Exercise 2

Discrete Mathematics

October 22, 2020

Example 11

A graph $H = (V', E')$ is an induced subgraph of $G = (V, E)$ if $V' \subseteq V$ and any edge in G connecting two vertices a, b in V' is in E'.

Let G be a connected simple graph that does not have path or cycle with four vertices as an induced subgraph. Show that G has a vertex adjacent to all other vertices. (Hint: Consider a vertex of maximum degree u and assume u is not connected with some vertex w. Consider the shortest path between u and w and show that there exists a vertex in this path with a degree larger than that of u.)

Solution: In case of the complete graphs $G = K_1, G = K_2, G = K_3$ all vertices are connected to all other vertices. Consequently, any vertex can be the required vertex. In case of the path graph on 3 vertices, the middle vertex is the required vertex.

In any other case there are at least 4 vertices $u, v, w, x \in V(G)$. Any graph has at least one vertex of maximum degree $\Delta(G)$ by definition. Let u be such a vertex and w be an arbitrary vertex.

Proof of by contradiction. Any two vertices of any graph are either adjacent or not adjacent. Assume u and w are not adjacent: $(u, w) \notin E(G)$. As G is connected, there is some (shortest) path p between u and w. There cannot be less than 3 vertices (including u, w) on this path, as $(u, w) \notin E(G)$. There cannot be more than 3 vertices on this (or any other) path, as G does not have a path with 4 vertices as induced subgraph. Hence, there is exactly one vertex v between u and w on p. Any other node $y \in V(G)$ must also be connected in some way to this path, as G is connected. y cannot be adjacent to u because that would create the path (y, u, v, w) of length 4 and it cannot be adjacent to w because that would create the path (u, v, w, y) of length 4. We still have to connect x to something. The only node that we can still make x adjacent to is v. Then v has strictly more neighbors than u: $\deg(v) > \deg(u)$. However, this contradicts the assumption that u is a vertex of maximum degree $\Delta(G)$. As a consequence, u and w must be adjacent. w was chosen arbitrarily. Therefore, u is adjacent to all other vertices. This concludes the proof.

Example 12

Let G be a connected graph with an even number of vertices. Show that G has a spanning (but not necessarily connected) subgraph with all vertices of odd degree. Show that this is not necessarily the case for arbitrary graphs.

Solution: We call a parity designation σ of G a function that assigns to each vertex v of G a desired parity (odd or even) for its degree, with the only restriction that there is an even number of vertices with odd desired parity.

We prove that given a parity designation σ of G there is a spanning subgraph of G that satisfies that parity designation. We prove it by induction and via the use of spanning substrees.

Base case: If G has 2 vertices and is connected then G is K_2 and there are only two possible parity designations (even, even and odd, odd). These are satisfied by the subgraph with no edges and by K_2 itself.

Inductive step. G has $n+1$ vertices. Since G is connected it has a spanning subtree T. This tree has a leaf v. Notice that when we romve v from T the remaining graph is still connected, therefore when we remove v from G we obtain a connected graph G' with n vertices.

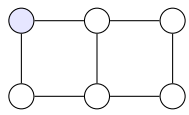


Figure 1: G of example 12

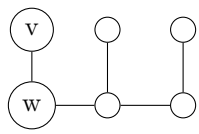


Figure 2: T of example 12

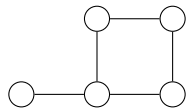


Figure 3: G' of example 12

Case 1 The parity designation φ requires that v is an even vertex. Consider φ restricted to G'. This is a parity designation (because v is supposed to be even, so there is still an even number of vertices in G' that should be odd (by handshaking lemma)). By induction hypothesis, there is a spanning subgraph H' of G' that

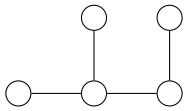


Figure 4: H' of example 12

satisfies φ in G'. Notice the edge of H' also satisfy φ in G' since they leave v with degree 0 which is even as desired.

Case 2 The parity designation φ requires that v is an odd vertex. In this case let w be the only neigbour of v in the spanning subtree T. We define φ', which is going to be a parity designation for G'. We let φ' be equal to φ for every value except w. In w, φ' is going to be the opposite of φ so if it was odd we change it to even and if it was even we change it to odd. By induction there is a subgraph H' of G' that satisifies φ'. If we take the edges of H' together with wv we obtain a spanning subgraph of G that satisfies φ. It is clear that it satisfies φ for all values other than u and v since φ'=φ in these values. In u it also holds because without edge uv the degree was the opposite of what φ indicated, however after adding edge uv it is the same as φ. w also has the desired parity since the order of w is 1, which is odd.

Now let φ be the parity designation that asks for every vertex to be odd. (Notice this is a parity designation since the order of G is even). This concludes the firts part.

The graph $G = (\{a, b\}, \{\})$ is an example that shows that it does not hold for arbitrary graphs.

Example 13

Let T be a tree and let n_d be the number of vertices of degree d in T. Show that the number of leaves of T equals

$$2 + \sum_{d \geq 3} (d-2) n_d$$

Solution: By definition, leaves are vertices degree $d = 1$. Therefore, the number of leaves is n_1.

It can be seen that the number of vertices of a graph is the number of leaves n_1 + the remaining nodes (internal nodes).

$$|V| = n_1 + \sum_{d \geq 2} n_d \tag{1}$$

From the handshake lemma $\sum_v \deg v = 2|E|$, the tree property $|V| = |E| + 1$ and explicitly setting two different notations for the same idea equal $\sum_v \deg v = \sum_{d \geq 1} (d \cdot$

n_d) we get

$$\sum_v \deg v = \sum_{d\geq 1}(d \cdot n_d) = 2|E| = 2(|V|-1) = 2|V|-2$$

From which follows

$$-2 = \sum_{d\geq 1}(d \cdot n_d) - 2|V| \tag{2}$$

We use equation 1 to replace $|V|$ in the right side of equation 2

$$\sum_{d\geq 1}(d \cdot n_d) - 2|V| = \sum_{d\geq 1}(d \cdot n_d) - 2\sum_{d\geq 1} n_d \tag{3}$$

Then we extract some elements out of each of the two sums

$$\begin{aligned}
&\sum_{d\geq 1}(d \cdot n_d) - 2\sum_{d\geq 1} n_d \\
&= 1n_1 + 2n_2 + \sum_{d\geq 3}(d \cdot n_d) - 2\left(n_1 + n_2 + \sum_{d\geq 3} n_d\right) \\
&= 1n_1 + 2n_2 + \sum_{d\geq 3}(d \cdot n_d) - 2n_1 - 2n_2 - 2\sum_{d\geq 3} n_d \\
&= -n_1 + \sum_{d\geq 3}(d \cdot n_d) - 2\sum_{d\geq 3} n_d
\end{aligned}$$

And put them together

$$-n_1 + \sum_{d\geq 3}(d \cdot n_d) - 2\sum_{d\geq 3} n_d = -n_1 + \sum_{d\geq 3}(d \cdot n_d - 2n_d) = -n_1 + \sum_{d\geq 3}(d-2)n_d$$

Note that everything was equal since equation 3, giving $\sum_{d\geq 1}(d \cdot n_d) - 2|V| = -n_1 + \sum_{d\geq 3}(d-2)n_d$. We now substitute this result in equation 2 and get the final implication

$$-2 = -n_1 + \sum_{d\geq 3}(d-2)n_d \implies n_1 = 2 + \sum_{d\geq 3}(d-2)n_d$$

By our initial definition of n_1, this concludes the proof.

Example 14

Show that the number of spanning trees of the complete graph on n vertices K_n is n^{n-2}, using the matrix tree theorem. Hint: To compute the determinant of the resulting matrix, add all rows except the first one to the first row. Then add the first row of this new matrix to the other rows.

Solution: The matrix tree theorem says that for undirected connected graph, the number of spanning trees is

$$\det(D(G) - A(G))' \tag{4}$$

where ' means deleting one row and one column of the matrix. For complete graphs the matrices $A(G), D(G)$ all have the same form, with n rows and n columns.

$$D = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix} A = \begin{pmatrix} n-1 & 0 & 0 & \dots & 0 \\ 0 & n-1 & 0 & \dots & 0 \\ 0 & 0 & n-1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & n-1 \end{pmatrix} \tag{5}$$

And therefore also their difference always looks equal. Here the matrix for D has n rows and n columns.

$$D = \begin{pmatrix} n-1 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & -1 & \dots & n-1 \end{pmatrix} \tag{6}$$

and the matrix for D' looks equal but with $n-1$ rows and $n-1$ columns. For example, for $n = 4$ we get

$$D = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \quad D' = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \tag{7}$$

If we now follow the hint about adding rows on the matrices D' we get the following new matrices D''

$$D'' = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & n & 0 & \dots & 0 \\ 0 & 0 & n & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & n \end{pmatrix} \tag{8}$$

$$D'' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{9}$$

We can observe that there is exactly $n-2$ times n in the main diagonal.

> **Theorem** Suppose $B = [b_{ij}]$ is an $n \times n$ matrix and choose any fixed $i, j \in \{1, 2, \dots, n\}$. Suppose i' is a fixed choice of $i \in \{1, 2, \dots, n\}$. Then

its determinant $|B| = \det(B)$ is given by:

$$\det(B) = [(-1)^{i'+1} b_{i'1} \det(M_{i'1})] + [(-1)^{i'+2} b_{i'2} \det(M_{i'2})] + \dots \\ + [(-1)^{i'+n} b_{1n} \det(M_{i'n})] \quad (10)$$

where $\det(M_{ij})$ is the minor of element B_{ij}, i.e. the determinant of the submatrix M_{ij} formed by removing the ith row and the jth column of matrix B.

For our previous example equation 9 we would calculate the following Laplace expansion:

$$|D''| = 1 \cdot \begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 0 \\ 0 & 4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix} \\ = 1 \cdot (4 \cdot 4 - 0 \cdot 0) - 1 \cdot (0 \cdot 4 + 0 \cdot 0) + 1 \cdot (0 \cdot 0 + 0 \cdot 4) = 16$$

This example makes visible that everything except the main diagonal becomes irrelevant because of 0 as factor in the multiplications.

Together with our observation from equation 8 the Laplace expansion (equation 10) shows that the number of spanning trees of K_n is n^{n-2}. Note that this is a special case of Kirchhoff's matrix tree theorem called Cayley's formula.

Example 16

Show that all bases of a matroid $M = (E, S)$ have the same cardinality.

Solution:

Definition 1 An independence system $M = (E, S)$ is called matroid if the existence of $A, B \in S$ such that $|B| = |A| + 1$ implies that the so called exchange property holds

$$\exists v \in B \setminus A \text{ with } A \cup \{v\} \in S \quad (11)$$

Definition 2 The maximal independent sets of a matroid are called bases.

Proof by contradiction:
Let A, B be arbitrary bases of a matroid. Then by definition 2 A, B are maximal independent sets. Assume wlog that $|A| < |B|$. Then, by definition 1, there exists an element $v \in B \setminus A$ with $A \cup \{v\} \in S$. This means $A \cup \{v\}$ is an independent set. As we added v, A is not maximal. Contradiction. Therefore, any two bases of a matroid have equal cardinality (rank).

Example 17

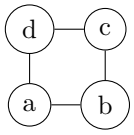
Let $G = (V, E)$ be an undirected graph. Set $M_k(G) = (E, S)$ where

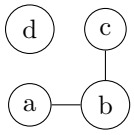
$$S = \{A \subseteq E : A = F \cup M, F \text{ acyclic and } |M| \leq k\}$$

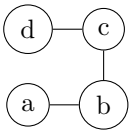
Show that $M_k(G)$ is a matroid.

Solution:

Example: $E = \{ab, ad, bc, cd\}$, $F = \{ab, bc\}$, $M_0 = \{\}$, $M_1 = \{cd\}$, $M_2 = \{ad, cd\}$

$G = A_2$ 

, $A_0 = F \cup M_0$ 

, $A_1 = F \cup M_1$ 

We have two show three properties (the first two make it an independence system):

1. $\emptyset \neq S$
2. If $A \in S$ and $B \subseteq A$ then $B \in S$. This means it is closed under inclusion.
3. The existence of $A, B \in S$ such that $|B| = |A| + 1$ implies that the so called exchange property holds

$$\exists v \in B \setminus A \text{ with } A \cup \{v\} \in S$$

We now show those three properties.

1. For any undirected graph it is possible to let $M = F = A = \emptyset$. Then the empty set is an element in S, that means $\{\} \in S$ or $\{\{\}\} \subseteq S$. Hence, $S \neq \emptyset$
2. Assume $B = F \cup M \in S$ and $A \subseteq B$. First of all, we proof the part before the colon :. As $B \in S$ we get by the task description $B \subseteq E$. We assumed $A \subseteq B$ and by transitivity of $\subseteq$ we get $A \subseteq E$.

   By substitution $A \subseteq (F \cup M)$. It is a property of the subset operation that $A \subseteq (F \cup M) \Leftrightarrow A \cap (F \cup M) = A$. Since intersection distributes over union $A \cap (F \cup M) = (A \cap F) \cup (A \cap M)$ we get $(A \cap F) \cup (A \cap M) = A$. This is exactly the required form for the equation in the task description.

   F is acyclic by assumption and clearly $|A \cap F| \leq |F|$. However, a graph can only get cyclic by adding edges. Therefore, $A \cap F$ must be acyclic aswell. Furthermore, $|M| \leq k$ by assumption and clearly $|A \cap M| \leq |M|$, and by transitivity of $\leq$ we get $|A \cap M| \leq k$.

   This proves that $M_k(G)$ is closed under inclusion.
3. Assume there exist $A = F_A \cup M_A, B = F_B \cup M_B \in S$ such that $|B| = |A| + 1$. By the equation in the task description there are then two possible cases:

a) $|M_A| < |M_B|$. Then we can take an arbitrary element $v \in B \setminus A$ and let $M'_A = M_A \cup \{v\}$. Additionally, let $A' = F_A \cup M'_A$. As $|M'_A| - |M_A| = 1$ we get $|M'_A| \leq |M_B| \leq k$. Furthermore, F_A did not change and remains acyclic. As a consequence, $A' \in S$.

b) $|M_A| \geq |M_B|$. By assumption we have $|F_A| + |M_A| = |F_B| + |M_B| + 1$. By the equation that defines this case and addition & substraction we get $|F_B| - |F_A| - 1 = |M_A| - |M_B| \geq 0$. Therefore, $|F_B| - |F_A| \geq 1$.

   A forest is an undirected graph without cycles. Consequently, F_a, F_b are forests. We know that in a forest with n vertices and m edges the number of connected components is $n - m$. Therefore, the number of components of F_A is $C_A = |V| - |F_A|$ and the number of components of F_B is $C_B = |V| - |F_B|$. Hence, we can first see $C_A + |F_A| = C_B + |F_B|$ and then $C_A - C_B = |F_B| - |F_A| \geq 1$ that F_A has more connected components than F_B.

   Therefore, there is a connected component C in F_B with vertices of at least two connected components of F_A. That means that there is an edge $e \in C$ that connects two distinct connected components of F_A. Then $F'_A = F_A \cup \{e\}$ is still a forest, or rather acyclic. Furthermore, M_A did not change and its cardinality remains equal. As a consequence, $A' = F'_A \cup M_A \in S$.

In either case, we have shown that $\exists v \in B \setminus A$ with $A \cup \{v\} \in S$. This concludes the proof of the exchange property.

As all properties are fulfilled, we have proven that $M_k(G)$ is a matroid.

Example 19

Let J be the set of jobs $\{0, 1, 2, 3, 4\}$ and W be the set of workers $\{04, 0, 0123, 12\}$. Suppose that a job can be done if its number appears in the name of the worker.

List all maximal sets of jobs that can be done simultaneously, i.e., the bases of the matroid considered in the lecture. Then use the greedy algorithm to find an optimal job assignment, where the priority of a job is given by its number.

Solution: There are 4 workers so at most 4 parallel jobs are possible. This means sets of jobs are maximal if they are of cardinality 4. There are $\binom{5}{4} = 5$ such possible assignments. 3 assignments are valid assignments (showing workers):

$$\{0(0), 1(12), 2(0123), 4(04)\}, \{0(0), 1(12), 3(0123), 4(04)\}, \{0(0), 2(12), 3(0123), 4(04)\}$$

In both invalid assignments $\{0, 1, 2, 3\}, \{1, 2, 3, 4\}$, it is necessary, that 3 is done by 0123, but then 12 would have to do 1 and 2 simultaneously.

With input set E, independence system S, weight function w and output element F, applying GREEDY (E, S, w, F):

a) Order the elements of E according to their weight: $(e_1 = 4, e_2 = 3, e_3 = 2, e_4 = 1, e_5 = 0)$

b) $F := \emptyset$

c) for k=1 to m do
    if $F \cup \{e_k\} \in S$ then $F := F \cup \{e_k\}$
end

1. $F = \{4\}$
2. $F = \{4, 3\}$
3. $F = \{4, 3, 2\}$
4. $F = \{4, 3, 2\}$ $\quad e_4 = 1$ cannot be done simultaneously with 2
5. $F = \{4, 3, 2, 0\}$