TECHNISCHE
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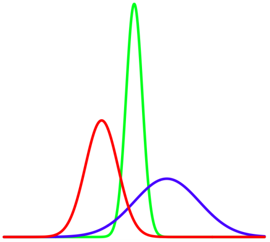
WIEN

Common Families of Distributions



Please send your comments to: T. Levajković, tijana.levajkovic@tuwien.ac.at

Some known distributions

- Discrete distributions
  - Bernoulli
  - Binomial
  - Geometric
  - Poisson

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...will be introduced later in the course ..
... also important in Statistics

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cdf	$F(x)$	$1 - p$	1

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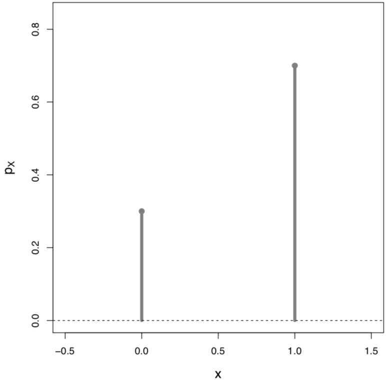
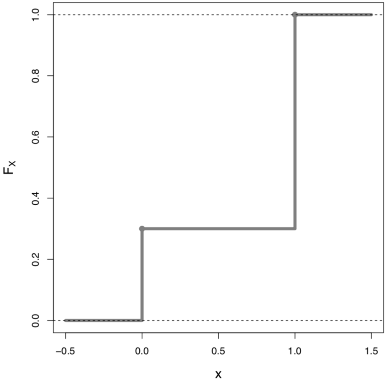
  - Expectation/Variance $E(X) = p, \qquad Var(X) = p(1 - p)$

Example

- $X \sim ber(0.7)$



Probability mass function



Cumulative distribution function

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$$\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x \cdot (1-p)^{n-x} = (p + (1-p))^n = 1$$

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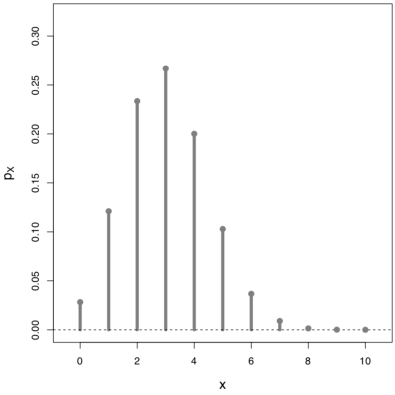
  - Expectation/Variance: $E(X) = np, \qquad Var(X) = np(1-p)$

Example

- $X \sim B(10, 0.3)$



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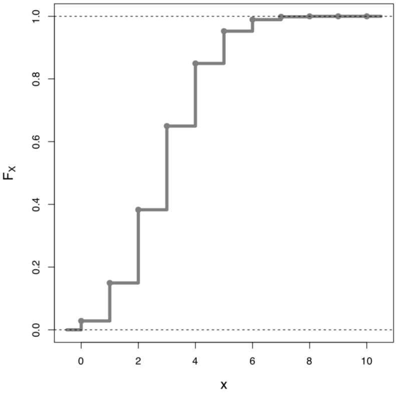

Cumulative distribution function

Example

Dice probabilities

(a) Find the probability of obtaining at least one 6 in four rolls of a die.

- This experiment can be modeled as a equence of four Bernulli trials with success probablity $p = \frac{1}{6} = P(\text{die shows } 6)$.
  Define the random variable

$$X = \text{Total number of sixes in four rolls}$$

- Then, $X \sim B(4, \frac{1}{6})$ and

$$\begin{aligned} P(\text{at least one } 6) &= P(X \geqslant 1) = 1 - P(X = 0) \\ &= 1 - \binom{4}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^4 = 0.518 \end{aligned}$$

  - In R: use `dbinom()`

```
1-dbinom(0,4,1/6)
[1] 0.5177469
```

Beispiel

(b) Consider another game; throw a pair of dice 24 times and ask for the probability of at least two double 6.

- This experiment can be modeled by the binomial distribution with success probability p, with $p = P(\text{roll two sixes}) = \frac{1}{36}$.
- If $Y =$ number of double 6s in 24 rolls, then $Y \sim B(24, \frac{1}{36})$ and

$$P(\text{at least two double sixes}) = P(Y \geqslant 2) = 1 - P(Y=0) - P(Y=1)$$

$$= 1 - \binom{24}{0}\left(\frac{1}{36}\right)^{0}\left(\frac{35}{36}\right)^{24} - \binom{24}{1}\left(\frac{1}{36}\right)^{1}\left(\frac{35}{36}\right)^{23} \approx 0.1427$$

- In R: use `dbinom()`

```
1-sum(dbinom(0:1,24,1/36))
[1] 0.1426522
```

oder `pbinom()`

```
pbinom(1,24,1/36,lower.tail = FALSE)
[1] 0.1426522
```

More examples

HW Standardized tests provide an interesting application of probability theory. Suppose that a test consists of 20 multiple-choice questions, each with 4 possible answers, of which exactly one is correct. If 17 questions are answered correctly, the exam is passed. A student comes unprepared for this test and randomly crosses one of the four possible answers. (If the student guesses on each question, then the taking of the exam can be modeled as a sequence of 20 independent events.) Use R to compute the probability that the student will pass the test.

HW From a list of 15 households, 9 are homeowners and 6 households live in rental housing. Four households are randomly selected from these 15. Find the probability that the number of households that do not own a home is at least three.

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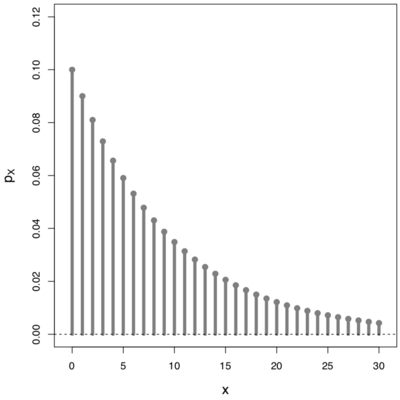
  - Expectation/Variance $\quad E(X) = \frac{1-p}{p}, \qquad Var(X) = \frac{1-p}{p^2}$

Example

- $X \sim$ Geometric distribution with $p = 0.1$



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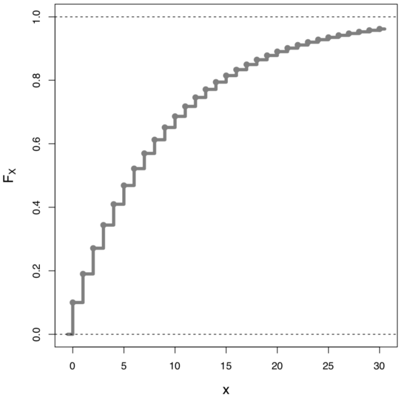

Cumulative distribution function

Example

- Failure times

  Geometric distribution is used to model lifetimes of devices or components.

  For example, if the probability is 0.001 that a light bulb will fail on any given day, then the probability that it will last more than 30 days is

$$P(X > 30) = \sum_{x=31}^{\infty} 0.001(1-0.001)^{x-1} = 0.970$$

  - In R: use `dgeom()`

```
1-sum(dgeom(1:29, 0.001))
[1] 0.9704605
```

Example

HW Suppose that the inhabitants of an island plan their families by having babies until the first girl is born. Assume the probability of having a girl with each pregnancy is 0.4 independent of other pregnancies, that all babies survive and there are no multiple births. What is the probability that a family has 6 boys?

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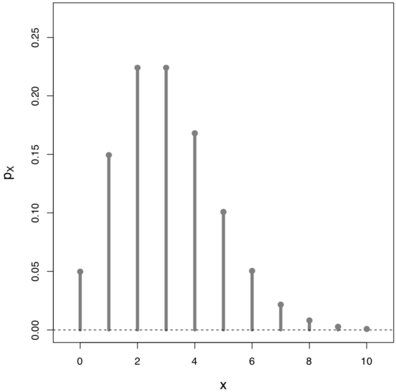
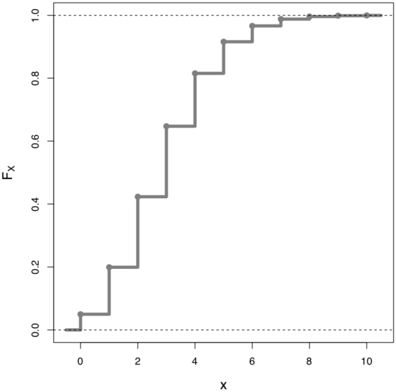
Example

- $X \sim \mathcal{P}(3)$





Probability mass function

Cumulative distribution function

Example

- Waiting time

  Consider a telephone operator who, on the average, handles five calls every 3 minutes. What is the probability that there will be no calls in the next minute? At least three calls?

  - Let X = number of calls in a minute. Then $X \sim \mathcal{P}(\lambda)$, where $EX = \lambda = \frac{5}{3}$. Then,

$$P(\text{no calls in the next minute}) = P(X = 0) = \frac{e^{-\frac{5}{3}}\left(\frac{5}{3}\right)^0}{0!} = 0.189$$

$$P(\text{at least three calls in the next minute}) = P(X \geqslant 3)$$
$$= 1 - \sum_{x=0}^{2} P(X = k) = 1 - \sum_{x=0}^{2} \frac{e^{-\frac{5}{3}}\left(\frac{5}{3}\right)^x}{x!} = 0.234.$$

  - In R: we use `dpois()`

```
1-sum(dpois(0:2, 5/3))
[1] 0.2340045
```

Example

HÜ Web visitors

A website manager has noticed that during the evening hours, about three people per minute check out from their shopping cart and make an online purchase. She believes that each purchase is independent of the others and wants to model the number of purchases per minute.

(1) What model might you suggest to model the number of purchases per minute?
(2) What is the probability that in any one minute at least one purchase is made?
(3) What is the probability that no one makes a purchase in the next two minutes?

Binomial-Poisson relationship

- $\mathcal{P}(\lambda)$ distribution as a limit of the $B(n, p)$ distribution
  - For $X_n \sim B(n, \frac{\lambda}{n})$ it holds

$$\lim_{n \to \infty} P(X_n = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

  - Interpretation: If there are many independent and identical Bernoulli experiments with a low probability of success, the number of successes can be approximated with a Poisson distribution

  - Rule of thumb

    For $n \geqslant 50, p \leqslant \frac{1}{10}$ and $np \leqslant 10$ a random variable $X \sim B(n, p)$ can be approximated

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... we will return to this later in the course...

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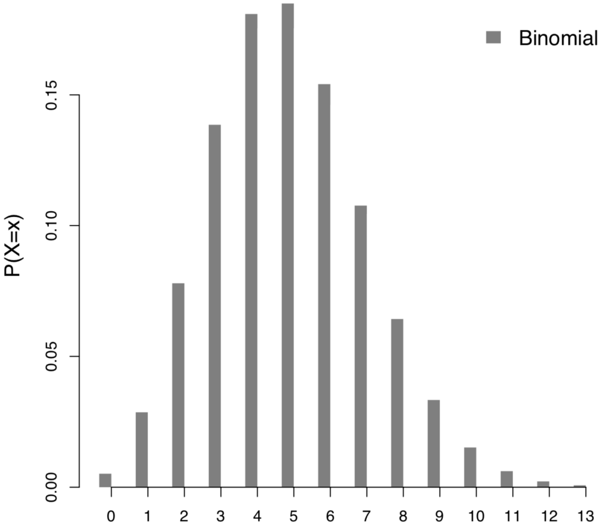
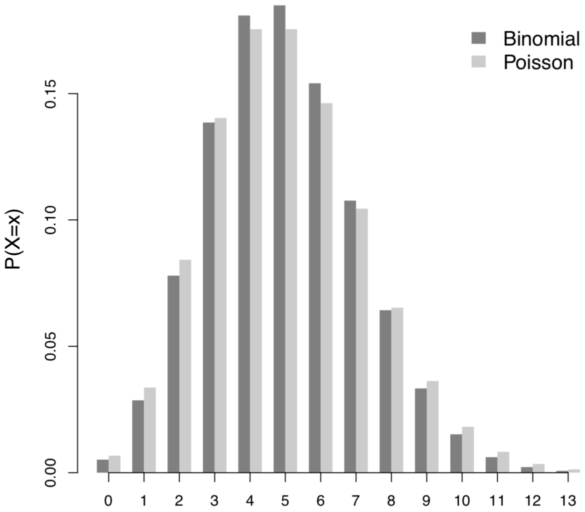


Example

- $X \sim B(50, \frac{1}{10}) \approx \mathcal{P}(5)$



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  - X is a random variable with uniform distribution over the interval (a, b) ($a < b, a, b \in \mathbb{R}$) if its pdf is of the form

$$f(x) = \begin{cases} \frac{1}{b-a}, & x \in (a, b) \\ 0, & \text{sonst} \end{cases}$$

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  - Expectation/Variance: $E(X) = \frac{a+b}{2}, \quad Var(X) = \frac{(b-a)^2}{12}$

Example

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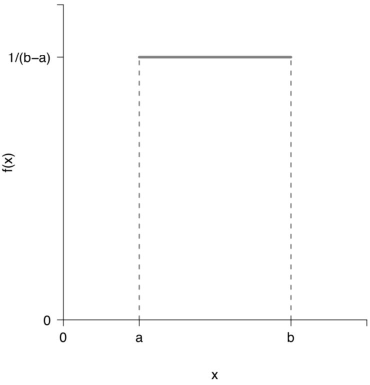
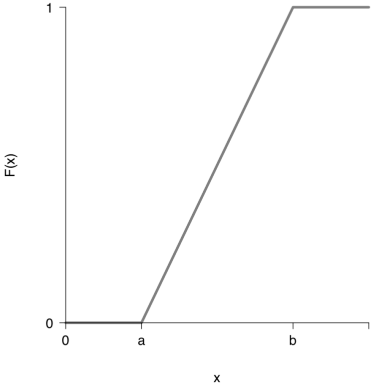


Probability density function



Cumulative distribution function

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(continuous analogue of geometric distribution)

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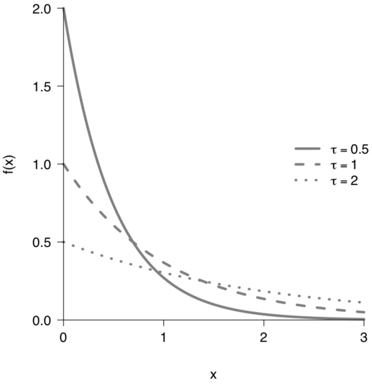
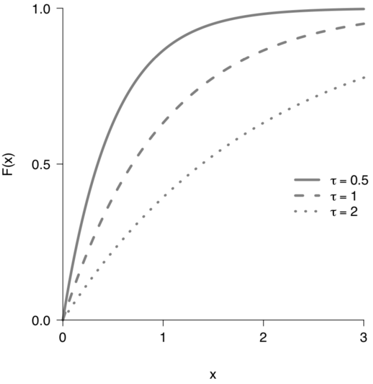
  - Expectation/Variance: $\quad E(X) = \tau = \frac{1}{\lambda}, \quad Var(X) = \tau^2 = \frac{1}{\lambda^2}$

Example

- $X \sim \exp(\tau)$, for $\tau = 0.5$, $\tau = 1$ and $\tau = 2$



Probability density function



Cumulative distribution function

Example

- Waiting time

  Anna noticed that taxies drive past her street on the average of once every 10 minutes. Suppose time spent waiting for a taxi is modeled by an exponential random variable

$$X \sim \exp(\frac{1}{10}); \qquad f(x) = \frac{1}{10}e^{-\frac{x}{10}}, \quad \text{for } x \geqslant 0$$

  Calculate the probability of waiting for a taxi between 3 and 7 minutes.

  - $P(3 < X < 7) = \int_3^7 f(x)\,dx = F(7) - F(3) \approx 0.244$
  - In R: we use `pexp()`

```
pexp(7,0.1)-pexp(3,0.1)
[1] 0.2442329
```

Normal (Gaussian) distribution

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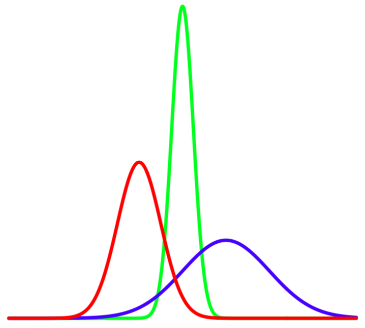

Normal distributions are symmetric with relatively more values at the center of the distribution and relatively few in the tails.

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$$f(x) = \frac{1}{\sqrt{2\pi\,\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for} \quad x \in \mathbb{R}$$

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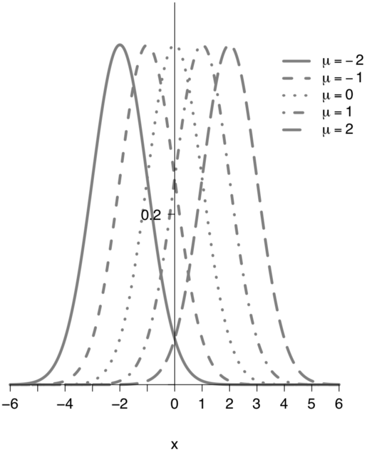
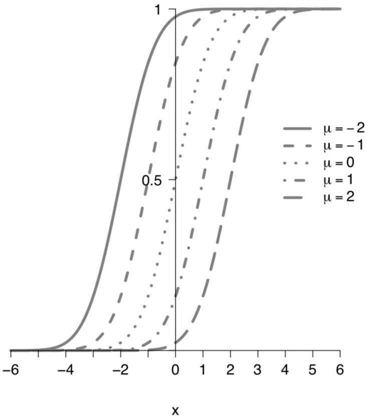
  - Expectation/Variance $\qquad E(X) = \mu, \quad Var(X) = \sigma^2$

Examples

- $X \sim \mathcal{N}(\mu, 1)$, for $\mu = -2$, $\mu = -1$, $\mu = 0$, $\mu = 1$ and $\mu = 2$



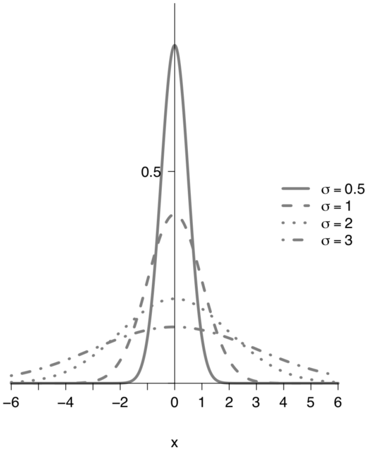
Probability density function



Cumulative distribution function

Examples

- $X \sim \mathcal{N}(0, \sigma^2)$, for $\sigma = 0.5$, $\sigma = 1$, $\sigma = 2$ and $\sigma = 3$



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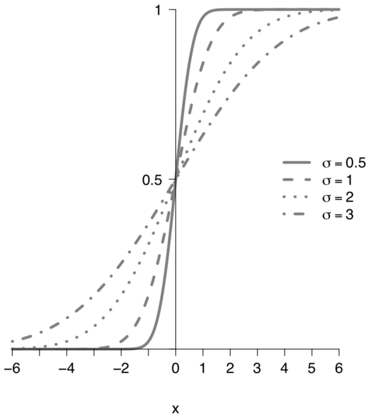
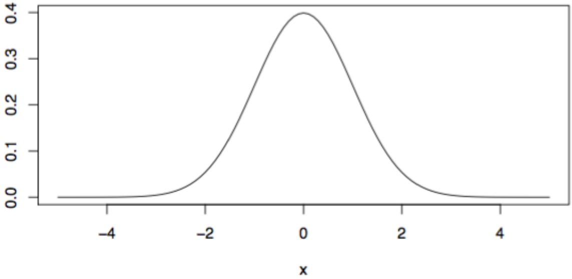


Cumulative distribution function

Standard Normal distribution

- In particular,
  - $Z \sim \mathcal{N}(0,1)$ is the Standard Normal distribution



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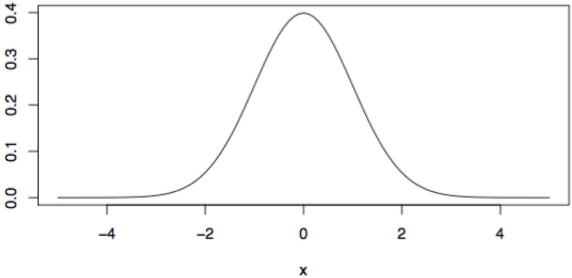

  - the cdf

$$\Phi(z) = P(Z \leqslant z) = \int_{-\infty}^{z} f(t)\,dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}}\,dt$$

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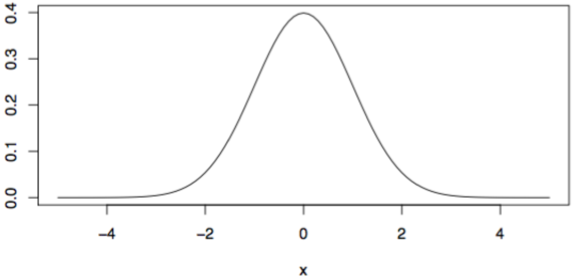

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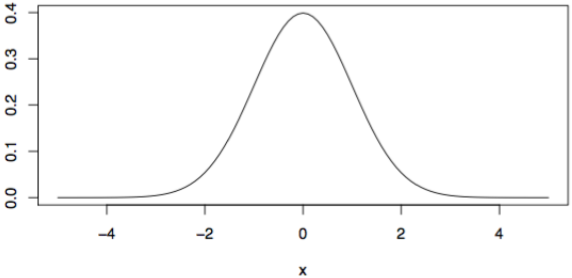

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  - Moreover,

$$\Phi(-z) = 1 - \Phi(z), \quad \text{for } z \in \mathbb{R}$$

... the values of Φ can be read from the `Table`
or obtained in R by applying `pnorm()`

$\Phi(z) = P(Z \leqslant z)$

- The Table of $\mathcal{N}(0,1)$-distribution

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9958	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

Table of $\Phi(z)$: Examples

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
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0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051

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z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0
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- $\Phi(-0.43) = 1 - 0.6664 = 0.3336$ $\qquad$ $\Phi(-0.76) = 1 - 0.7764 = 0.2236$
- $P(-0.76 \leqslant Z \leqslant 0.43) = \Phi(0.43) - \Phi(-0.76) = 0.4428$

Table of $\Phi(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
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- Compute $P(|Z| \leqslant \frac{1}{2})$

Table of $\Phi(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051

- Compute $P(|Z| \leqslant \frac{1}{2})$

$$\begin{aligned} P(|Z| \leqslant \frac{1}{2}) &= P(-0.5 \leqslant Z \leqslant 0.5) \\ &= \Phi(0.5) - \Phi(-0.5) \\ &= \Phi(0.5) - (1 - \Phi(0.5)) \\ &= 2 \cdot \Phi(0.5) - 1 \\ &= 2 \cdot 0.6915 - 1 = 0.3830 \end{aligned}$$

$\Phi(z)$ in R

pnorm(x, mean, sd)

... pnorm (x, μ, σ) uses σ while in the notation of the normal distribution $\mathcal{N}(\mu, \sigma^2)$ is used σ^2...

```
#Values of the cdf of  N(0,1) distribution
pnorm(0.5)
[1] 0.6914625
# it is the same as
pnorm(0.5, mean=0, sd=1)
[1] 0.6914625
pnorm(1.3)
[1] 0.9031995
# Values of the cdf of  N(3,4) distribution
pnorm(2.5, mean=3, sd=2)
[1] 0.4012937
# it is the same as pnorm((2.5-3)/2)
pnorm(-0.25)
[1] 0.4012937
```

- We compute $P(|Z| \leqslant \frac{1}{2}) \approx 0.3830$ using R:

```
diff(pnorm(c(-0.5,0.5)))
[1] 0.3829249
```

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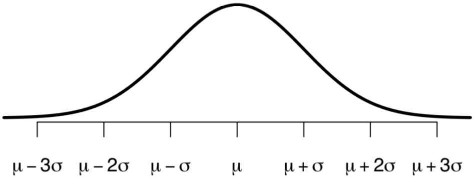

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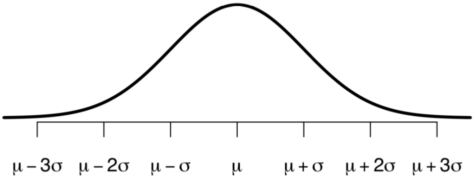

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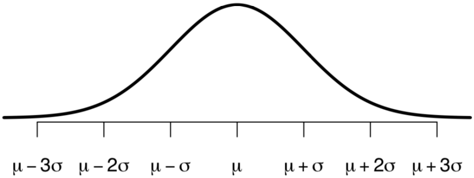

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$$P(X \in [\mu - \sigma, \mu + \sigma])$$
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Probabilities of the normal distribution

(1) Let $Z \sim \mathcal{N}(0,1)$. Then,

$$P(-1 \leqslant Z \leqslant 1) \approx 0.68$$
$$P(-2 \leqslant Z \leqslant 2) \approx 0.95$$
$$P(-3 \leqslant Z \leqslant 3) \approx 0.997$$

```
# one sigma-away from mu=0
diff(pnorm(c(-1,1)))
[1] 0.6826895
# two sigmas-away from mu=0
diff(pnorm(c(-2,2)))
[1] 0.9544997
# three sigmas-away from mu=0
diff(pnorm(c(-3,3)))
[1] 0.9973002
```

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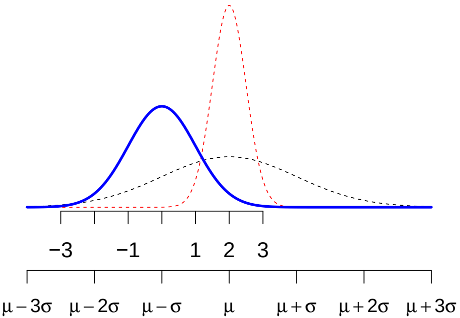

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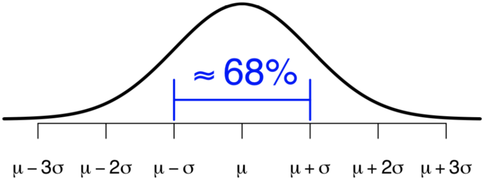

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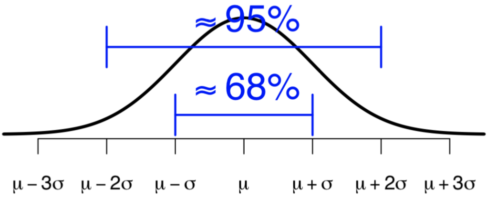

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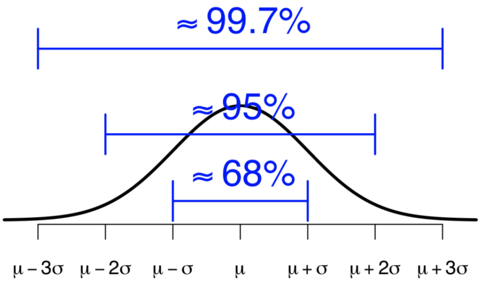

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    - in R: `qnorm()`

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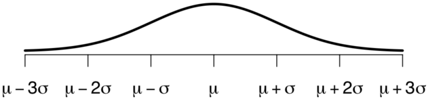

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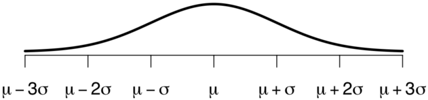

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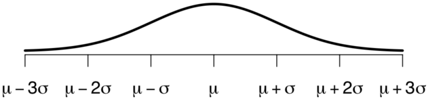

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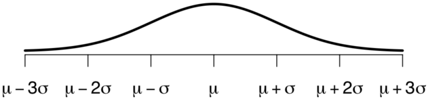

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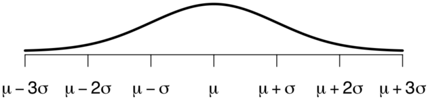

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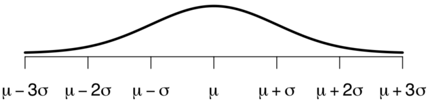

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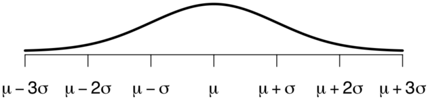

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... we are going to prove this!

The mean of iid normally distributed rvs

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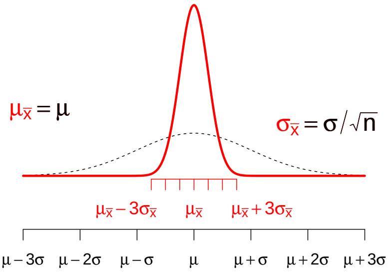

Examples

(1) As a group, the Dutch are among the tallest people in the world. The average Dutsch man is 184 cm tall. If a normal model is appropriate and the standard deviation for men is about 8 cm, what percentage of all Dutsch men will be over two meters?

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  We also know

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 184}{8} \sim \mathcal{N}(0, 1)$$

  I way: We use the table and calculate

$$\begin{aligned} P(X > 200) &= P\left(\frac{X - 184}{8} > \frac{200 - 184}{8}\right) \\ &= P(Z > 2) \\ &= 1 - P(Z \leqslant 2) \\ &= 1 - \Phi(2) \\ &= 1 - 0.9772 = 0.0228 = 2.28\% \end{aligned}$$

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```
pnorm(184,200,8)
[1] 0.02275013
# or
1-pnorm(184,200,8, lower.tail=FALSE)
[1] 0.02275013
 # or
pnorm(184,200,8, lower.tail=TRUE)
[1] 0.02275013
```

Beispiele

(2) Suppose you need 20 minutes, on average, to drive to work, with a standard deviation of 2 minutes. Suppose a Normal model is appropriate for the distribution of driving times.

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$$P(X < 22) = P\left(\frac{X - 20}{2} < \frac{22 - 20}{2}\right) = P\left(Z < 1\right) = \Phi(1) = 0.8413$$

$$P(X > 24) = P\left(Z > 2\right) = 1 - \Phi(2) = 1 - 0.9772 = 0.0228$$

Examples

(3) Let $Y \sim \mathcal{N}(10, 3^2)$.

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and

$$\frac{y_{0.7}-10}{3} = \Phi^{-1}(0.7) \quad \Longrightarrow \quad y_{0.7} = 3\cdot\Phi^{-1}(0.7) + 10$$

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- in R we use `qnorm()`
  $\Phi^{-1}(0.7)$ = `qnorm`(0.7, mean=0, sd=1) $= 0.5244$

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- We use R:

$$y_0 = 3 \cdot \texttt{qnorm (0.9, 0, 1)} + 10 = 13.8447$$

or

$$y_0 = \texttt{qnorm(0.1, 10,3, lower.tail = FALSE)} = 13.84465$$

Examples

HW Miraculin is a protein naturally produced in a rare tropical fruit. It can convert a sour taste into a sweet taste. Consequently, miraculin has the potential to be an alternative low-calorie sweetener.

A group of Japanese environmental scientists investigated the ability of a hybrid tomato plant to produce miraculin. For a particular generation of the potato plant, the amount X of miraculin produced (measured in micrograms per gram of fresh weight) had a mean 105.3 and a standard deviation of 8.0. Assume that X is normally distributed.

(a) Find $P(X > 120)$.
(b) Find $P(100 < X < 110)$.
(c) Find the values a for which $P(X < a) = 0.25$.

R-functions

- The following functions in R may be used:

`pbinom()`	`dbinom()`	`qbinom()`	`rbinom()`
`pexp()`	`dexp()`	`qexp()`	`rexp()`
`pnorm()`	`dnorm()`	`qnorm()`	`rnorm()`
`punif()`	`dunif()`	`qunif()`	`runif()`
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    - For example, `dnorm` is the height of the density of a normal curve while `dbinom` gives the probability of an outcome of a binomial distribution.

Various examples

HW High temperatures in Vienna for the month of August follow a uniform distribution over the interval 22 to 27 degrees Celsius. Find the temperature which 90% of the August days exceed.

HW A construction zone on a highway has a posted speed limit of 40 miles per hour. The speeds of vehicles passing through this construction zone are normally distributed with a mean of 46 miles per hour and a standard deviation of 4 miles per hour. Find the percentage of vehicles passing through this construction zone that is exceeding the posted speed limit.

HW The length of a workpiece is normally distributed with $\mu = 3$ and $\sigma = 0.2$. All workpieces that are shorter than 2.8 or longer than 3.2 are considered to be rejects. What is the probability that a workpiece is rejected?

Various examples

HW It is known from a certain mailbox advertisement that in two out of 1000 people a purchase contract is concluded based on this advertisement. What is the probability that out of 800 people who find the advertisement in your mailbox,

(1) no one
(2) at most three
(3) at least four

will conclude a purchase contract?

HW Let Z a standard normal random variable and let $X = 5Z + 1$.

(1) Calculate $P(|X| \leqslant 1)$.
(2) Recall that the probability that Z is within one standard deviation of its mean is approximately 68%. What is the probability that X is within one standard deviation of its mean?

Various examples

HW It is known that the time X (in hours), that a technician needs to repair a machine, follows an exponential distribution with parameter $\lambda = 2$.

(1) Calculate the associated distribution function F and sketch it as well as its density.

(2) What is the probability that the technician needs

- (i) at most half an hour
- (ii) between 0.2 and 0.4 hours
- (iii) more than 12 minutes

for the repair?

(3) How many hours are required on average for the repair of a machine? Also determine the variance of the repair time.

Questions

A few multiple-choice questions

(1) Jan wants to compute in R the probability of obtaining at least one 6 when rolling a fair dice 4 times. He should use the command

a. `1-pbinom(0,4,1/6)`
b. `dbinom(0,4,1/6)`
c. `qbinom(0,4,1/6)`
d. `rbinom(0,4,1/6)`

(2) The distribution of cholesterol levels for patients in a cardiology practice follows a normal distribution with a mean of 220 and a standard deviation of 40. In this practice, the probability that a patient has a cholesterol level less than 140 is the same as the probability that a patient has a cholesterol level reading more than

______________.

a. 320
b. 300
c. 340
d. 400

A few multiple-choice questions

(3) Let Z be a standard normal random variable and let $X = -4Z + 0.5$. Calculate

$$P(|X| \leqslant 0.5).$$

Use the values given in the table below.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
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Table 1: Cumulative distribution function of the standard normal distribution

a. 0.0987
b. 0.1915
c. 0.1987
d. 0.4013

A few multiple-choice questions

(4) In a list of 15 households, 9 own homes and 6 do not own homes. Four households are randomly selected from these 15 households. Find the probability that the number of households in these four who own homes is at most one.

a. 0.1536
b. 0.1792
c. 0.3456
d. 0.4752

(5) A literature professor decides to give a 20-question true-false quiz. Each question is worth 1 point. He wants to choose the passing grade so that the probability of passing a student who guesses on every question is less than 0.1. What score should be set as the lowest passing grade?

a. 12
b. 13
c. 14
d. 15

A few multiple-choice questions

(6) A coating machine coats film between 120 and 210 microns with a uniform random distribution. If any section of film with a coating greater than 200 microns cannot be sold, approximately how much of the product must be scrapped?

a. 1/3
b. 1/2
c. 1/7
d. 1/9

Thank you for your attention!