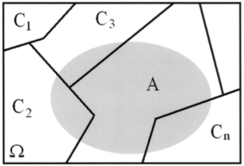
TECHNISCHE UNIVERSITÄT WIEN

Probability

Conditional probability and the Bayes theorem



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  - Discrete sample space - countable (listable) and it can be finite or infinite
    - Example: $\{H, T\}$, $\{1, 3, 5, \ldots\}$ are discrete sample spaces
      The interval $0 \leqslant x \leqslant 1$ is not discrete, but continuous
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  $P(A \cup B) = P(A) + P(B)$ for all disjoint A, B
- $(\Omega, \mathcal{P}(\Omega), P)$ is called a probability space

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- Probability of an event A

$$P(A) = \sum_{\omega_i \in A} p_i$$

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- Probability: each outcome is equally likely with probability $\frac{1}{4}$

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$$B = \{HH, HT, TH\} \qquad \text{and} \qquad P(B) = \frac{3}{4}$$

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  $C = \emptyset$ and $P(C) = 0$

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- Event B: head appears in at least 3 throws

$$B = \Omega \backslash \{H, TH\} \quad \text{and} \quad P(B) = 1 - (\frac{1}{2} + \frac{1}{2^2}) = 0.25$$

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HW What is the probability that 2 taxis will pass by? 5? at least 8?

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		1	2	3	4	5	6
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	2	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
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two dimensional probability table (Table 1)

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- A: one gets the same number on both dice

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	2	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	4	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

two dimensional probability table (Table 1)

- A: one gets the same number on both dice
  - $A = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$ and $P(A) = \frac{6}{36} = \frac{1}{6}$

Simple examples

(5) Two dice (choice of sample space)

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- probability: these 11 outcomes are *not* all equally likely.

ω_i	2	3	4	5	6	7	8	9	10	11	12	
p_i	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1

Table 2

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HW

- B: The total is even
- C: the sum is at most 10
- D: the sum is 1
- Are A and C disjoint? Compute $A \cap B$, $B \cup C$ and $B^c \cap D$.

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- Question: What is the relationship between the two probability tables above?

Simple examples

HW The statistical experiment

- Experiment: random selection of k objects from a set of n distinguishable objects $\mathcal{M} = \{1, 2, \ldots, n\}$, $k \leqslant n$, assuming an object that has been selected cannot be selected a second time.

  Describe the result sets and determine their size when selected a second time.

  - the order of selection does not matter
  - the order of selection is important.

Simple examples

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(Real probability space)

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We will talk about it later.

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  We can visualize these rules using Venn diagrams.

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    - Since an integer is divisible by 6 if and only if it is divisible by both 2 and 3, we have $C = A \cap B$. Then,
      $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(C) = 0.6 + 0.3 - 0.2 = 0.7.$

Examples

HW If $P(A^c) = \frac{1}{4}$ and $P(B) = \frac{1}{3}$, can A and B be disjoint? Justify your answer.

HW The 24h express delivery has increased significantly in the last ten years. Customer service has consistently been shown to be the greatest influence on a company's success. A study was conducted to examine customer satisfaction with an express delivery service. In addition to their satisfaction, the customers were asked how often they had used the overnight shipping option. The results are shown below

Frequency of Use	**Satisfaction**	**level**		**Total**
	High	**Medium**	**Low**	
<2 per month	250	140	10	400
2-5 per month	140	55	5	200
>5 per month	70	25	5	100
Total	460	220	20	700

- What is the probability that a randomly selected person will use the company two to five times a month or have moderate satisfaction?

Another example

(1) Playing poker
52 cards: 13 ranks $2, \ldots 9, 10$, J, Q, K, A and 4 suits $\heartsuit$, $\spadesuit$, $\diamondsuit$, $\clubsuit$
Poker hands consist of 5 cards

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$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1098240$$

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... note, a full house is <u>not</u> a hand with exactly one pair

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(b) B: we get four kings

- How many different hands are there with four kings? What is the probability of having four kings?

  - If we specify that four of the cards are kings, there are 48 different ways to determine the fifth card. Thus,

$$P(\text{four kings}) = \frac{48}{\binom{52}{5}} = \frac{1}{54145} < \frac{1}{50000}.$$

- We can also calculate the probability through an "updating" argument, by calculating conditional probabilities.

  - The probability that the first card is a king is $\frac{4}{52}$. Assuming the first card is a king, the probability that the second card is a king is $\frac{3}{51}$. If we continue this argument, we get that the probability is four kings

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This is called the conditional probability because additional conditions are taken into account.

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HW Draw two cards from a deck of cards. Let the events be
D_1 = "first card is a diamond" and D_2 = "second card is a diamond".
What are $P(D_2|D_1)$, $P(D_1|D_2)$ und $P(D_2|D_1^c)$?

Multiplication theorem and the Bayes theorem

- We rewrite the definition of conditional probability and obtain

$$P(A \cap B) = P(A|B) \cdot P(B) \quad \text{for } P(B) > 0 \qquad \text{and}$$
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- We equate the two expressions and obtain

$$\boxed{P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} \qquad \text{and} \qquad P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}}$$

which gives the formula for reversing ("turning around") conditional probabilities. The previous equation is called the Bayes theorem (for two events).

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HW Let A and B be independent. Knowning that $P(A|B) = 0.2$ and $P(B|A) = 0.5$, compute the probability $P(A^c \cap B)$.

Example

- We toss a fair coin four times. Consider the events
  $A =$ " at least three heads" and
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  What are $P(A|B)$ and $P(B|A)$? Are A and B independent?

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  - Since $P(A|B) = \frac{1}{8} \neq \frac{5}{16} = P(A)$, we conclude that A and B are not independent.

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$A =$ "first die is 3"
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Answer: (c)

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- If only the conditional probabilities and the probabilities of the conditional event are known, the total probability of A results from

$$P(A) = P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c),$$

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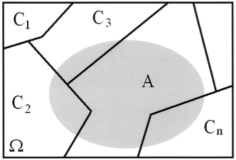

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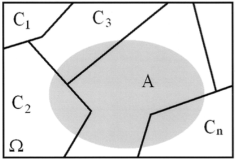

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Example

- An urn contains 5 red and 2 green balls. A random ball is selected and replaced with a ball of the other color, then a second ball is drawn.
  (1) What is the probability the second ball is red?
  (2) What is the probability the first ball was red given the second ball was red?

  Use the probability tree to solve the problems.

Answer:

- Denote by R_1 = "the first ball is red" and R_2= the second ball is red.
- R_1^c="the first ball is green" and R_2^c="the second ball is green"
- $P(R_1) = \frac{5}{7}$ and $P(R_1^c) = 1 - \frac{5}{7} = \frac{2}{7}$
- $P(R_2|R_1) = \frac{4}{7}$ and $P(R_2^c|R_1) = \frac{3}{7}$
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Example cont.

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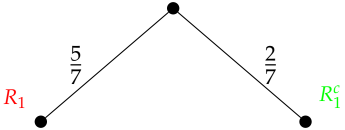

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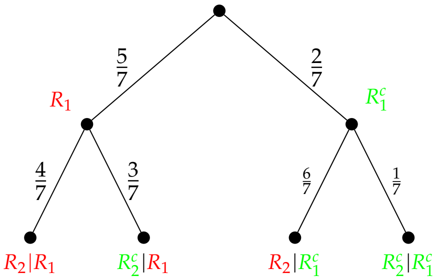

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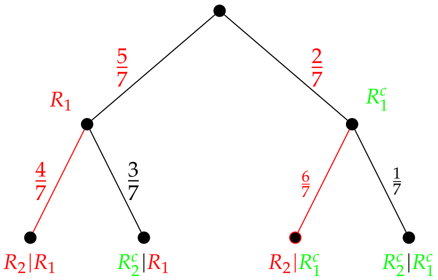

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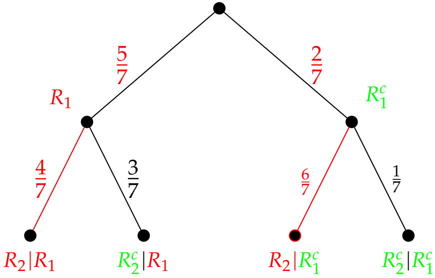

(1) From the Law of total probability we obtain

$$P(R_2) = P(R_2|R_1) \cdot P(R_1) + P(R_2|R_1^c) \cdot P(R_1^c) = \frac{5}{7} \cdot \frac{4}{7} + \frac{2}{7} \cdot \frac{6}{7} = \frac{32}{49}$$

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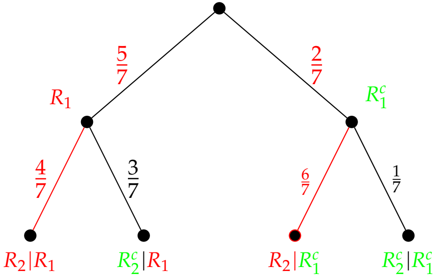

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$$P(R_1|R_2) = \frac{P(R_1 \cap R_2)}{P(R_2)} = \frac{P(R_2|R_1) \cdot P(R_1)}{P(R_2)} = \frac{20/49}{32/49} = \frac{20}{32}$$

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  What is the best strategy for winning a car?

  (a) switch (b) do not switch (c) it does not matter

Example: Monty Hall problem

- In a TW show, a candidate can win a car if he guesses correctly behind which of the three doors the car is.
  - There are three doors, one door hides a car, two hide goats.
  - The participant chooses any door.
  - The presenter always opens a door with a goat (because he knows where the car is).
  - The candidate can then change doors if he wants.

  What is the best strategy for winning a car?

  (a) switch (b) do not switch (c) it does not matter

  Answer:

  - We build a tree representing the switching strategy:
    first the candidate chooses a door (then Monty shows a goat),
    then the candidate switches doors or not

Example: Monty Hall problem

- Denote the doors by T_1, T_2 and T_3. Let the car be behind the first door T_1.

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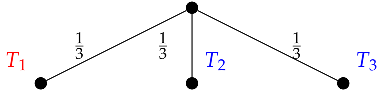

Candidate chooses the door

Example: Monty Hall problem

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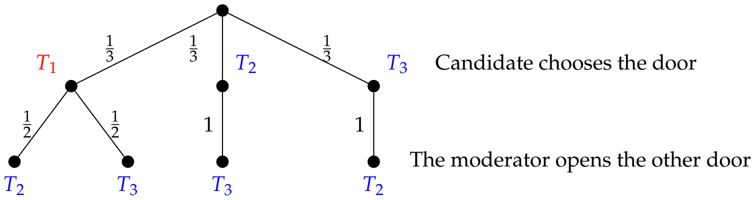

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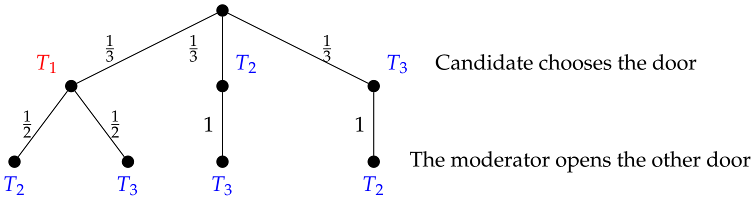

$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	probability
				the decision:
T_1	T_1	T_2	T_3	stick to the first choice
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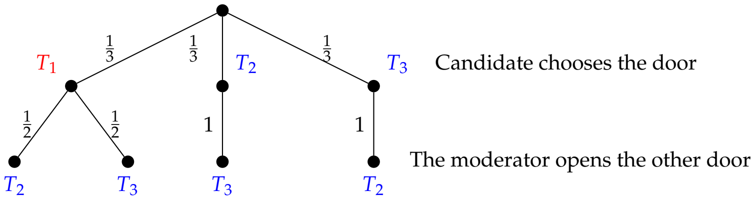

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				the decision:
T_1	T_1	T_2	T_3	stick to the first choice
T_3	T_2	T_1	T_1	switch

- The probability of winning the car for both scenarios is

$$P(\text{win the car}|\text{stick to the first choice}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$P(\text{win the car}|\text{switch the first choice}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Example: Batteries

- A container contains three types of batteries Type 1, Type 2 and Type 3 in the ratio of $20 : 30 : 50$. A battery is good if it lasts more than 100 hours. The probability that a Type 1 battery is good is 0.7, the probability that a Type 2 battery is good is 0.4 and the probability that a Type 3 battery is good is 0.3. A battery is randomly taken from the container.
  (1) What is the probability the chosen battery is good?
  (2) What is the probability the battery is of Type 3 given it was not good?

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  - Let $G =$ 'battery is good", $H_i =$ "battery is of Type i", where $i = 1, 2, 3$.

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  (1) Then,

$$
\begin{aligned}
P(G) &= P(G|H_1) \cdot P(H_1) + P(G|H_2) \cdot P(H_2) + P(G|H_3) \cdot P(H_3) \\
&= 0.2 \cdot 0.7 + 0.3 \cdot 0.4 + 0.5 \cdot 0.3 = 0.41
\end{aligned}
$$

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(2)

$$P(H_3|G^c) = \frac{P(G^c|H_3) \cdot P(H_3)}{P(G^c)} = \frac{P(G^c|H_3) \cdot P(H_3)}{1 - P(G)} = \frac{0.5 \cdot 0.7}{0.59} = \frac{35}{59} = 0.593$$

More examples

HW A blood test reacts 95% positive if there is a disease. It also reacts to 1% "false positive". It is assumed that 0.5% of the population are ill.

(1) What is the probability a randomly selected person whose blood test is positive is actually sick?
(2) What is the probability of being sick if the blood test is positive?

HW The Randomizer holds the 6-sided die in one fist and the 8-sided die in the other. The Roller selects one of the Randomizer's fists and takes the die face down. The Roller rolls the die in secret and reports the result to the table.

- Given the reporeted number, what is the probability that the 6-sided die was selected? (Find the probability for each possible number reported.)

Questions

A few multiple-choice questions

(1) A medical treatment has a success rate of 0.8. Two patients will be treated with this treatment. Assuming the results are independent for the two patients, what is the probability that at least one of them will be successfully cured?

(a) 0.96
(b) 0.32
(c) 0.64
(d) 0.04

(2) In a list of 15 households, 9 own homes and 6 do not own homes. Four households are randomly selected from these 15 households. Find the probability that the number of households in these four who own homes is at most one.

(a) 0.1536
(b) 0.1792
(c) 0.3456
(d) 0.4752

A few multiple-choice questions

(3) Suppose that there are 4 women and 8 men. How many 5 person committees can be formed with exactly 2 women and 3 men?

(a) $\binom{12}{5}$
(b) $\binom{4}{2} \cdot \binom{8}{2}$
(c) $\binom{4}{2} \cdot \binom{8}{3}$
(d) $\binom{12}{5} - \binom{4}{1} \cdot \binom{8}{4}$

(4) Suppose box A contains 4 red and 5 blue coins and box B contains 6 red and 3 blue coins. A coin is chosen at random from the box A and placed in box B. Finally, a coin is chosen at random from among those now in box B. What is the probability a blue coin was transferred from box A to box B given that the coin chosen from box B is red?

(a) $\frac{15}{29}$
(b) $\frac{14}{29}$
(c) $\frac{1}{2}$
(d) $\frac{7}{10}$

A few multiple-choice questions

(5) A fair die is rolled. Find the probability of getting an even number or a number bigger than 2.

(a) $\frac{1}{3}$

(b) $\frac{7}{12}$

(c) $\frac{2}{3}$

(d) $\frac{5}{6}$

(6) Anna has an unfair coin. She knows that when fliping this coin, head is obtained twice as often as a tail. Anna rolls this coin until she obtains a tail. What is the probability that she rolled the coin at most three times?

(a) $\frac{19}{27}$

(b) $\frac{26}{27}$

(c) $\frac{5}{9}$

(d) $\frac{4}{27}$

Thank you for your attention!