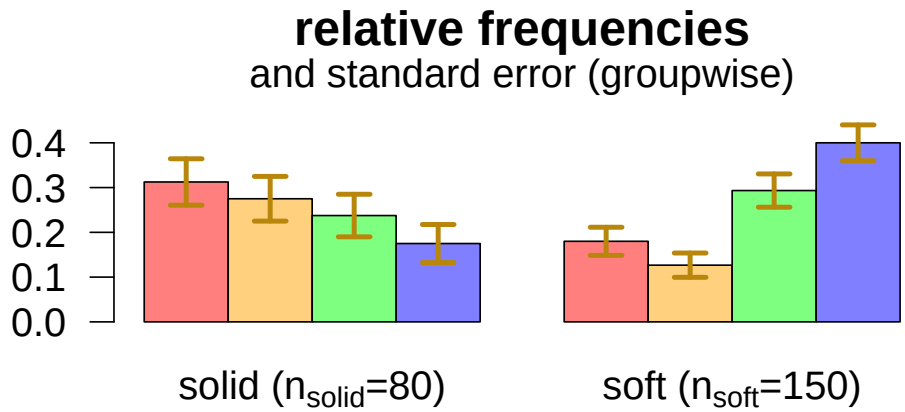
# The $\chi^2$-test (for independence)



All examples are fictitious. All data are simulated and the graphics were created with the statistical program package R.



Sämtliche Beispiele sind frei erfunden. Alle Daten sind simuliert und die Grafiken wurden mit statistischen Programmpaket R erstellt.

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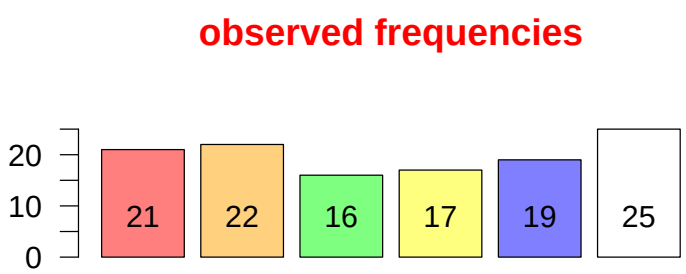

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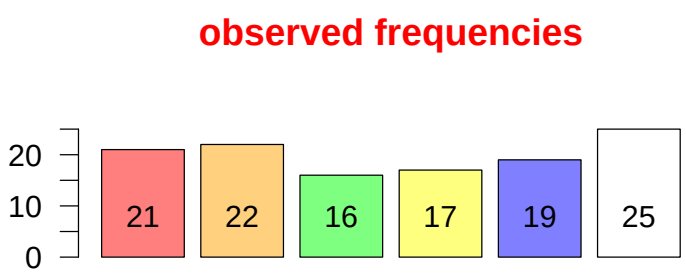

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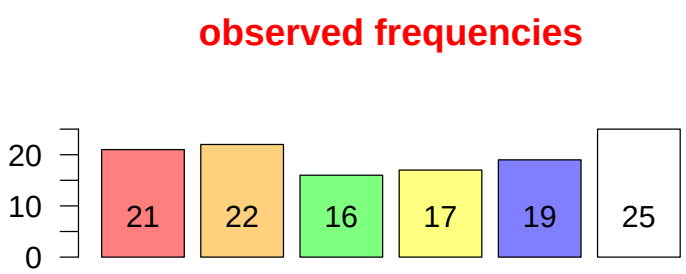

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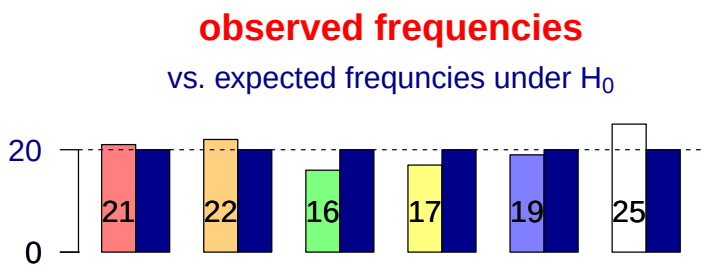

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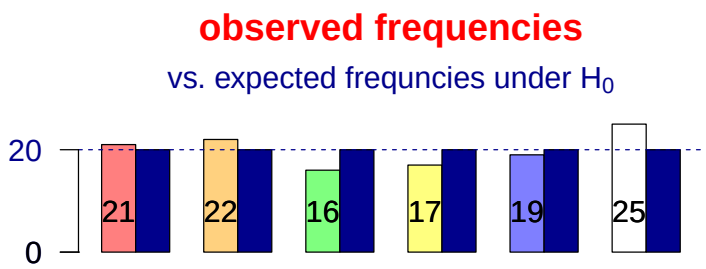

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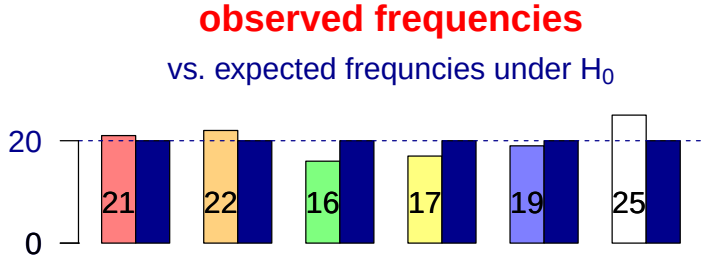

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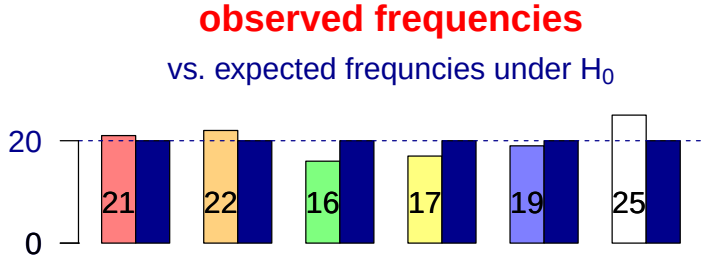

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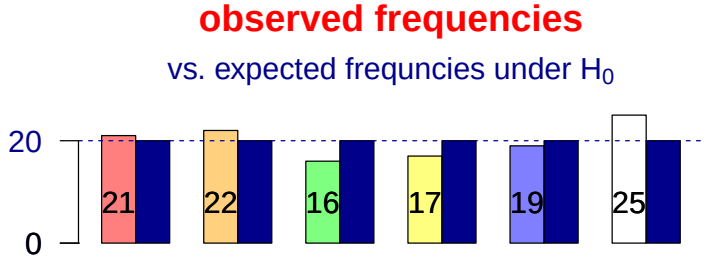

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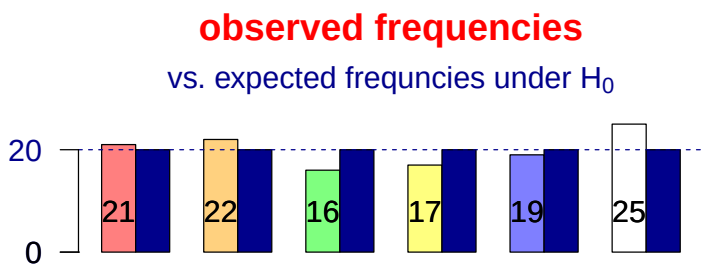

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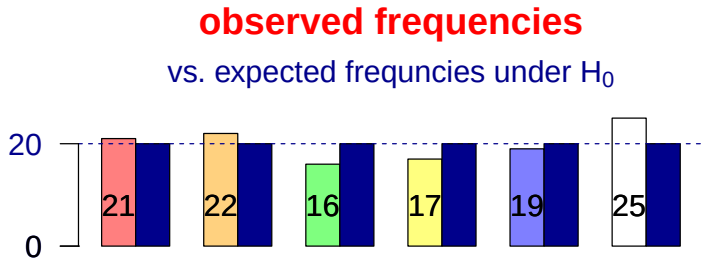

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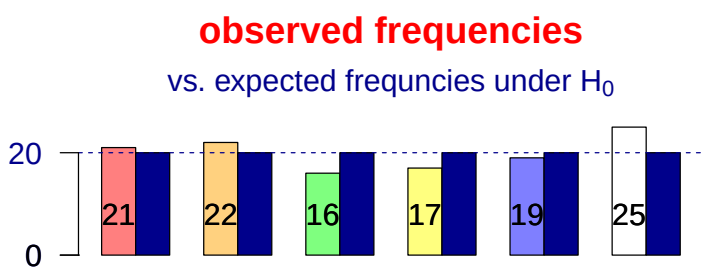


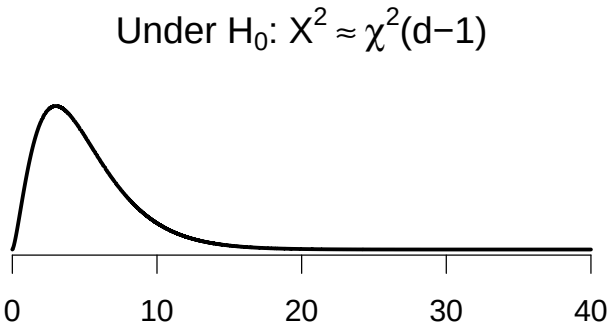

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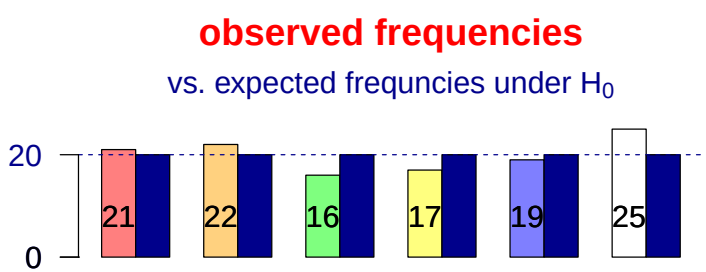


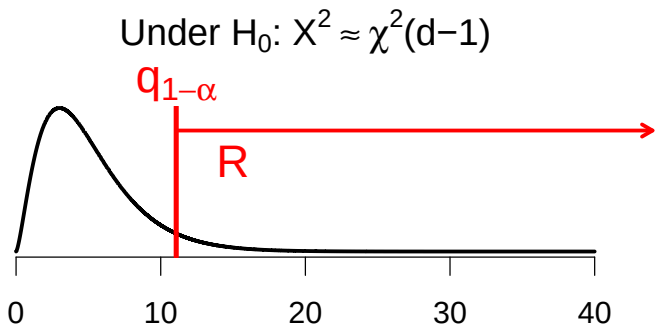

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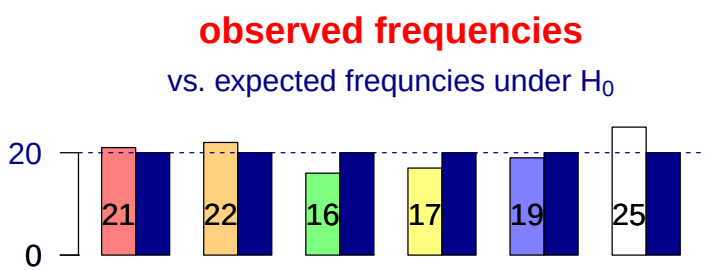


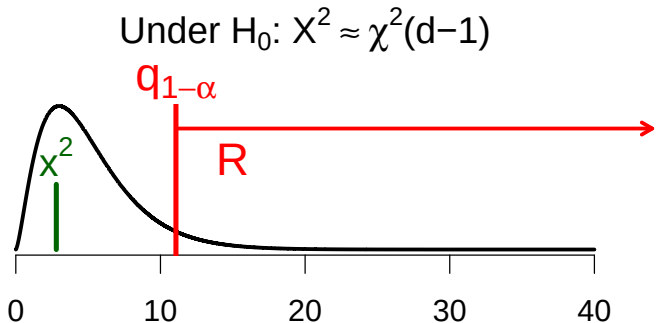

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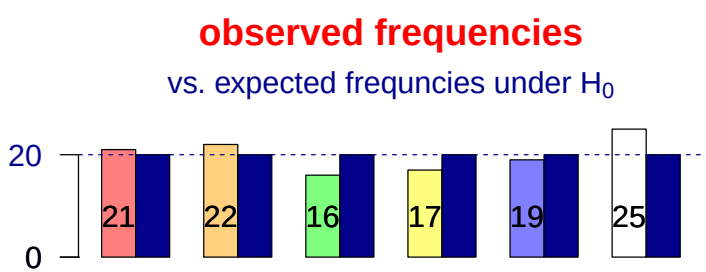


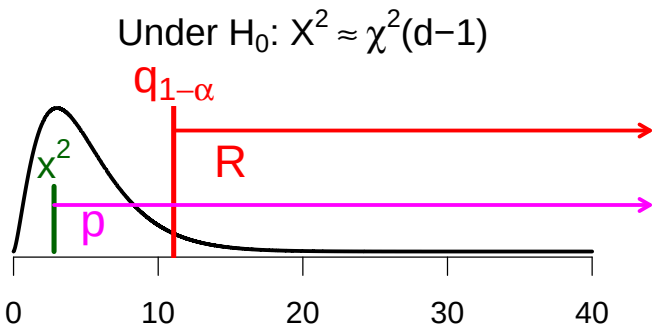

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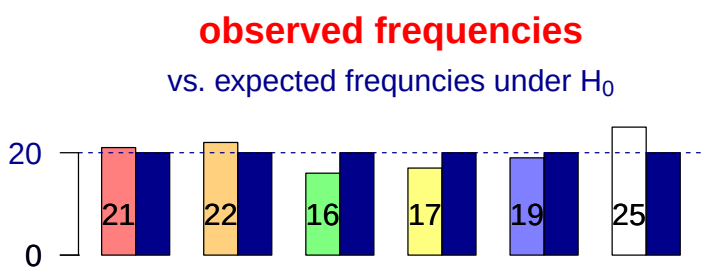
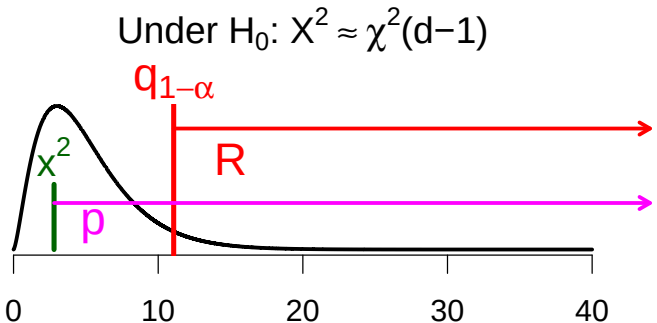
  $\rightarrow x^2 \notin R \Leftrightarrow p > \alpha = 5\% \rightarrow$ 'If $H_0$ holds true, then the discrepancy is not unlikely'



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Message: On the one hand there is a different question (and setup)...
...on the other hand we will 'technically' work with the same statistics

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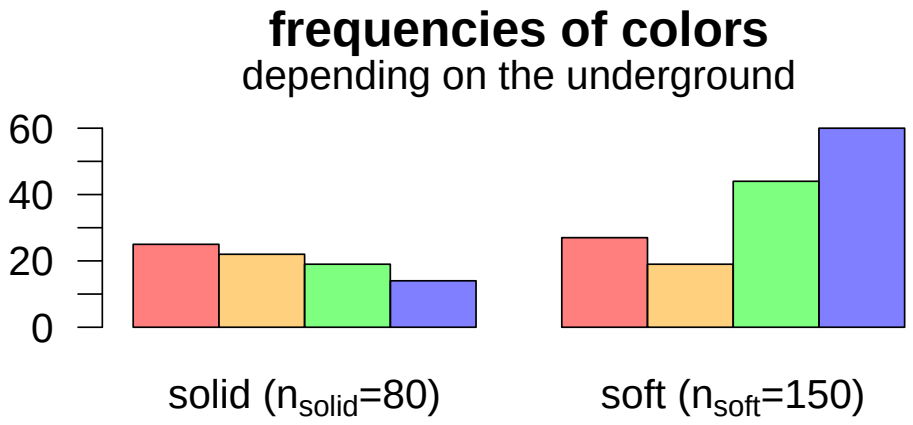


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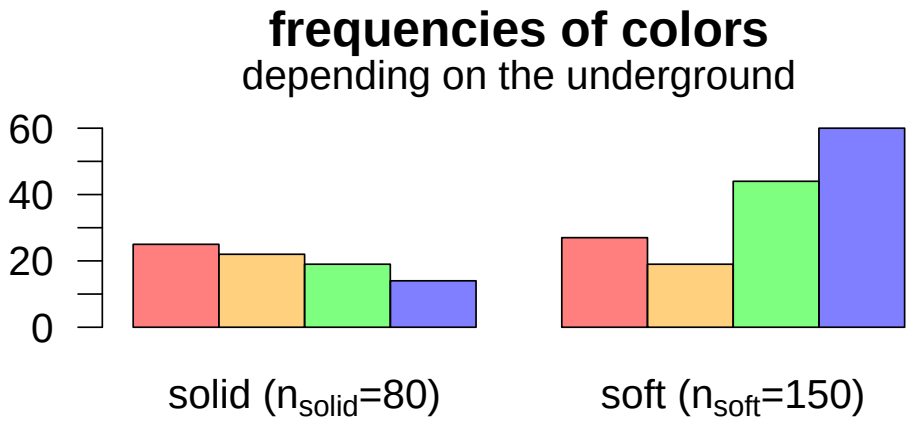


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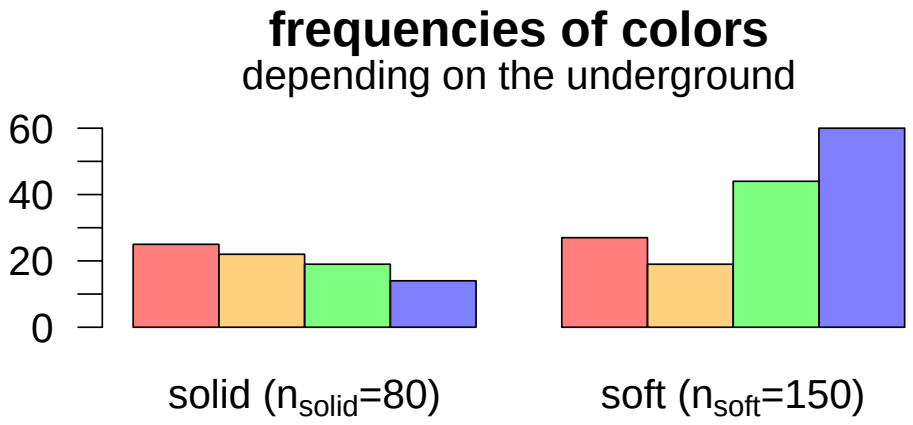


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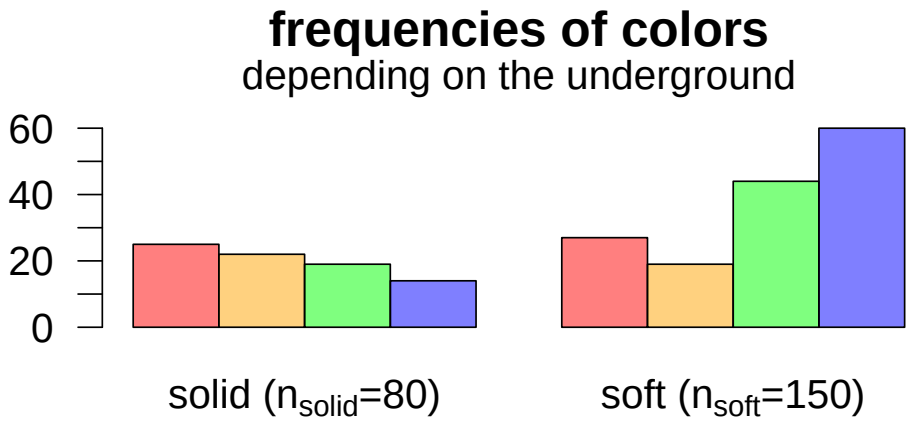


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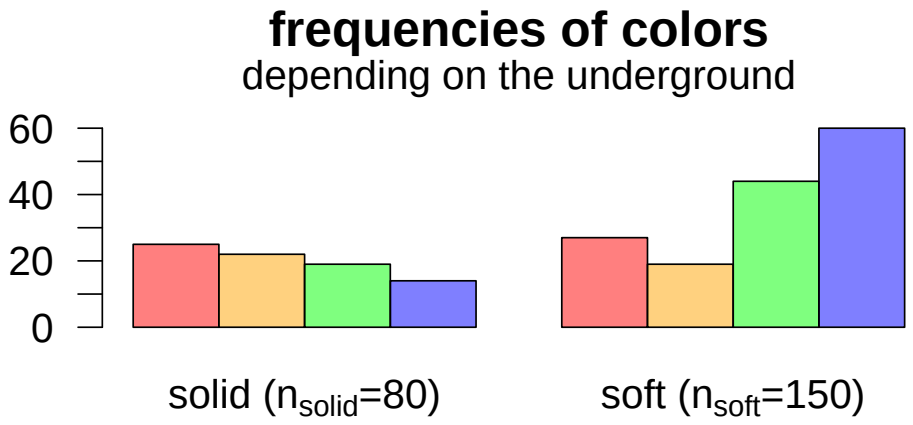


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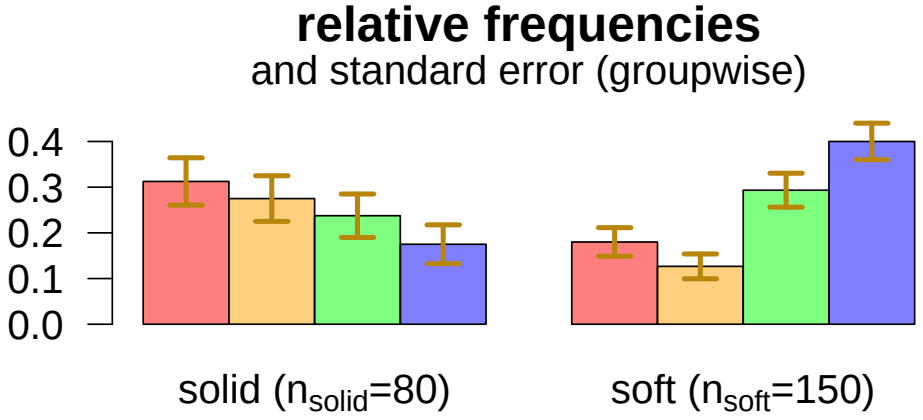


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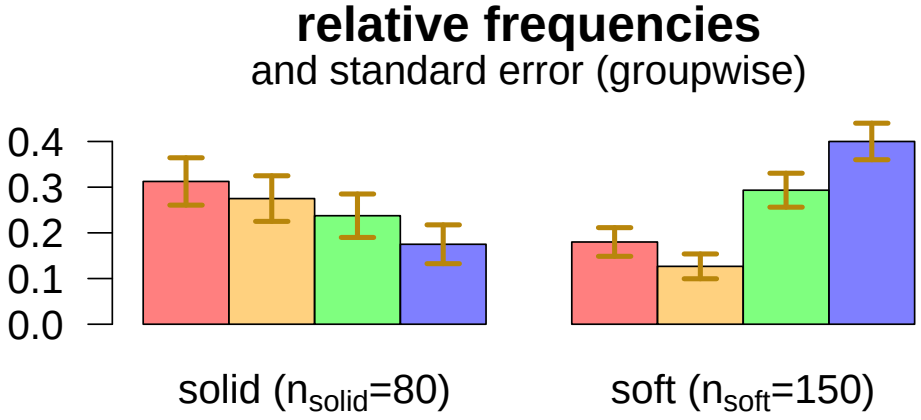


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- Can this difference be explained easily by chance under independence?
- Not really, when considering the standard errors
  $\rightarrow$ more precisely: $\chi^2$-*test for independence*

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- $n = 230$ data in $d = d_1 \cdot d_2 = 8$ categories

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| solid | $p_{1,1}$ | $p_{1,2}$ | $p_{1,3}$ | $p_{1,4}$ | $p_{1,\cdot}$ |
| soft | $p_{2,1}$ | $p_{2,2}$ | $p_{2,3}$ | $p_{2,4}$ | $p_{2,\cdot}$ |
| $p_{\cdot,k}$ | $p_{\cdot,1}$ | $p_{\cdot,2}$ | $p_{\cdot,3}$ | $p_{\cdot,4}$ | |

- while $p_{j,k}$ denotes the probability to fall into row $j$ and column $k$
- and the *row sums* $p_{j,\cdot}$, the *column sums* $p_{\cdot,k}$

# Model

- $n = 230$ data in $d = d_1 \cdot d_2 = 8$ categories
  feature 1 (underground) has $d_1 = 2$ categories
  and feature 2 (color) has $d_2 = 4$ categories
- observed frequencies: $x_{1,1}, x_{1,2}, \cdots, x_{2,4}$ ($x_{j,k} \to$ row $j$, column $k$)
- Model: Let $\mathfrak{X} = (X_{1,1}, \ldots, X_{d_1,d_2})^t \sim mult(n, p)$
  with $p = (p_{1,1}, \ldots, p_{d_1,d_2})^t \in (0,1)^{d_1 \cdot d_2}$ and $\sum_{j,k} p_{j,k} = 1$

| $x_{j,k}$ | side | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|---|
| underground | solid | 25 | 22 | 19 | 14 | 80 |
| | soft | 27 | 19 | 44 | 60 | 150 |
| | $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| occupation probabilities | red | orange | grün | blue | $p_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | $p_{1,1}$ | $p_{1,2}$ | $p_{1,3}$ | $p_{1,4}$ | $p_{1,\cdot}$ |
| soft | $p_{2,1}$ | $p_{2,2}$ | $p_{2,3}$ | $p_{2,4}$ | $p_{2,\cdot}$ |
| $p_{\cdot,k}$ | $p_{\cdot,1}$ | $p_{\cdot,2}$ | $p_{\cdot,3}$ | $p_{\cdot,4}$ | $\sum = 1$ |

- while $p_{j,k}$ denotes the probability to fall into row $j$ and column $k$
- and the *row sums* $p_{j,\cdot}$, the *column sums* $p_{\cdot,k}$, and *total sum* $\sum_{j,k} p_{j,k} = 1$

# Model

- $n = 230$ data in $d = d_1 \cdot d_2 = 8$ categories
  feature 1 (underground) has $d_1 = 2$ categories
  and feature 2 (color) has $d_2 = 4$ categories
- observed frequencies: $x_{1,1}, x_{1,2}, \cdots, x_{2,4}$ ($x_{j,k} \to$ row $j$, column $k$)
- Model: Let $\mathfrak{X} = (X_{1,1}, \ldots, X_{d_1,d_2})^t \sim mult(n, p)$
  with $p = (p_{1,1}, \ldots, p_{d_1,d_2})^t \in (0,1)^{d_1 \cdot d_2}$ and $\sum_{j,k} p_{j,k} = 1$

| $x_{j,k}$ | side | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|---|
| underground | solid | 25 | 22 | 19 | 14 | 80 |
| | soft | 27 | 19 | 44 | 60 | 150 |
| | $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| occupation probabilities | red | orange | grün | blue | $p_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | $p_{1,1}$ | $p_{1,2}$ | $p_{1,3}$ | $p_{1,4}$ | $p_{1,\cdot}$ |
| soft | $p_{2,1}$ | $p_{2,2}$ | $p_{2,3}$ | $p_{2,4}$ | $p_{2,\cdot}$ |
| $p_{\cdot,k}$ | $p_{\cdot,1}$ | $p_{\cdot,2}$ | $p_{\cdot,3}$ | $p_{\cdot,4}$ | $\sum = 1$ |

- while $p_{j,k}$ denotes the probability to fall into row $j$ and column $k$
- and the *row sums* $p_{j,\cdot}$, the *column sums* $p_{\cdot,k}$, and *total sum* $\sum_{j,k} p_{j,k} = 1$
- Independence means, that

$$p_{j,k} = \mathbb{P}(\text{row } j \text{ and column } k) = \mathbb{P}(\text{row } j) \cdot \mathbb{P}(\text{column } k) = p_{j,\cdot} \cdot p_{\cdot,k}$$

# Model

- $n = 230$ data in $d = d_1 \cdot d_2 = 8$ categories
  feature 1 (underground) has $d_1 = 2$ categories
  and feature 2 (color) has $d_2 = 4$ categories
- observed frequencies: $x_{1,1}, x_{1,2}, \cdots, x_{2,4}$ ($x_{j,k} \to$ row $j$, column $k$)
- Model: Let $\mathfrak{X} = (X_{1,1}, \ldots, X_{d_1,d_2})^t \sim mult(n, p)$
  with $p = (p_{1,1}, \ldots, p_{d_1,d_2})^t \in (0,1)^{d_1 \cdot d_2}$ and $\sum_{j,k} p_{j,k} = 1$

| $x_{j,k}$ | side | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|---|
| underground | solid | 25 | 22 | 19 | 14 | 80 |
| | soft | 27 | 19 | 44 | 60 | 150 |
| | $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| occupation probabilities | red | orange | grün | blue | $p_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | $p_{1,1}$ | $p_{1,2}$ | $p_{1,3}$ | $p_{1,4}$ | $p_{1,\cdot}$ |
| soft | $p_{2,1}$ | $p_{2,2}$ | $p_{2,3}$ | $p_{2,4}$ | $p_{2,\cdot}$ |
| $p_{\cdot,k}$ | $p_{\cdot,1}$ | $p_{\cdot,2}$ | $p_{\cdot,3}$ | $p_{\cdot,4}$ | $\sum = 1$ |

- while $p_{j,k}$ denotes the probability to fall into row $j$ and column $k$
- and the *row sums* $p_{j,\cdot}$, the *column sums* $p_{\cdot,k}$, and *total sum* $\sum_{j,k} p_{j,k} = 1$
- Independence means, that (e.g., $p_{1,2} = p_{1,\cdot} \cdot p_{\cdot,2}$)

$$p_{j,k} = \mathbb{P}(\text{row } j \text{ and column } k) = \mathbb{P}(\text{row } j) \cdot \mathbb{P}(\text{column } k) = p_{j,\cdot} \cdot p_{\cdot,k}$$

# Model and null hypothesis

- observed frequencies: $x_{1,1}, x_{1,2}, \cdots, x_{2,4}$
- Model: Let $\mathfrak{X} = (X_{1,1}, \ldots, X_{d_1,d_2})^t \sim mult(n, p)$
  with $p = (p_{1,1}, \ldots, p_{d_1,d_2})^t \in (0,1)^{d_1 \cdot d_2}$ and $\sum_{j,k} p_{j,k} = 1$

| $x_{j,k}$ | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\mathbb{P}_{H_0}$ | red | orange | green | blue | $p_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | $p_{1,\cdot} \cdot p_{\cdot,1}$ | $p_{1,\cdot} \cdot p_{\cdot,2}$ | $p_{1,\cdot} \cdot p_{\cdot,3}$ | $p_{1,\cdot} \cdot p_{\cdot,4}$ | $p_{1,\cdot}$ |
| soft | $p_{2,\cdot} \cdot p_{\cdot,1}$ | $p_{2,\cdot} \cdot p_{\cdot,2}$ | $p_{2,\cdot} \cdot p_{\cdot,3}$ | $p_{2,\cdot} \cdot p_{\cdot,4}$ | $p_{2,\cdot}$ |
| $p_{\cdot,k}$ | $p_{\cdot,1}$ | $p_{\cdot,2}$ | $p_{\cdot,3}$ | $p_{\cdot,4}$ | $\sum = 1$ |

- Independence means that

$$p_{j,k} = \mathbb{P}(\text{row } j \text{ and column } k) = \mathbb{P}(\text{row } j) \cdot \mathbb{P}(\text{column } k) = p_{j,\cdot} \cdot p_{\cdot,k}$$

# Model and null hypothesis

- observed frequencies: $x_{1,1}, x_{1,2}, \cdots, x_{2,4}$
- Model: Let $\mathfrak{X} = (X_{1,1}, \ldots, X_{d_1,d_2})^t \sim mult(n,p)$
  with $p = (p_{1,1}, \ldots, p_{d_1,d_2})^t \in (0,1)^{d_1 \cdot d_2}$ and $\sum_{j,k} p_{j,k} = 1$
- Null hypothesis:
  $H_0 : p = p_0 := (p_{1,\cdot} \cdot p_{\cdot,1}, \ldots, p_{d_1,\cdot} \cdot p_{\cdot,d_2})^t \in (0,1)^{d_1 \cdot d_2}$
  and $\sum_{j=1}^{d_1} p_{j,\cdot} = \sum_{k=1}^{d_2} p_{\cdot,k} = 1$

| $x_{j,k}$ | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\mathbb{P}_{H_0}$ | red | orange | green | blue | $p_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | $p_{1,\cdot} \cdot p_{\cdot,1}$ | $p_{1,\cdot} \cdot p_{\cdot,2}$ | $p_{1,\cdot} \cdot p_{\cdot,3}$ | $p_{1,\cdot} \cdot p_{\cdot,4}$ | $p_{1,\cdot}$ |
| soft | $p_{2,\cdot} \cdot p_{\cdot,1}$ | $p_{2,\cdot} \cdot p_{\cdot,2}$ | $p_{2,\cdot} \cdot p_{\cdot,3}$ | $p_{2,\cdot} \cdot p_{\cdot,4}$ | $p_{2,\cdot}$ |
| $p_{\cdot,k}$ | $p_{\cdot,1}$ | $p_{\cdot,2}$ | $p_{\cdot,3}$ | $p_{\cdot,4}$ | $\sum = 1$ |

- Independence means that

$$p_{j,k} = \mathbb{P}(\text{row } j \text{ and column } k) = \mathbb{P}(\text{row } j) \cdot \mathbb{P}(\text{column } k) = p_{j,\cdot} \cdot p_{\cdot,k}$$

# Model and null hypothesis

- observed frequencies: $x_{1,1}, x_{1,2}, \cdots, x_{2,4}$
- Model: Let $\mathfrak{X} = (X_{1,1}, \ldots, X_{d_1,d_2})^t \sim mult(n, p)$
  with $p = (p_{1,1}, \ldots, p_{d_1,d_2})^t \in (0,1)^{d_1 \cdot d_2}$ and $\sum_{j,k} p_{j,k} = 1$
- Null hypothesis:
  $H_0 : p = p_0 := (p_{1,\cdot} \cdot p_{\cdot,1}, \ldots, p_{d_1,\cdot} \cdot p_{\cdot,d_2})^t \in (0,1)^{d_1 \cdot d_2}$
  and $\sum_{j=1}^{d_1} p_{j,\cdot} = \sum_{k=1}^{d_2} p_{\cdot,k} = 1$

| $x_{j,k}$ | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\mathbb{P}_{H_0}$ | red | orange | green | blue | $p_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | $p_{1,\cdot} \cdot p_{\cdot,1}$ | $p_{1,\cdot} \cdot p_{\cdot,2}$ | $p_{1,\cdot} \cdot p_{\cdot,3}$ | $p_{1,\cdot} \cdot p_{\cdot,4}$ | $p_{1,\cdot}$ |
| soft | $p_{2,\cdot} \cdot p_{\cdot,1}$ | $p_{2,\cdot} \cdot p_{\cdot,2}$ | $p_{2,\cdot} \cdot p_{\cdot,3}$ | $p_{2,\cdot} \cdot p_{\cdot,4}$ | $p_{2,\cdot}$ |
| $p_{\cdot,k}$ | $p_{\cdot,1}$ | $p_{\cdot,2}$ | $p_{\cdot,3}$ | $p_{\cdot,4}$ | $\sum = 1$ |

- Independence means that

$$p_{j,k} = \mathbb{P}(\text{row } j \text{ and column } k) = \mathbb{P}(\text{row } j) \cdot \mathbb{P}(\text{column } k) = p_{j,\cdot} \cdot p_{\cdot,k}$$

- Expectations in the categories under $H_0$: $\mathbb{E}_{H_0}[\cdot] = n \cdot p_{j,\cdot} \cdot p_{\cdot,k}$

# Observed and expected frequencies

| $x_{j,k}$ | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\mathbb{E}_{H_0}[\cdot]$ | red | orange | green | blue |
|---|---|---|---|---|
| solid | $n \cdot p_{1,\cdot} \cdot p_{\cdot,1}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,2}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,3}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,4}$ |
| soft | $n \cdot p_{2,\cdot} \cdot p_{\cdot,1}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,2}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,3}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,4}$ |

# Observed and expected frequencies

| $x_{j,k}$ | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\mathbb{E}_{H_0}[\cdot]$ | red | orange | green | blue |
|---|---|---|---|---|
| solid | $n \cdot p_{1,\cdot} \cdot p_{\cdot,1}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,2}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,3}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,4}$ |
| soft | $n \cdot p_{2,\cdot} \cdot p_{\cdot,1}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,2}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,3}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,4}$ |

- Now we can compare 'observed' and 'expected under $H_0$'

# Observed and expected frequencies

| $x_{j,k}$ | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\mathbb{E}_{H_0}[\cdot]$ | red | orange | green | blue |
|---|---|---|---|---|
| solid | $n \cdot p_{1,\cdot} \cdot p_{\cdot,1}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,2}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,3}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,4}$ |
| soft | $n \cdot p_{2,\cdot} \cdot p_{\cdot,1}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,2}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,3}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,4}$ |

- Now we can compare 'observed' and 'expected under $H_0$'

$$\sum_{j,k} \frac{(x_{j,k} - n \cdot p_{j,\cdot} \cdot p_{\cdot,k})^2}{n \cdot p_{j,\cdot} \cdot p_{\cdot,k}} \qquad (*)$$

- Problem: products $p_{j,\cdot} \cdot p_{\cdot,k}$ unknown in practice

# Observed and expected frequencies

| $x_{j,k}$ | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\mathbb{E}_{H_0}[\cdot]$ | red | orange | green | blue |
|---|---|---|---|---|
| solid | $n \cdot p_{1,\cdot} \cdot p_{\cdot,1}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,2}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,3}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,4}$ |
| soft | $n \cdot p_{2,\cdot} \cdot p_{\cdot,1}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,2}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,3}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,4}$ |

- Now we can compare 'observed' and 'expected under $H_0$'

$$\sum_{j,k} \frac{(x_{j,k} - n \cdot p_{j,\cdot} \cdot p_{\cdot,k})^2}{n \cdot p_{j,\cdot} \cdot p_{\cdot,k}} \qquad (*)$$

- Problem: products $p_{j,\cdot} \cdot p_{\cdot,k}$ unknown in practice
- Solution: Estimate marginal probabilities via marginal frequencies

# Observed and expected frequencies

| $x_{j,k}$ | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\mathbb{E}_{H_0}[\cdot]$ | red | orange | green | blue |
|---|---|---|---|---|
| solid | $n \cdot p_{1,\cdot} \cdot p_{\cdot,1}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,2}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,3}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,4}$ |
| soft | $n \cdot p_{2,\cdot} \cdot p_{\cdot,1}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,2}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,3}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,4}$ |

- Now we can compare 'observed' and 'expected under $H_0$'

$$\sum_{j,k} \frac{(x_{j,k} - n \cdot p_{j,\cdot} \cdot p_{\cdot,k})^2}{n \cdot p_{j,\cdot} \cdot p_{\cdot,k}} \qquad (*)$$

- Problem: products $p_{j,\cdot} \cdot p_{\cdot,k}$ unknown in practice
- Solution: Estimate marginal probabilities via marginal frequencies
- More precisely: row proportions $x_{j,\cdot}/n$ estimate row probabilities $p_{j,\cdot}$ and column proportions $x_{\cdot,k}/n$ estimates column probabilities $p_{\cdot,k}$

# Observed and expected frequencies

| $x_{j,k}$ | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|
| solid | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\mathbb{E}_{H_0}[\cdot]$ | red | orange | green | blue |
|---|---|---|---|---|
| solid | $n \cdot p_{1,\cdot} \cdot p_{\cdot,1}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,2}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,3}$ | $n \cdot p_{1,\cdot} \cdot p_{\cdot,4}$ |
| soft | $n \cdot p_{2,\cdot} \cdot p_{\cdot,1}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,2}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,3}$ | $n \cdot p_{2,\cdot} \cdot p_{\cdot,4}$ |

- Now we can compare 'observed' and 'expected under $H_0$'

$$\sum_{j,k} \frac{(x_{j,k} - n \cdot p_{j,\cdot} \cdot p_{\cdot,k})^2}{n \cdot p_{j,\cdot} \cdot p_{\cdot,k}} \qquad (*)$$

- Problem: products $p_{j,\cdot} \cdot p_{\cdot,k}$ unknown in practice
- Solution: Estimate marginal probabilities via marginal frequencies
- More precisely: row proportions $x_{j,\cdot}/n$ estimate row probabilities $p_{j,\cdot}$ and column proportions $x_{\cdot,k}/n$ estimates column probabilities $p_{\cdot,k}$
  i.e., $(x_{j,\cdot} \cdot x_{\cdot,k})/n$ estimates $n \cdot p_{j,\cdot} \cdot p_{\cdot,k}$

# Observed and expected frequencies

| $x_{j,k}$ | red | orange | green | blue | $x_{j,.}$ |
|---|---|---|---|---|---|
| hard | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{.,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\hat{\mathbb{E}}_{H_0}[\cdot]$ | rot | orange | green | blue |
|---|---|---|---|---|
| hard | $(80 \cdot 52)/n$ | $(80 \cdot 41)/n$ | $(80 \cdot 63)/n$ | $(80 \cdot 74)/n$ |
| soft | $(150 \cdot 52)/n$ | $(150 \cdot 41)/n$ | $(150 \cdot 63)/n$ | $(150 \cdot 74)/n$ |

- Now we can compare 'observed' and 'expected under $H_0$'

$$\sum_{j,k} \frac{(x_{j,k} - n \cdot p_{j,.} \cdot p_{.,k})^2}{n \cdot p_{j,.} \cdot p_{.,k}} \qquad (*)$$

- Estimate marginal probabilities via marginal frequencies
  i.e., $(x_{j,.} \cdot x_{.,k})/n$ estimates $n \cdot p_{j,.} \cdot p_{.,k}$

# Observed and expected frequencies

| $x_{j,k}$ | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|
| hard | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\hat{\mathbb{E}}_{H_0}[\cdot]$ | rot | orange | green | blue |
|---|---|---|---|---|
| hard | $(80 \cdot 52)/n$ | $(80 \cdot 41)/n$ | $(80 \cdot 63)/n$ | $(80 \cdot 74)/n$ |
| soft | $(150 \cdot 52)/n$ | $(150 \cdot 41)/n$ | $(150 \cdot 63)/n$ | $(150 \cdot 74)/n$ |

- Now we can compare 'observed' and 'expected under $H_0$'

$$\sum_{j,k} \frac{(x_{j,k} - n \cdot p_{j,\cdot} \cdot p_{\cdot,k})^2}{n \cdot p_{j,\cdot} \cdot p_{\cdot,k}} \qquad (*)$$

- Estimate marginal probabilities via marginal frequencies
  i.e., $(x_{j,\cdot} \cdot x_{\cdot,k})/n$ estimates $n \cdot p_{j,\cdot} \cdot p_{\cdot,k}$
- plugging the estimator into $(*)$ yields the $\chi^2$-statistic

$$x^2 = \sum_{j,k} \frac{\left(x_{j,k} - \frac{x_{j,\cdot} \cdot x_{\cdot,k}}{n}\right)^2}{\frac{x_{j,\cdot} \cdot x_{\cdot,k}}{n}}$$

# Observed and expected frequencies

| $x_{j,k}$ | red | orange | green | blue | $x_{j,.}$ |
|---|---|---|---|---|---|
| hard | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{.,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\hat{\mathbb{E}}_{H_0}[\cdot]$ | rot | orange | green | blue |
|---|---|---|---|---|
| hard | $(80 \cdot 52)/n$ | $(80 \cdot 41)/n$ | $(80 \cdot 63)/n$ | $(80 \cdot 74)/n$ |
| soft | $(150 \cdot 52)/n$ | $(150 \cdot 41)/n$ | $(150 \cdot 63)/n$ | $(150 \cdot 74)/n$ |

- Now we can compare 'observed' and 'expected under $H_0$'

$$\sum_{j,k} \frac{(x_{j,k} - n \cdot p_{j,.} \cdot p_{.,k})^2}{n \cdot p_{j,.} \cdot p_{.,k}} \qquad (*)$$

- Estimate marginal probabilities via marginal frequencies i.e., $(x_{j,.} \cdot x_{.,k})/n$ estimates $n \cdot p_{j,.} \cdot p_{.,k}$
- plugging the estimator into $(*)$ yields the $\chi^2$-statistic

$$x^2 = \sum_{j,k} \frac{\left(x_{j,k} - \frac{x_{j,.} \cdot x_{.,k}}{n}\right)^2}{\frac{x_{j,.} \cdot x_{.,k}}{n}} \approx 19.3$$

# Observed and expected frequencies

| $x_{j,k}$ | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|
| hard | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\hat{\mathbb{E}}_{H_0}[\cdot]$ | rot | orange | green | blue |
|---|---|---|---|---|
| hard | $(80 \cdot 52)/n$ | $(80 \cdot 41)/n$ | $(80 \cdot 63)/n$ | $(80 \cdot 74)/n$ |
| soft | $(150 \cdot 52)/n$ | $(150 \cdot 41)/n$ | $(150 \cdot 63)/n$ | $(150 \cdot 74)/n$ |

- Now we can compare 'observed' and 'expected under $H_0$'

$$\sum_{j,k} \frac{(x_{j,k} - n \cdot p_{j,\cdot} \cdot p_{\cdot,k})^2}{n \cdot p_{j,\cdot} \cdot p_{\cdot,k}} \qquad (*)$$

- Estimate marginal probabilities via marginal frequencies
  i.e., $(x_{j,\cdot} \cdot x_{\cdot,k})/n$ estimates $n \cdot p_{j,\cdot} \cdot p_{\cdot,k}$
- plugging the estimator into $(*)$ yields the $\chi^2$-statistic

$$x^2 = \sum_{j,k} \frac{\left(x_{j,k} - \frac{x_{j,\cdot} \cdot x_{\cdot,k}}{n}\right)^2}{\frac{x_{j,\cdot} \cdot x_{\cdot,k}}{n}} \approx 19.3$$

...is this a large value?

# Observed and expected frequencies

| $x_{j,k}$ | red | orange | green | blue | $x_{j,\cdot}$ |
|---|---|---|---|---|---|
| hard | 25 | 22 | 19 | 14 | 80 |
| soft | 27 | 19 | 44 | 60 | 150 |
| $x_{\cdot,k}$ | 52 | 41 | 63 | 74 | $n = 230$ |

| $\hat{\mathbb{E}}_{H_0}[\cdot]$ | rot | orange | green | blue |
|---|---|---|---|---|
| hard | $(80 \cdot 52)/n$ | $(80 \cdot 41)/n$ | $(80 \cdot 63)/n$ | $(80 \cdot 74)/n$ |
| soft | $(150 \cdot 52)/n$ | $(150 \cdot 41)/n$ | $(150 \cdot 63)/n$ | $(150 \cdot 74)/n$ |

- Now we can compare 'observed' and 'expected under $H_0$'

$$\sum_{j,k} \frac{(x_{j,k} - n \cdot p_{j,\cdot} \cdot p_{\cdot,k})^2}{n \cdot p_{j,\cdot} \cdot p_{\cdot,k}} \qquad (*)$$

- Estimate marginal probabilities via marginal frequencies
  i.e., $(x_{j,\cdot} \cdot x_{\cdot,k})/n$ estimates $n \cdot p_{j,\cdot} \cdot p_{\cdot,k}$
- plugging the estimator into $(*)$ yields the $\chi^2$-statistic

$$x^2 = \sum_{j,k} \frac{\left(x_{j,k} - \frac{x_{j,\cdot} \cdot x_{\cdot,k}}{n}\right)^2}{\frac{x_{j,\cdot} \cdot x_{\cdot,k}}{n}} \approx 19.3$$

...is this a large value?

yes, as the comparison with the $\chi^2$-distribution reveals...

# The $\chi^2$-test for independence

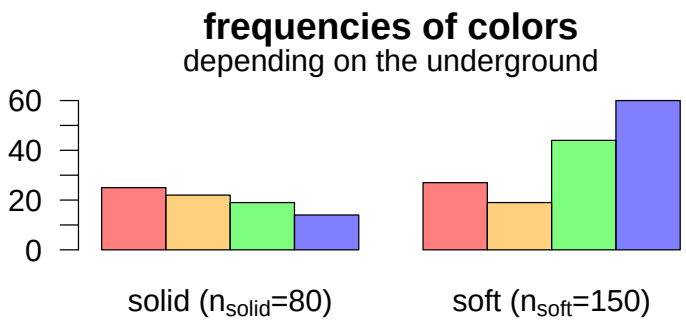
- Model: Let $\mathfrak{x} = (X_{1,1}, \ldots, X_{d_1,d_2})^t \sim mult(n,p)$
  with $p = (p_{1,1}, \ldots, p_{d_1,d_2})^t \in (0,1)^{d_1 \cdot d_2}$ and $\sum_{j,k} p_{j,k} = 1$

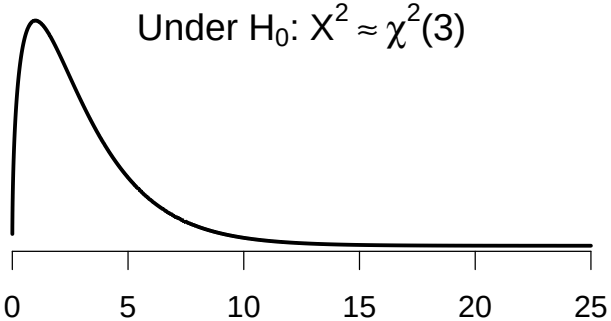
Under $H_0 : p = p_0 := (p_{1,\cdot} \cdot p_{\cdot,1}, \ldots, p_{d_1,\cdot} \cdot p_{\cdot,d_2})^t \in (0,1)^{d_1 \cdot d_2}$
and $\sum_{j=1}^{d_1} p_{j,\cdot} = \sum_{k=1}^{d_2} p_{\cdot,k} = 1$

it holds (approximately)

$$X^2 := \sum_{j,k} \frac{\left(X_{j,k} - \frac{X_{j,\cdot} \cdot X_{\cdot,k}}{n}\right)^2}{\frac{X_{j,\cdot} \cdot X_{\cdot,k}}{n}} \overset{d}{\approx} \chi^2((d_1 - 1) \cdot (d_2 - 1))$$

in fact, it holds that $X^2 \xrightarrow{d} \chi^2((d_1 - 1)(d_2 - 1))$ as $n \to \infty$



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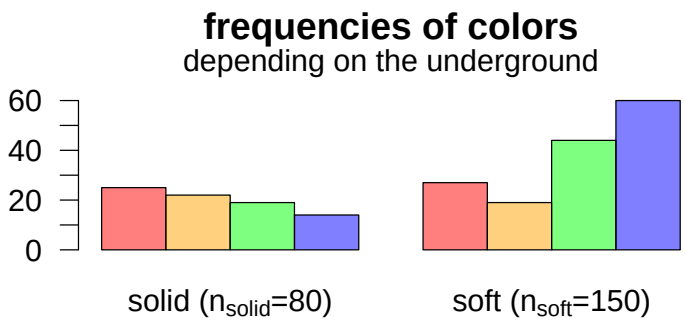


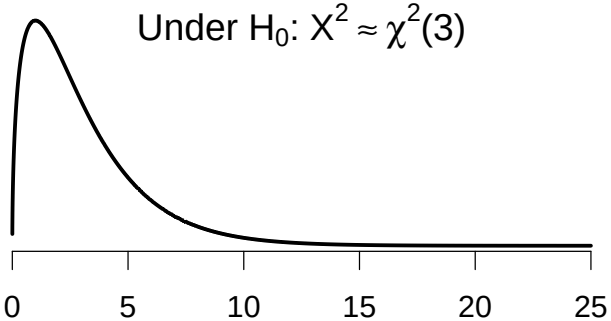

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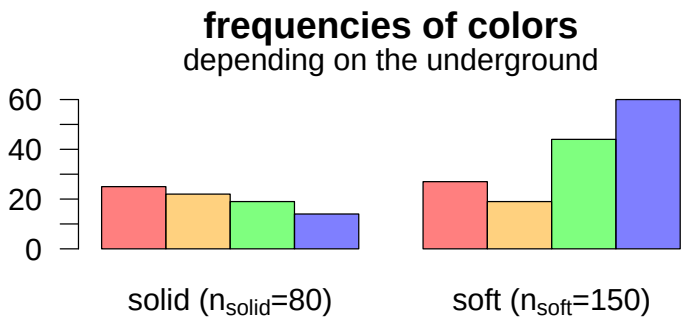


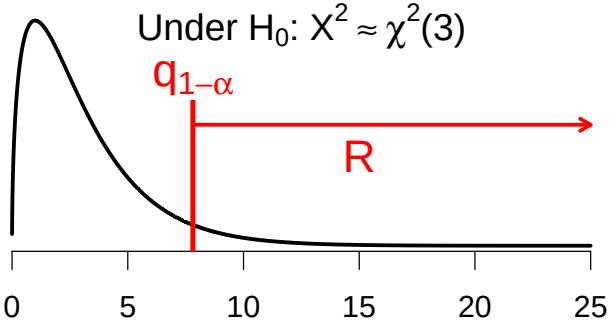

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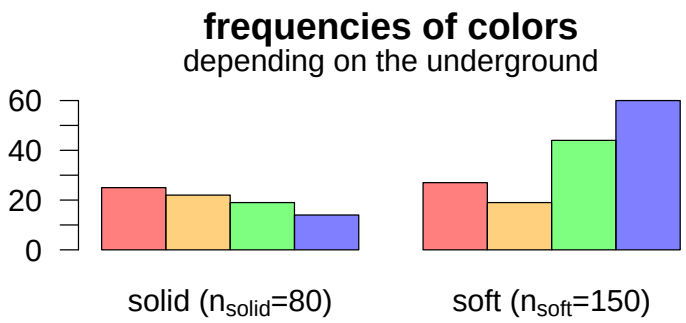


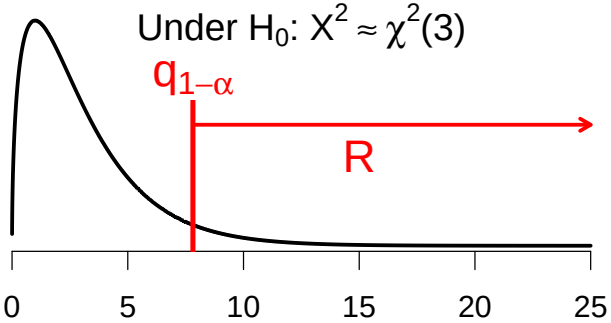

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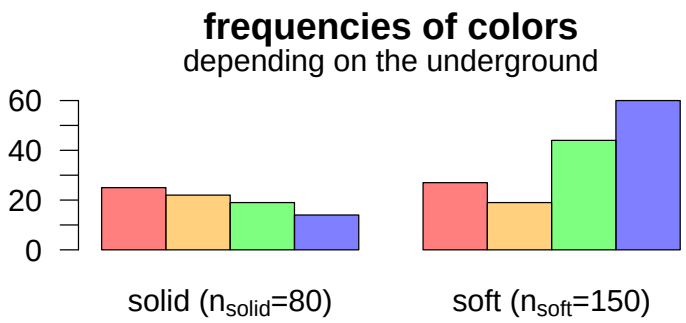


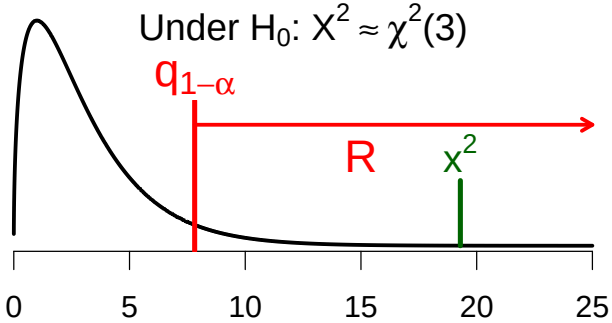

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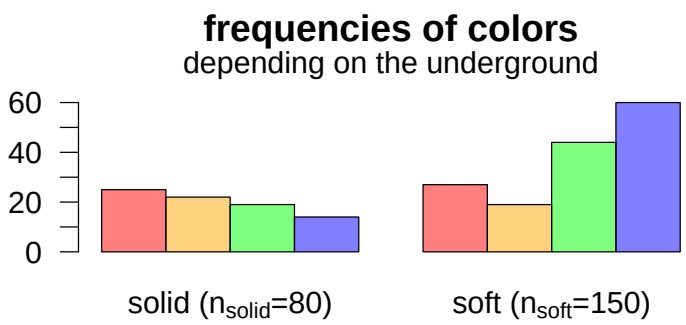


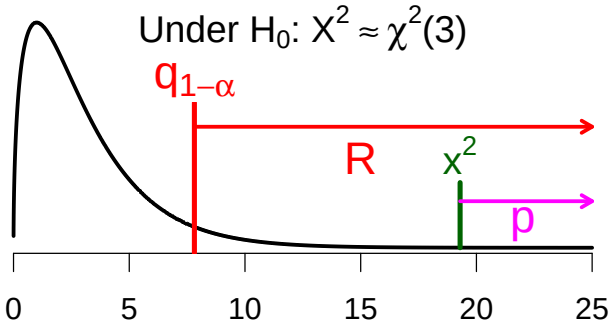

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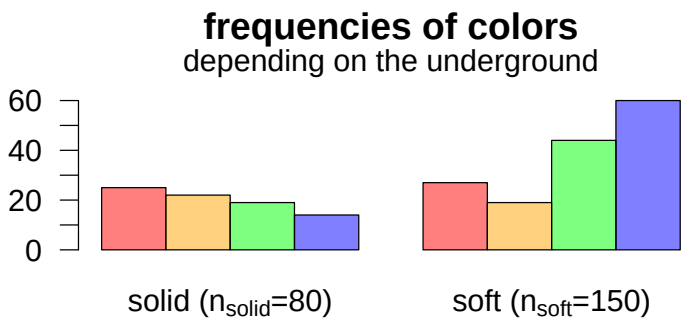
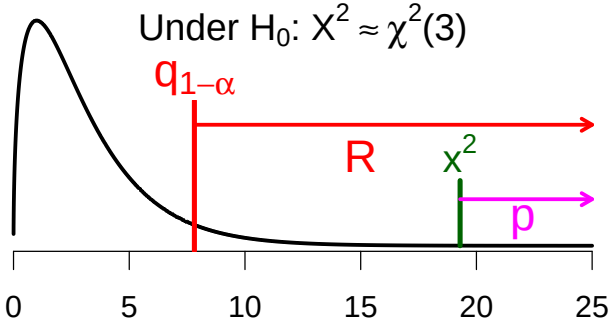
- data: $x^2 = 19.3 \in R$, $\rightarrow$ reject $H_0$
- $p \approx 2.4 \cdot 10^{-4}$. If the features are independent, then in less than one of 4000 cases we observe a discrepancy, which is at least as extreme as in the data



# Nutshell

| $x_{j,k}$ | | | | |
|---|---|---|---|---|
| | $x_{1,1}$ | $\cdots$ | $x_{1,d_2}$ | |
| | $\vdots$ | $\ddots$ | $\vdots$ | |
| | $x_{d_1,1}$ | $\cdots$ | $x_{d_1,d_2}$ | |
| | | | | $n$ |

- 2 features, with $d_1$ and $d_2$ categories, $\rightarrow d_1 \cdot d_2$ 'cells'

# Nutshell

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| | $\vdots$ | $\ddots$ | $\vdots$ | |
| | $x_{d_1,1}$ | $\cdots$ | $x_{d_1,d_2}$ | |
| | | | | $n$ |

- 2 features, with $d_1$ and $d_2$ categories, $\rightarrow d_1 \cdot d_2$ 'cells'
- observed frequencies

# Nutshell

| $x_{j,k}$ | | | |
|---|---|---|---|
| | $x_{1,1}$ | $\cdots$ | $x_{1,d_2}$ |
| | $\vdots$ | $\ddots$ | $\vdots$ |
| | $x_{d_1,1}$ | $\cdots$ | $x_{d_1,d_2}$ |

$n$

| $\hat{\mathbb{E}}_{H_0}[\cdot]$ | | | |
|---|---|---|---|
| | $(x_{1,\cdot} \cdot x_{\cdot,1})/n$ | $\cdots$ | $(x_{1,\cdot} \cdot x_{\cdot,d_2})/n$ |
| | $\vdots$ | $\ddots$ | $\vdots$ |
| | $(x_{d_1,\cdot} \cdot x_{\cdot,1})/n$ | $\cdots$ | $(x_{d_1,\cdot} \cdot x_{\cdot,d_2})/n$ |

- 2 features, with $d_1$ and $d_2$ categories, $\rightarrow d_1 \cdot d_2$ 'cells'
- observed frequencies are compared to the expected frequencies under the null hypothesis

# Nutshell

| $x_{j,k}$ | | | | |
|---|---|---|---|---|
| | $x_{1,1}$ | $\cdots$ | $x_{1,d_2}$ | $x_{1,\cdot}$ |
| | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| | $x_{d_1,1}$ | $\cdots$ | $x_{d_1,d_2}$ | $x_{d_1,\cdot}$ |
| | $x_{\cdot,1}$ | $\cdots$ | $x_{\cdot,d_2}$ | $n$ |

| $\hat{\mathbb{E}}_{H_0}[\cdot]$ | | | |
|---|---|---|---|
| | $(x_{1,\cdot} \cdot x_{\cdot,1})/n$ | $\cdots$ | $(x_{1,\cdot} \cdot x_{\cdot,d_2})/n$ |
| | $\vdots$ | $\ddots$ | $\vdots$ |
| | $(x_{d_1,\cdot} \cdot x_{\cdot,1})/n$ | $\cdots$ | $(x_{d_1,\cdot} \cdot x_{\cdot,d_2})/n$ |

- 2 features, with $d_1$ and $d_2$ categories, $\rightarrow d_1 \cdot d_2$ 'cells'
- observed frequencies are compared to the expected frequencies under the null hypothesis (estimated from the marginal frequencies)

# Nutshell

| $x_{j,k}$ | | | | |
|---|---|---|---|---|
| | $x_{1,1}$ | $\cdots$ | $x_{1,d_2}$ | $x_{1,\cdot}$ |
| | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| | $x_{d_1,1}$ | $\cdots$ | $x_{d_1,d_2}$ | $x_{d_1,\cdot}$ |
| | $x_{\cdot,1}$ | $\cdots$ | $x_{\cdot,d_2}$ | $n$ |

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- observed frequencies are compared to the expected frequencies under the null hypothesis (estimated from the marginal frequencies)
- Comparison ('over all cells') through the $\chi^2$-statistic

$$x^2 = \sum_{j,k} \frac{\left(x_{j,k} - \frac{x_{j,\cdot} \cdot x_{\cdot,k}}{n}\right)^2}{\frac{x_{j,\cdot} \cdot x_{\cdot,k}}{n}}$$

# Nutshell

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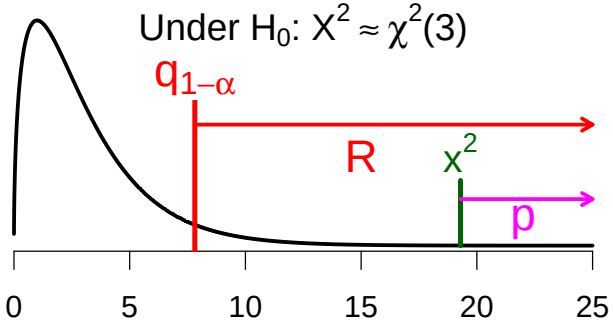


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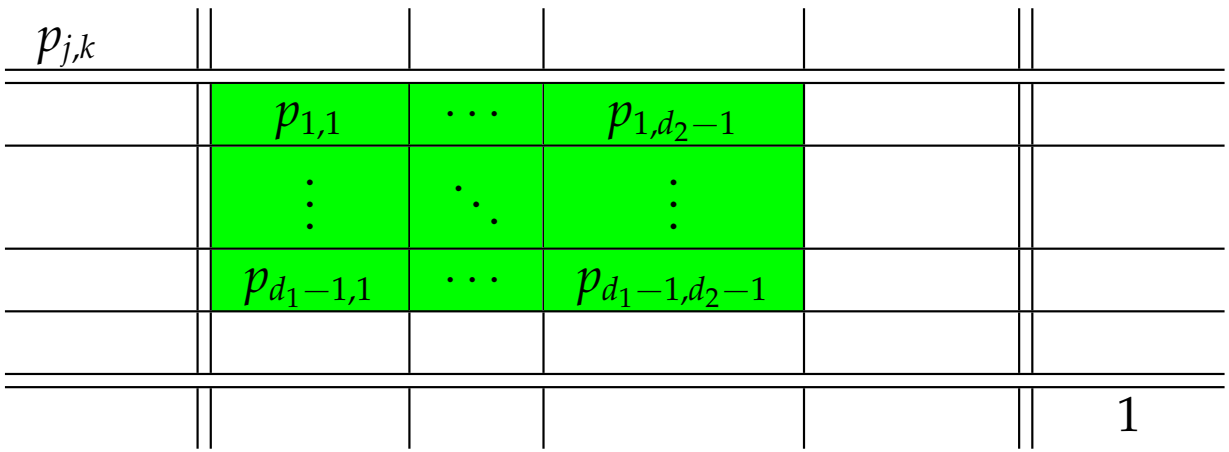
- Judgment of the discrepancy according to the $\chi^2((d_1 - 1) \cdot (d_2 - 1))$-dist.

# Remarks

- It holds: $X^2 \xrightarrow{d} \chi^2((d_1 - 1) \cdot (d_2 - 1))$ under $H_0$ as $n \to \infty$
- Why $(d_1 - 1) \cdot (d_2 - 1)$ degrees of freedom?
- Intuition: only $(d_1 - 1) \cdot (d_2 - 1)$ probabilities can be chosen 'freely'



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- The marginal probabilities

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|---|---|---|---|---|---|
| | $p_{1,1}$ | $\cdots$ | $p_{1,d_2-1}$ | | $p_{1,\cdot}$ |
| | $\vdots$ | $\ddots$ | $\vdots$ | | $\vdots$ |
| | $p_{d_1-1,1}$ | $\cdots$ | $p_{d_1-1,d_2-1}$ | | $p_{d_1-1,\cdot}$ |
| | | | | | $p_{d_1,\cdot}$ |
| | $p_{\cdot,1}$ | $\cdots$ | $p_{\cdot,d_2-1}$ | $p_{\cdot,d_2}$ | 1 |

# Remarks

- It holds: $X^2 \xrightarrow{d} \chi^2((d_1-1)\cdot(d_2-1))$ under $H_0$ as $n \to \infty$
- Why $(d_1-1)\cdot(d_2-1)$ degrees of freedom?
- Intuition: only $(d_1-1)\cdot(d_2-1)$ probabilities can be chosen 'freely'
- The marginal probabilities already fix the other probabilities e.g.,

$$p_{j,d_2} = p_{j,\cdot} - \sum_{k=1}^{d_2-1} p_{j,k}$$

| $p_{j,k}$ | | | | | |
|---|---|---|---|---|---|
| | $p_{1,1}$ | $\cdots$ | $p_{1,d_2-1}$ | $p_{1,d_2}$ | $p_{1,\cdot}$ |
| | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| | $p_{d_1-1,1}$ | $\cdots$ | $p_{d_1-1,d_2-1}$ | $p_{d_1-1,d_2}$ | $p_{d_1-1,\cdot}$ |
| | $p_{d_1,1}$ | $\cdots$ | $p_{d_1,d_2-1}$ | $p_{d_1,d_2}$ | $p_{d_1,\cdot}$ |
| | $p_{\cdot,1}$ | $\cdots$ | $p_{\cdot,d_2-1}$ | $p_{\cdot,d_2}$ | 1 |

# Remarks

- It holds: $X^2 \xrightarrow{d} \chi^2((d_1-1)\cdot(d_2-1))$ under $H_0$ as $n \to \infty$
- Why $(d_1-1)\cdot(d_2-1)$ degrees of freedom?
- Intuition: only $(d_1-1)\cdot(d_2-1)$ probabilities can be chosen 'freely'
- The marginal probabilities already fix the other probabilities e.g.,

$$p_{j,d_2} = p_{j,\cdot} - \sum_{k=1}^{d_2-1} p_{j,k}$$

| $p_{j,k}$ | | | | | |
|---|---|---|---|---|---|
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| | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ |
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- The convergence of $X^2$ to the $\chi^2$-distribution is again reasonable, as according to the central limit theorem we find every summand approx distributed like a square of a $N(0,1)$-distributed random variable (and just $(d_1-1)\cdot(d_2-1)$ summands are 'free').

# $\chi^2$-test in R

```
# Enter data
die_solid       <- c("red","blue","blue",...)
die_soft        <- c("blue","green","blue",...)
# Compute frequencies, e.g., via
x_solid <- table(die_solid)
x_soft  <- table(die_soft)
# Combine frequencies, e.g., as a matrix
x <- rbind(x_solid,x_soft)
x
     1  2  3  4
[1,] 25 22 19 14
[2,] 27 19 44 60
# Perform chi^2-test
chisq.test(x)
# Output

        Pearson's Chi-squared test

data:  x
X-squared = 19.295, df = 3, p-value = 0.0002376
```

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  $\rightarrow$ ANOVA, metric data

Thank you!