X(\omega) = x, \ \omega \in \Omega \}$$

Random variables

- Two different types of random variables:
  - Discrete
    - If the image space is finite or countably infinite, the random variable X is discrete.
  - Continuous
    - A random variable X that can take values in one or more intervals, i.e. values that are infinite and uncountable are called continuous.

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  - Example:
    - The number of cash registers opened in a grocery store is a discrete random variable.

      The time spent waiting in the queue is a continuous random variable.

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  - M is a discrete random variable. We describe it by listing its possible values and the probabilities associated with those values.

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pmf	$p(a)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

$$\sum_{a=1}^{6} p(a) = 1$$

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- A random variable X is continuous if there exists a function f such that for any $c \leqslant d$ it holds

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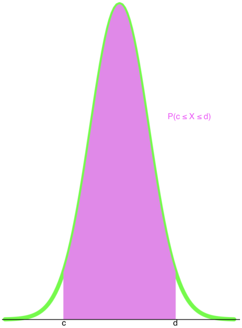
  - Properties:
    - f is nonnegative: $\quad f(x) \geqslant 0$
    - The area under f is one: $\quad \int_{-\infty}^{+\infty} f(x)\, dx = 1$

Continuous random variables

- Visualization

$$P(c \leqslant X \leqslant d) = \int_c^d f(x)\,dx$$
$$= \text{Area under } f \text{ between } c \text{ and } d$$



Probability density function and probability

Continuous vs. discrete random variables

- The probability density function $f(x)$ of a continuous random variables (pdf) is the analogue to the probability mass function $p(x)$ of a discrete random variable (pmf).

- Important differences:

  - In contrast to $p(x)$, the probability density function $f(x)$ is not a probability. We have to integrate it to get probability.

  - Since $f(x)$ is not a probability, there is no restriction that $f(x)$ is less than or equal to 1.

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- A cumulative distribution function satisfies the following properties:
  (1) $0 \leqslant F(x) \leqslant 1$ for all $x \in \mathbb{R}$
  (2) F is monotonically increasing, i.e. from $x \leqslant y$ it follows $F(x) \leqslant F(y)$
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  - $P(a \leqslant X \leqslant b) = F(b) - F(a)$.

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  we get, for example

  $F(9) = 1, \quad F(-2) = 0, \quad F(2.5) = \frac{4}{36} \quad \text{and} \quad F(\pi) = \frac{9}{36}.$

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  - We summarize, the cdf of M is of the following form

$$F(a) = F_M(a) = \begin{cases} 0 & a < 1 \\ \frac{1}{36} & 1 \leqslant a < 2 \\ \frac{4}{36} & 2 \leqslant a < 3 \\ \frac{9}{36} & 3 \leqslant a < 4 \\ \frac{16}{36} & 4 \leqslant a < 5 \\ \frac{25}{36} & 5 \leqslant a < 6 \\ 1 & a \geqslant 6 \end{cases}.$$

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cdf	$F(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{25}{36}$	1	

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- e.g. $F(5.3) = P(X \leqslant 5.3)$
  $= P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) = \frac{10}{36}$

Example: Sum of two dice

- The probability mass function

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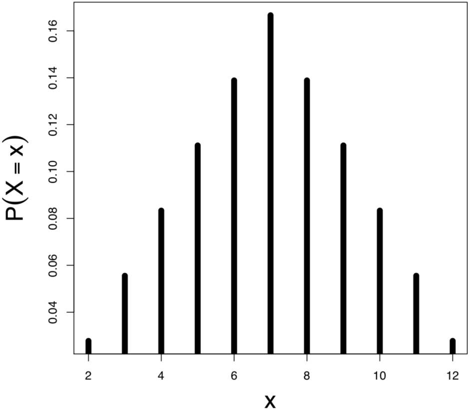

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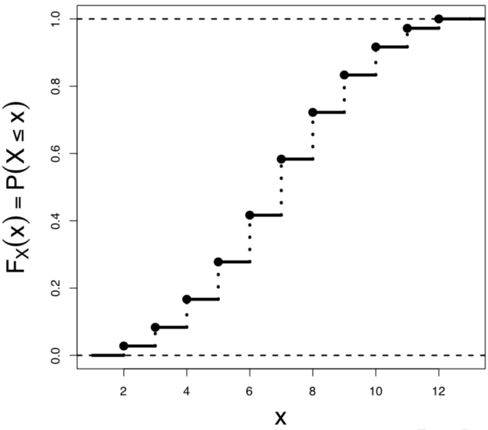
- The graph of the cumulative distribution function is a step function.



Continuous random variable: Cumulative distribution function

- The cumulative distribution function of a continuous random variable X is given by

$$F(x) = P(X \leqslant x) = \int_{-\infty}^{x} f(t)\,dt$$

- $P(a \leqslant X \leqslant b) \ = \ F(b) \ - \ F(a)$
$$= \int\limits_{-\infty}^{b} f(x)dx - \int\limits_{-\infty}^{a} f(x)dx = \int_a^b f(x)\,dx$$

- It also holds $F'(x) = f(x)$ for all x

Example

- Let X has the density function

$$f(x) = \begin{cases} 3, & x \in [0, \frac{1}{3}] \\ 0, & \text{else} \end{cases}$$

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  (b) The cumulative distribution function $F(x) = P(X \leqslant x)$ has the form

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More examples

HW Let X has range $[0, 1]$ and probability density function $f(x) = ax^2$.

(a) What is the value of a?

(b) Compute the cumulative distribution function (cdf) $F_X(x)$.

(c) Compute $P(1 \leqslant X \leqslant 2)$.

HW Let Y has range $[0, b]$ and its cdf is given by

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{y^2}{9}, & 0 \leqslant y < b \\ 1, & y \geqslant b \end{cases}$$

(a) What is the value of b?

(b) Find the density function $f(y)$ of Y.

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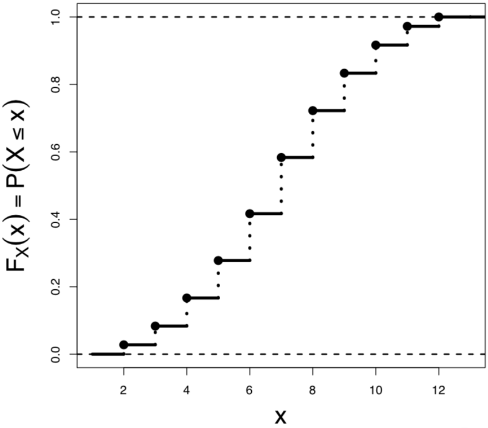
- A function that assigns the value $F^{-1}(p)$ to every $p \in (0,1)$ is called quantile function

$$x_p = F^{-1}(p) \quad \text{for} \quad p \in (0,1) \iff \qquad F(x_p) = p \quad \text{for} \quad p \in (0,1)$$

  - x_p is a p-quantil of F

Example: Sum of two dice

- The graph of the cumulative distribution function is a step function.



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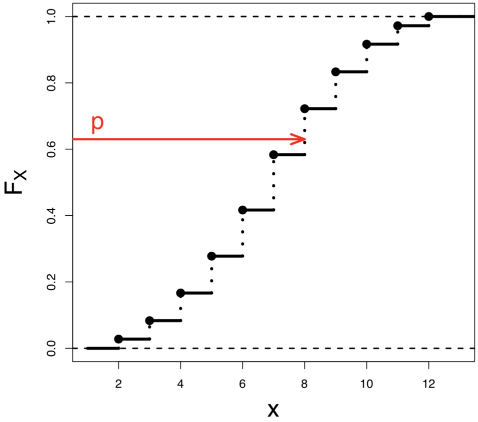

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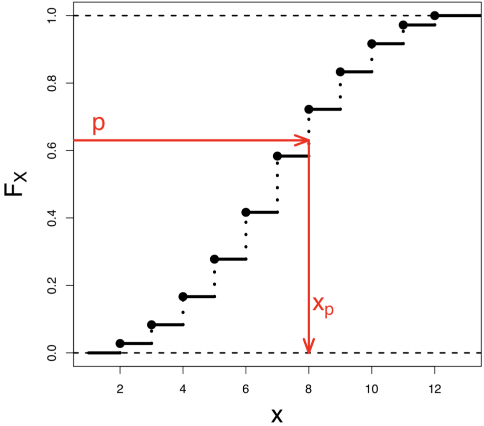

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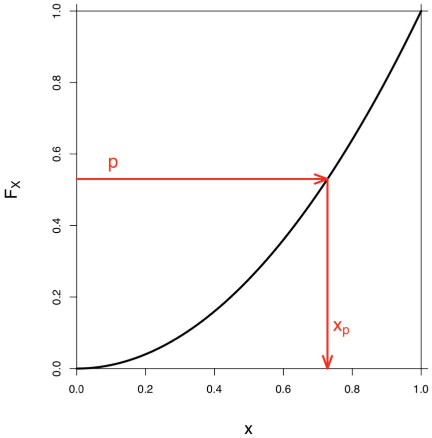

Quartiles

- A 50%-quan<u>n</u>til $x_{0.5}$ of F is called median.
- A 25%-quan<u>n</u>til $x_{0.25}$ of F is called lower qua<u>r</u>tile
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It is given

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Example cont.

(b) $E(X^2) = \frac{9}{4} + \frac{16}{2} + \frac{25}{8} + \frac{36}{8} = 18$

$Var(X) = E(X^2) - (E(X))^2 = 18 - (\frac{33}{8})^2.$

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  - Method: Use the cumulative distribution function of X
    - Determine F_X.
    - Determine F_Y for $Y = g(X)$.
    - Find $f_Y(y) = F'_Y(y)$.

Another example

Let X has the density function

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(a) Find a.

(b) Compute $E(X)$ and $Var(X)$.

(c) Find the median value of X.

(d) Let $X_1, X_2, \ldots, X_{25}$ are independentand identically distributed (i.i.d.) copies of X. Let $\bar{X}$ be their average (mean), i.e.

$$\bar{X} = \frac{1}{25} \sum_{i=1}^{25} X_i.$$

Compute the standard deviation of $\bar{X}$.

(e) Let $Y = 2X^3 + 1$. Find the density of Y.

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Median $x_{0.5}$ solves the equation

$$F_X(x_{0.5}) = 0.5$$

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(b) The expectation and variance are

$$\mu = E(X) = \int_0^1 5x^5\,dx = \frac{5}{6}$$

$$Var(X) = \int_0^1 (x - \frac{5}{6})^2\,5x^4 dx = 5\int_0^1 (x^6 - \frac{5}{3}x^5 + \frac{25}{36}x^4)dx = \frac{5}{252} \approx 0.02$$

$$\sigma = \sqrt{Var(X)} \approx 0.14$$

(c) First we calculate the cumulative distribution function

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^5, & 0 \leqslant x < 1 \\ 1, & x \geqslant 1 \end{cases}$$

Median $x_{0.5}$ solves the equation

$$F_X(x_{0.5}) = 0.5$$

Then, $F_X(x_{0.5}) = x_{0.5}^5 = 0.5$ and $x_{0.5} = \sqrt[5]{0.5} \approx 0.87$.

Another example: Solution

(d) First we compute the variance of $E(\bar{X})$ and $Var(\bar{X})$ of $\bar{X} = \frac{1}{25}(X_1 + X_2 + \cdots + X_{25})$.

$$E(\bar{X}) = \frac{1}{25}E(X_1 + \cdots + X_{25}) = E(X) = \frac{5}{6}$$

$$\begin{aligned} Var(\bar{X}) &= \frac{1}{25^2}Var(X_1 + X_2 + \cdots + X_{25}) = \frac{1}{25^2}(Var(X_1) + \cdots + Var(X_{25})) \\ &= \frac{1}{25^2} \cdot 25Var(X) = \frac{Var(X)}{25} < Var(X). \end{aligned}$$

Another example: Solution

(d) First we compute the variance of $E(\bar{X})$ and $Var(\bar{X})$ of $\bar{X} = \frac{1}{25}(X_1 + X_2 + \cdots + X_{25})$.

$$E(\bar{X}) = \frac{1}{25}E(X_1 + \cdots + X_{25}) = E(X) = \frac{5}{6}$$

$$Var(\bar{X}) = \frac{1}{25^2}Var(X_1 + X_2 + \cdots + X_{25}) = \frac{1}{25^2}(Var(X_1) + \cdots + Var(X_{25}))$$

$$= \frac{1}{25^2} \cdot 25Var(X) = \frac{Var(X)}{25} < Var(X).$$

Also, $\sigma_{\bar{X}} = \frac{\sigma}{5} \approx 0.028$.

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$$= \frac{1}{25^2} \cdot 25Var(X) = \frac{Var(X)}{25} < Var(X).$$

Also, $\sigma_{\bar{X}} = \frac{\sigma}{5} \approx 0.028$.

(e) In order to find pdf of $Y = 2X^3 + 1$, we use the cdf of X

$$F_Y(y) = P(Y \leqslant y) = P(2X^3 + 1 \leqslant y) = P(X \leqslant \sqrt[3]{\frac{y-1}{2}})$$

$$= F_X(\sqrt[3]{\frac{y-1}{3}}) = \begin{cases} 0, & y < 1 \\ (\frac{y-1}{2})^{\frac{5}{3}}, & 1 \leqslant y < 3 \\ 1, & y \geqslant 3 \end{cases}.$$

Also,

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{5}{6} \cdot (\frac{y-1}{2})^{\frac{2}{3}}, & 1 < y < 3 \\ 0, & \text{else} \end{cases}.$$

Examples

HW Let X be a random variable that takes on values 0, 2 and 3 with probabilities 0.3, 0.1 and 0.6 respectively. Let $Y = 3(X-1)^2$.

(a) Compute $E(X)$ and $Var(X)$.
(b) Compute $E(Y)$.
(c) Let $F_Y(y)$ be the cdf of X. What is $F_Y(7)$?

HW Let X be a random variable with the cdf

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x(2-x) & \text{for } 0 \leqslant x < 1 \\ 1 & \text{for } x > 1 \end{cases}.$$

(a) Compute $P(X < 0.4)$.
(b) Compute $E(X)$.
(c) Find the moment of order four of X.
(d) Find the median, upper and lower quartiles.

Examples

HW Let X be a random variable with the range $[0, 1]$ and cumulative distribution function

$$F(x) = 2x^2 - x^4, \qquad \text{for } 0 \leqslant x \leqslant 1.$$

(a) Verify that F is a cumulative distribution function.
(b) Compute $P(\frac{1}{4} \leqslant X \leqslant \frac{3}{4})$.
(c) What is the probability density of X?

HW Let X be a random variable with the cumulative distribution function F_X. Let X_1 and X_2 be independent and identically distributed (i.i.d.) copies from X. Let $Y = \max\{X_1, X_2\}$. Find the distribution function F_Y of Y with respect to F_X.

HW An enthusiastic football fan gives away Tototips every week, using the digits 0 (draw), 1 (home win), 2 (victory) with the help of the probability function

$$P(X = k) = \begin{cases} \frac{1}{4} + ak + bk^2 & k = 0, 1, 2 \\ 0 & \text{else} \end{cases}$$

with unknown values a and b. However, it is known that for his tips it holds $P(X = 1) = 1/4$. Determine a and b and the corresponding distribution function.

Questions

A few multiple-choice questions

(1) Let X be a random variable with probability density function of the form

$$f(x) = \begin{cases} -2x, & -1 \leqslant x \leqslant 0 \\ 0, & \text{else} \end{cases}.$$

Compute $P(-\frac{3}{4} \leqslant X < -\frac{1}{2})$.

a. $5/16$
b. $1/2$
c. $7/8$
d. $19/64$

(2) Let

$$F(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \leqslant x < 0.5 \\ 1 & x \geqslant 0.5 \end{cases}$$

be the cumulative distribution function of a random variable X and let $Y = 2X + 1$. Then, the expectation $E(Y)$ equals

(a) 0.25
(b) 0.5
(c) 1.5
(d) 0.75

A few multiple-choice questions

(3) A random variable X has a probability distribution as follows:

X	0	1	2	3
$P(X)$	2k	3k	13 k	2k

where k is a positive constant. The probability $P(X < 2.0)$ is equal to

a. 0.90
b. 0.25
c. 0.65
d. 0.15

(4) Let X be a random variable that takes values $-2, -1, 0, 1$ and 2, each with probability $1/5$. Let $Y = X^2$. Then,

a. $Cov(X, Y) > 0$
b. $Cov(X, Y) < 0$
c. $VarY < 2VarX$
d. $VarY = 2VarX$

... Note: We will learn soon what is the covariance $Cov(X, Y)$!

A few multiple-choice questions

(5) Suppose a bookie will give you \$6 for every \$1 you risk if you pick the winners in three ballgames. Thus, for every \$1 you bet you will either lose \$1 or gain \$5. What is the bookie's expected earnings per dollar wagered?

a. $-\$2/8$
b. $\$34/8$
c. $\$2/8$
d. $\$21/27$

Thank you for your attention!