Homework - Serie 12

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LaTeX

Problem 1. *Lemma von Lax-Milgram* in the finite dimensional case reads: Let X be a finite dimensional vector space over $\mathbf{R}$ with the basis $\{v_1, \ldots, v_n\}$, $F : X \to \mathbf{R}$ linear and $a(\cdot, \cdot) : X \times X \to \mathbf{R}$ a bilinear form on X, i.e. $a(\cdot, \cdot)$ is linear in both components. Further, we assume $a(v, v) > 0$ for all $v \in X$. Then there exists a unique $u \in X$ with $a(u, v) = F(v)$ for all $v \in X$. To prove this, one uses the approach $u = \sum_{k=1}^{n} x_k v_k$ and shows that the coefficient vector $x \in \mathbf{R}^n$ is unique. Formulate the Lemma of Lax-Milgram as theorem with proof in LaTeX and extend the document of the previous exercises. All appearing references should be realized via `\label` and `\ref` etc.

Problem 2. Write the following definition of the characteristic polynomial of a matrix $A \in \mathbf{R}^{n \times n}$

$$p(t) = \det(A - t \cdot \mathrm{Id}) = \begin{vmatrix} a_{11} - t & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - t & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - t \end{vmatrix}$$

in LaTeX. Note the symbol Id instead of Id for the identity matrix. **Hint:** To generate the right hand side you can use the `array` or `matrix` environment.

Problem 3. Formulate the following assertion as a theorem, prove it with techniques of linear algebra and write the theorem with its proof in LaTeX, where all appearing references should be realized via `\label` and `\ref` etc. If $A \in \mathbf{R}^{n \times n}$ is a matrix with $\sum_{j,k=1}^{n} x_j A_{jk} x_k > 0$ for all $x \in \mathbf{R}^n$, then A is regular.

Problem 4. Let I be a nonempty open interval. Then it holds for $f, g \in C^\infty(I)$ and $n \in \mathbf{N}$

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

Write a LaTeX-file which includes the assertion and the (detailed) proof of the product rule for the n-th derivative. All appearing references should be realised via `\label` and `\ref` etc.

Problem 5. Formulate the following result as theorem including its with proof in LaTeX. All appearing references should be realised via `\label` and `\ref` etc.
Let $n \in \mathbf{N}$. It holds:

$$\sqrt{n} \in \begin{cases} \mathbb{N}, & \text{if } n \text{ is a square number,} \\ \mathbb{R} \setminus \mathbb{Q}, & \text{otherwise.} \end{cases}$$

Write a LaTeX-file which includes the assertion (formulated as theorem) and the (detailed) proof of this assertion.
Hint. You may use the fact that each natural number x has a unique prime factorisation, i.e., there exists a unique finite sequence of prime numbers $2 \le p_1 \le \cdots \le p_k$ with $x = \prod_{j=1}^k p_j$.

Problem 6. Write a LaTeX-code which produces

$$\begin{aligned}
\int_{-1}^{1} \sqrt{1-x^2}\,dx &= \Big[x\sqrt{1-x^2}\Big]_{x=-1}^{1} - \int_{-1}^{1} \frac{x(-2x)}{2\sqrt{1-x^2}}\,dx \\
&= \Big[x\sqrt{1-x^2}\Big]_{x=-1}^{1} + \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} - \int_{-1}^{1} \frac{1-x^2}{\sqrt{1-x^2}}\,dx \\
&= \Big[x\sqrt{1-x^2} + \arcsin x\Big]_{x=-1}^{1} - \int_{-1}^{1} \sqrt{1-x^2}\,dx.
\end{aligned}$$

Problem 7. Formally a triangle T with vertices $x, y, z \in \mathbf{R}^2$ is defined as convex hull of these points

$$\operatorname{conv}(x,y,z) := \{ax + by + cz : \ a,b,c \ge 0 \text{ with } a+b+c = 1\}.$$

The triangle T is called non-degenerated if the vectors $y - x$ and $z - x$ are linearly independent. Formulate the following result with proof in LaTeX. Let $T = \operatorname{conv}(x,y,z)$ and $\widetilde{T} = (\widetilde{x}, \widetilde{y}, \widetilde{z})$ be two non-degenerated triangles. Then, there exists an affine bijection $\Phi : T \to \widetilde{T}$, i.e., a bijective mapping of the form $\Phi(v) = Av + b$ with a matrix $A \in \mathbf{R}^{2\times 2}$ and a vector $b \in \mathbf{R}^2$. Here, the symbol $\widetilde{T}$ is obtained by `\widetilde T`. The symbol $\times$ is obtained by `\times`. Note the symbol conv instead of *conv* for the convex hull.

Problem 8. Write the following text in LaTeX: Let $\Omega \subseteq \mathbf{R}^d$ (with $d \ge 3$) be a bounded domain with Lipschitz-boundary and $u \in C^2(\Omega)$ a solution of the Laplace equation $\Delta u = 0$ in Ω.[1] Then, there holds the representation formula

$$\forall x \in \Omega : \quad u(x) = \frac{1}{4\pi} \int_{\partial\Omega} \frac{1}{|x-y|} \frac{\partial}{\partial\nu(y)} u(y)\,dy - \frac{1}{4\pi} \int_{\partial\Omega} \Big(\frac{\partial}{\partial\nu(y)} \frac{1}{|x-y|}\Big) u(y)\,dy.$$

[1] Recall that the Laplace operator Δ is defined for all $x \in \Omega$ by $(\Delta u)(x) := \sum_{i=1}^d \frac{\partial}{\partial x_i} u(x) = 0$.