Homework - Serie 11

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LaTeX

Problem 1. Write a LaTeX-file in which the following theorem of Brezzi is formulated.

Theorem (Brezzi 1974). Let X and Y be Hilbert spaces. Further, let $a : X \times X \to \mathbf{R}$ and $b : X \times Y \to \mathbf{R}$ be continuous bilinear forms and $X_0 := \big\{x \in X \,:\, b(x, \cdot) = 0 \in Y^*\big\}$. Under the assumptions

- $\alpha := \inf\limits_{v \in X_0 \setminus \{0\}} \dfrac{a(v,v)}{\|v\|_X^2} > 0$, i.e., $a(\cdot,\cdot)$ is coercive auf X_0,
- $\beta := \inf\limits_{\substack{y \in Y\\ y \neq 0}} \sup\limits_{\substack{x \in X\\ x \neq 0}} \dfrac{b(x,y)}{\|x\|_X \|y\|_Y} > 0$

there holds the assertion: For each $(x^*, y^*) \in X^* \times Y^*$ there is a unique solution $(x, y) \in X \times Y$ of the so-called saddle point problem

$$\begin{array}{lcll} a(x,\widetilde{x}) \;+\; b(\widetilde{x},y) & = & x^*(\widetilde{x}) & \text{for all } \widetilde{x} \in X, \\ b(x,\widetilde{y}) & = & y^*(\widetilde{y}) & \text{for all } \widetilde{y} \in Y. \end{array} \tag{SP}$$

Problem 2. Write the following text in LaTeX: The Gamma function is defined as

$$\Gamma(x) := \lim_{n \to \infty} \frac{n! n^x}{x(x+1)\cdots(x+n)}.$$

There holds the Weierstraß product representation

$$\frac{1}{\Gamma(x)} = x \cdot e^{Cx} \cdot \prod_{k=1}^{\infty} \Big(1 + \frac{x}{k}\Big) e^{-x/k} \quad \text{mit} \quad C := \lim_{n \to \infty} \bigg(\sum_{k=1}^{n} \frac{1}{k} - \ln n \bigg).$$

Here, `\infty` is the symbol ∞, and $\cdot$ resp. $\cdots$ is obtained by `\cdot` resp. `\cdots`.

Problem 3. Write the following text in LaTeX, where the symbol $\pm$ is generated by `\pm`: For given ***basis*** $b \in \mathbf{N}_{\geq 2}$, ***mantissa length*** $t \in \mathbf{N}$ and ***exponential bounds*** $e_{\min} < 0 < e_{\max}$ we define the set of ***normalized floating point numbers*** $F := F(b, t, e_{\min}, e_{\max}) \subset \mathbf{R}$ by

$$F = \{0\} \cup \bigg\{ \Big(\sigma \sum_{k=1}^{t} a_k b^{-k} \Big) b^e \,\bigg|\, \sigma \in \{\pm 1\}, a_j \in \{0, \ldots, b-1\}, a_1 \neq 0, e \in \mathbb{Z}, e_{\min} \leq e \leq e_{\max} \bigg\}.$$

The finite sum $a = \sum_{k=1}^{t} a_k b^{-k}$ is called ***normalized mantissa*** of a floating point number.

Problem 4. Write the following formula in LaTeX-file: For $q \in \mathbf{R}$, it holds that

$$\lim_{n\to\infty} q^n = \begin{cases} +\infty & \text{falls } q > 1, \\ 1 & \text{falls } q = 1, \\ 0 & \text{falls } -1 < q < 1, \\ \nexists & \text{falls } q \leq -1. \end{cases}$$

The symbol $\nexists$ is generated via `\nexists` or `\not\exists`.

Problem 5. Write the following definition of a Vandermonde matrix:

$$V := \begin{pmatrix} 1 & x_1 & x_2^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} \in \mathbf{R}^{n\times n}$$

in LaTeX. The dots are generated by `\cdots`, `\vdots` and `\ddots`, the symbol $\times$ by `\times`.

Problem 6. In sympy a matrix can be defined with `sympy.Matrix` (syntax is the same as for numpy arrays). Define the matrix V from the previous exercise for $n = 6$ in sympy and compute its determinant. Use `sympy.simplify` to simplify the expression. Now use `sympy.latex` to convert the expression into latex code and put this into a LaTeX file.

Problem 7. The matrix $L \in \mathbf{R}^{n\times n}$ has the following form

$$L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}$$

with $L_{11} \in \mathbf{R}^{k\times k}$ and $0 < k < n$. If L_{11} and L_{22} are regular, then L is regular as well, and the inverse is given by

$$L^{-1} = \begin{pmatrix} L_{11}^{-1} & 0 \\ -L_{22}^{-1}L_{21}L_{11}^{-1} & L_{22}^{-1} \end{pmatrix}.$$

Formulate the result with its proof in LaTeX.

Problem 8. Use `\newtheorem`, to generate a new *theorem*-environment. Write as well a *proof*-environment. The proof should start (as part of the environment) with bold-italic ***Proof***. The end of the proof (as part of the environment) should be indicated with a right-aligned `\blacksquare` $\blacksquare$, i.e., there is a right-aligned $\blacksquare$ at the end of the proof. Formulate and prove the following theorem in LaTeX. All appearing references should be realised via `\label` and `\ref` etc. Write a suitable macro for norms and $\operatorname{dist}(\cdot,\cdot)$.

Let $A, B \subset \mathbf{R}$ open intervals with compact closure $\overline{A}, \overline{B}$ and $A \cap B = \emptyset$. We define the boundary of the sets as $\partial A := \overline{A} \setminus A$ and $\partial B := \overline{B} \setminus B$ (the symbol ∂ is generated by `\partial`). Then, there holds for the distances of the two sets that $\operatorname{dist}(A, B) = \operatorname{dist}(\partial A, \partial B)$, where we define for arbitrary sets $C, D \subset \mathbf{R}$

$$\operatorname{dist}(C, D) := \inf\{\|x - y\|_2 : x \in C, y \in D\} \tag{1}$$

Hint. Show that $\operatorname{dist}(A, B) = \operatorname{dist}(\overline{A}, \overline{B})$. Next, note that the infimum in (1) is a minimum for compact sets C, D.
Remark. The theorem also holds for open subsets $A, B \subset \mathbf{R}^n$ with $n \in \mathbf{N}$.