5. Übung

Aufgabe 2.9:

Sie sollen ein zeitdiskretes System untersuchen, von dem die Beziehung zwischen Eingangssignal $\boldsymbol{x[n]}$ und Ausgangssignal $\boldsymbol{y[n]}$ gegeben ist:

$$\boldsymbol{y[n] = \sum_{k=n-1}^{n+1} (x[k+1] - x[k] + x[k-1])}$$

a) **Prüfen Sie (mit Beweis), ob das System die folgenden Eigenschaften besitzt:**

Linearität

$$\boldsymbol{x[n] = a \cdot x_1[n] + b \cdot x_2[n]}$$
$$\downarrow$$
$$\boldsymbol{y[n] = a \cdot y_1[n] + b \cdot y_2[n]}$$

Kausalität

$$\boldsymbol{y[n] = \cdots + x[n+k] \cdot \ldots, \qquad k \leq 0}$$

Stabilität

$$\boldsymbol{(\ |x[n]| < M < \infty\ )}$$
$$\boldsymbol{|y[n]| < C \cdot M < \infty}$$

Zeitinvarianz

$$\boldsymbol{x[n+n_0] \rightarrow y[n+n_0]}$$

$$y[n] = \sum_{k=n-1}^{n+1} (x[k+1] - x[k] + x[k-1])$$
$$= x[(n-1)+1] - x[n-1] + x[(n-1)-1] + x[n+1] - x[n] + x[n-1]$$
$$+ x[(n+1)+1] - x[n+1] + x[(n+1)-1]$$
$$= x[n] - x[n-1] + x[n-2] + x[n+1] - x[n] + x[n-1] + x[n+2] - x[n+1] + x[n]$$
$$= x[n+2] + x[n] + x[n-2]$$

Linearität

$$y[n] = (a \cdot x_1[n+2] + b \cdot x_2[n+2]) + (a \cdot x_1[n] + b \cdot x_2[n]) + (a \cdot x_1[n-2] + b \cdot x_2[n-2])$$
$$= a \cdot (x_1[n+2] + x_1[n] + x_1[n-2]) + b \cdot (x_2[n+2] + x_2[n] + x_2[n-2])$$
$$= a \cdot y_1[n] + b \cdot y_2[n]$$
$$\rightarrow \text{linear}$$

Kausalität

$$y[n] = x[n+2] + x[n] + x[n-2]$$

$$\rightarrow \text{akausal}$$

Stabilität

$$y[n] = x[n+2] + x[n] + x[n-2]$$

$$y[n] < 3 \cdot M$$

$$\rightarrow \text{stabil}$$

Zeitinvarianz

$$y[n+n_0] = x[(n+n_0)+2] + x[n+n_0] + x[(n+n_0)-2]$$

$$(n' = n + n_0)$$

$$y[n'] = x[n'+2] + x[n'] + x[n'-2]$$

$$\rightarrow \text{zeitinvariant}$$

b) Berechnen und skizzieren Sie die Impulsantwort und die Übertragungsfunktion des Systems.

$$y[n] = x[n+2] + x[n] + x[n-2]$$

$$\downarrow$$

$$h[n] = \delta[n+2] + \delta[n] + \delta[n-2]$$

$$\downarrow$$

Formelsammlung

$$\delta[n - N_0] \leftrightarrow e^{-j\theta N_0}$$

Herleitung der Formel:

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\theta n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n - N_0] \cdot e^{-j\theta n}$$

$$= e^{-j\theta N_0}$$

$$\downarrow$$

$$H(e^{j\theta}) = e^{j\theta \cdot 2} + e^{-j\theta \cdot 0} + e^{-j\theta \cdot 2}$$

$$= 1 + 2 \cdot \cos(2\theta)$$

c) Welches Ausgangssignal liefert das System für $x[n] = (-1)^n \quad \forall n$?

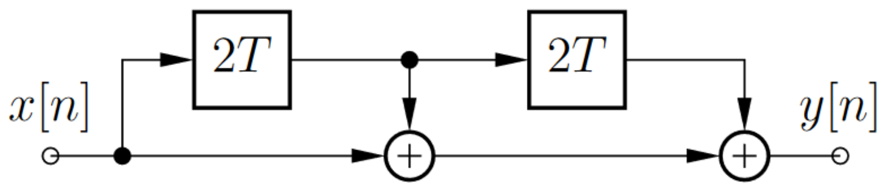
$$y[n] = x[n+2] + x[n] + x[n-2]$$
$$= 3 \cdot (-1)^n$$

d) Nun wird das System mit dem Signal $x[n] = \lambda^n$ angeregt. Wie ist λ zu wählen, damit $y[n] \equiv 0$ ist?

$$y[n] = x[n+2] + x[n] + x[n-2]$$
$$= \lambda^{n+2} + \lambda^n + \lambda^{n-2}$$
$$\downarrow$$
$$\lambda^{n+2} + \lambda^n + \lambda^{n-2} = 0$$
$$(\lambda^2 + \lambda^0 + \lambda^{-2}) \cdot \lambda^n = 0$$
$$\lambda^2 + \lambda^0 + \lambda^{-2} = 0$$
$$\lambda^4 + \lambda^2 + 1 = 0$$
$$(x = \lambda^2)$$
$$x^2 + x + 1 = 0$$
$$x_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}$$
$$= -\frac{1}{2} \pm j \cdot \frac{\sqrt{3}}{2}$$
$$\lambda_{1,2,3,4} = \pm \sqrt{-\frac{1}{2} \pm j \cdot \frac{\sqrt{3}}{2}}$$

e) Geben Sie eine Realisierung des Systems an.

$$y[n] = x[n+2] + x[n] + x[n-2]$$



Aufgabe 3.1:

Berechnen Sie die Fouriertransformation $X\left(e^{j\theta}\right)$ der folgenden Signale:

a) $\boldsymbol{x[n] = \alpha^n \cdot sin(\theta_0 \cdot n) \cdot \sigma[n]}$ **für** $\boldsymbol{|\alpha| < 1}$

$$X\left(e^{j\theta}\right) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\theta n}$$

$$= \sum_{n=-\infty}^{\infty} \alpha^n \cdot sin(\theta_0 \cdot n) \cdot \sigma[n] \cdot e^{-j\theta n}$$

$$= \sum_{n=0}^{\infty} \alpha^n \cdot sin(\theta_0 \cdot n) \cdot e^{-j\theta n}$$

$$\left(\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}\right)$$

$$= \frac{1}{2j} \cdot \sum_{n=0}^{\infty} \alpha^n \cdot \left(e^{j\theta_0 n} - e^{-j\theta_0 n}\right) \cdot e^{-j\theta n}$$

$$= \frac{1}{2j} \cdot \left(\sum_{n=0}^{\infty} \alpha^n \cdot e^{j\theta_0 n} \cdot e^{-j\theta n} - \sum_{n=0}^{\infty} \alpha^n \cdot e^{-j\theta_0 n} \cdot e^{-j\theta n}\right)$$

$$= \frac{1}{2j} \cdot \left(\sum_{n=0}^{\infty} \alpha^n \cdot e^{j(\theta_0 - \theta)n} - \sum_{n=0}^{\infty} \alpha^n \cdot e^{-j(\theta_0 + \theta)n}\right)$$

$$= \frac{1}{2j} \cdot \left(\sum_{n=0}^{\infty} \left(\alpha \cdot e^{j(\theta_0 - \theta)}\right)^n - \sum_{n=0}^{\infty} \left(\alpha \cdot e^{-j(\theta_0 + \theta)}\right)\right)$$

↓

Geometrische Summenformel:

$$\sum_{n=0}^{N} q^n = \frac{1 - q^{N+1}}{1 - q}$$

↓

$$= \frac{1}{2j} \cdot \left(\frac{1 - \left(\alpha \cdot e^{j(\theta_0 - \theta)}\right)^\infty}{1 - (\alpha \cdot e^{j(\theta_0 - \theta)})} - \frac{1 - \left(\alpha \cdot e^{-j(\theta_0 + \theta)}\right)^\infty}{1 - (\alpha \cdot e^{-j(\theta_0 + \theta)})}\right)$$

$$(|\alpha| < 1)$$

$$= \frac{1}{2j} \cdot \left(\frac{1}{1 - \alpha \cdot e^{j(\theta_0 - \theta)}} - \frac{1}{1 - \alpha \cdot e^{-j(\theta_0 + \theta)}}\right)$$

$$= \frac{1}{2j} \cdot \left(\frac{1 - \alpha \cdot e^{-j(\theta_0 + \theta)} - 1 + \alpha \cdot e^{j(\theta_0 - \theta)}}{(1 - \alpha \cdot e^{j(\theta_0 - \theta)}) \cdot (1 - \alpha \cdot e^{-j(\theta_0 + \theta)})}\right)$$

$$= \frac{\alpha \cdot e^{-j\theta} \cdot \frac{e^{j\theta_0} - e^{-j\theta_0}}{2j}}{(1 - \alpha \cdot e^{j(\theta_0 - \theta)}) \cdot (1 - \alpha \cdot e^{-j(\theta_0 + \theta)})}$$

$$= \frac{\alpha \cdot e^{-j\theta} \cdot \sin(\theta_0)}{1 - \alpha \cdot e^{j(\theta_0 - \theta)} - \alpha \cdot e^{-j(\theta_0 + \theta)} + \alpha \cdot e^{j(\theta_0 - \theta)} \cdot \alpha \cdot e^{-j(\theta_0 + \theta)}}$$

$$= \frac{\alpha \cdot e^{-j\theta} \cdot \sin(\theta_0)}{1 - \alpha \cdot e^{j(\theta_0 - \theta)} - \alpha \cdot e^{-j(\theta_0 + \theta)} + \alpha^2 \cdot e^{-2j\theta}}$$

$$= \frac{\alpha \cdot e^{-j\theta} \cdot \sin(\theta_0)}{1 - \alpha \cdot e^{-j\theta} \cdot (e^{j\theta_0} + e^{-j\theta_0}) + \alpha^2 \cdot e^{-2j\theta}}$$

$$= \frac{\alpha \cdot e^{-j\theta} \cdot \sin(\theta_0)}{1 + \alpha^2 \cdot e^{-2j\theta} - 2\alpha \cdot e^{-j\theta} \cdot \cos(\theta_0)}$$

b) $\boldsymbol{x[n] = 2^n \cdot \sigma[-n]}$

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\theta n}$$

$$= \sum_{n=-\infty}^{\infty} 2^n \cdot \sigma[-n] \cdot e^{-j\theta n}$$

$$= \sum_{n=-\infty}^{0} 2^n \cdot e^{-j\theta n}$$

$$= \sum_{n=0}^{\infty} 2^{-n} \cdot e^{j\theta n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{j\theta}}{2}\right)^n$$

$$\downarrow$$

Geometrische Summenformel:

$$\sum_{n=0}^{N} q^n = \frac{1 - q^{N+1}}{1 - q}$$

$$\downarrow$$

$$= \frac{1 - \left(\frac{e^{j\theta}}{2}\right)^{\infty}}{1 - \left(\frac{e^{j\theta}}{2}\right)}$$

$$= \frac{1}{1 - \frac{1}{2} \cdot e^{j\theta}}$$

c) $\boldsymbol{x[n] = \frac{sin\left(\frac{\pi}{2}\cdot n\right)}{\pi n} \cdot \frac{sin\left(\frac{\pi}{4}\cdot n\right)}{\pi n}}$

Formelsammlung

$$x[n] \cdot y[n] \leftrightarrow \frac{1}{2\pi} \cdot (X * Y) \cdot e^{j\theta}$$

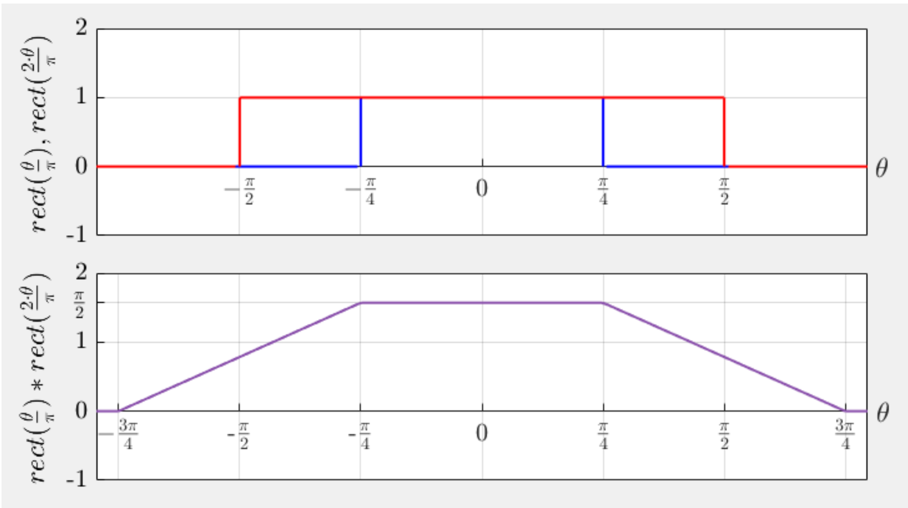
$$\frac{\sin(\alpha n)}{\pi n} \leftrightarrow X(e^{j\theta}) = \begin{cases} 1 & 0 \le |\theta| \le \alpha \\ 0 & \alpha < |\theta| < \pi \end{cases}$$

$$\downarrow$$

$$X(e^{j\theta}) = \frac{1}{2\pi} \cdot \left(\left[-\frac{\pi}{2};\frac{\pi}{2}\right] * \left[-\frac{\pi}{4};\frac{\pi}{4}\right]\right)$$

Grafisch Falten

$$rect\left(\frac{\theta}{\pi}\right) * rect\left(\frac{2\theta}{\pi}\right)$$



$$X(e^{j\theta}) = \frac{1}{2\pi} \cdot \begin{cases} \frac{\pi/2}{\pi/2} \cdot \theta + \frac{3\pi}{4} & -\frac{3\pi}{4} < \theta < -\frac{\pi}{4} \\ \frac{\pi}{2} & -\frac{\pi}{4} < \theta < \frac{\pi}{4} \\ -\frac{\pi/2}{\pi/2} \cdot \theta + \frac{3\pi}{4} & \frac{\pi}{4} < \theta < \frac{3\pi}{4} \\ 0 & sonst \end{cases}$$

$$= \begin{cases} \frac{1}{2\pi} \cdot \left(\theta + \frac{3\pi}{4}\right) & -\frac{3\pi}{4} < \theta < -\frac{\pi}{4} \\ \frac{1}{4} & -\frac{\pi}{4} < \theta < \frac{\pi}{4} \\ -\frac{1}{2\pi} \cdot \left(\theta - \frac{3\pi}{4}\right) & \frac{\pi}{4} < \theta < \frac{3\pi}{4} \\ 0 & sonst \end{cases}$$

d) $\boldsymbol{x[n] = \left(\frac{1}{2}\right)^{|n|}}$

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\theta n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} \cdot e^{-j\theta n}$$

$$= \sum_{n=-\infty}^{-1} \left(\left(\frac{1}{2}\right)^{-n} \cdot e^{-j\theta n}\right) + 1 + \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}\right)^{n} \cdot e^{-j\theta n}\right)$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{e^{j\theta}}{2}\right)^{-n} + \sum_{n=1}^{\infty} \left(\frac{e^{-j\theta}}{2}\right)^{n} + 1$$

$$= \sum_{n=\infty}^{1} \left(\frac{e^{j\theta}}{2}\right)^{n} + \sum_{n=1}^{\infty} \left(\frac{e^{-j\theta}}{2}\right)^{n} + 1$$

$$= \sum_{n=1}^{\infty} \left(\frac{e^{j\theta}}{2}\right)^{n} + \sum_{n=1}^{\infty} \left(\frac{e^{-j\theta}}{2}\right)^{n} + 1$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{j\theta}}{2}\right)^{n} + \sum_{n=0}^{\infty} \left(\frac{e^{-j\theta}}{2}\right)^{n} - 1$$

$$\downarrow$$

Geometrische Summenformel:

$$\sum_{n=0}^{N} q^n = \frac{1 - q^{N+1}}{1 - q}$$

$$\downarrow$$

$$= \frac{1 - \left(\frac{e^{j\theta}}{2}\right)^{\infty}}{1 - \left(\frac{e^{j\theta}}{2}\right)} + \frac{1 - \left(\frac{e^{-j\theta}}{2}\right)^{\infty}}{1 - \left(\frac{e^{-j\theta}}{2}\right)} - 1$$

$$= \frac{1}{1 - \frac{e^{j\theta}}{2}} + \frac{1}{1 - \frac{e^{-j\theta}}{2}} - 1$$

$$= \frac{1 - \frac{e^{-j\theta}}{2} + 1 - \frac{e^{j\theta}}{2}}{\left(1 - \frac{e^{j\theta}}{2}\right) \cdot \left(1 - \frac{e^{-j\theta}}{2}\right)} - 1$$

$$= \frac{2 - \frac{e^{j\theta} + e^{-j\theta}}{2}}{1 - \frac{e^{j\theta}}{2} - \frac{e^{-j\theta}}{2} + \frac{1}{4}} - 1$$

$$= \frac{2 - \cos(\theta)}{\frac{5}{4} - \frac{e^{j\theta} + e^{-j\theta}}{2}} - 1$$

$$= \frac{2 - \cos(\theta)}{\frac{5}{4} - \cos(\theta)} - 1$$

$$= \frac{8 - 4 \cdot \cos(\theta)}{5 - 4 \cdot \cos(\theta)} - 1$$

$$= \frac{8 - 4 \cdot \cos(\theta) - 5 + 4 \cdot \cos(\theta)}{5 - 4 \cdot \cos(\theta)}$$

$$= \frac{3}{5 - 4 \cdot \cos(\theta)}$$

e) $\boldsymbol{x[n] = n \cdot \left(\frac{1}{2}\right)^{|n|}}$

Formelsammlung

$$n \cdot x[n] \leftrightarrow j \cdot \frac{dX(e^{j\theta})}{d\theta}$$

$$\downarrow$$

$$y[n] = n \cdot x[n]$$

$$x[n] = \left(\frac{1}{2}\right)^{|n|}$$

$$X(e^{j\theta}) = \frac{3}{5 - 4 \cdot \cos(\theta)}$$

$$\downarrow$$

$$Y(e^{j\theta}) = j \cdot \frac{dX(e^{j\theta})}{d\theta}$$

$$= j \cdot \frac{0 \cdot (5 - 4 \cdot \cos(\theta)) - 3 \cdot \big(-4 \cdot (-\sin(\theta))\big)}{(5 - 4 \cdot \cos(\theta))^2}$$

$$= -j \cdot \frac{12 \cdot \sin(\theta)}{(5 - 4 \cdot \cos(\theta))^2}$$

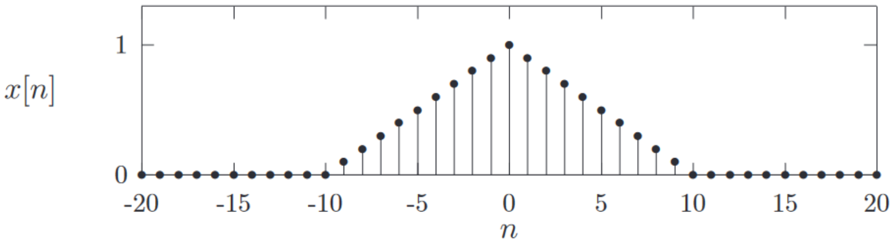
f) $\boldsymbol{x[n] = (-1)^n}$

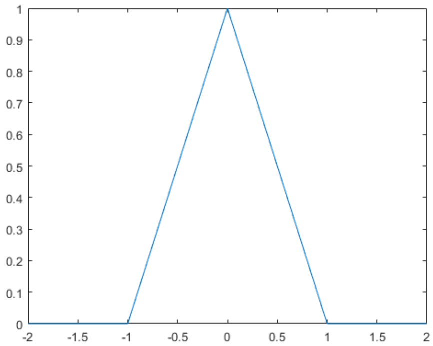
$$x[n] = (-1)^n$$
$$= e^{j\pi n}$$
$$\downarrow$$

Formelsammlung

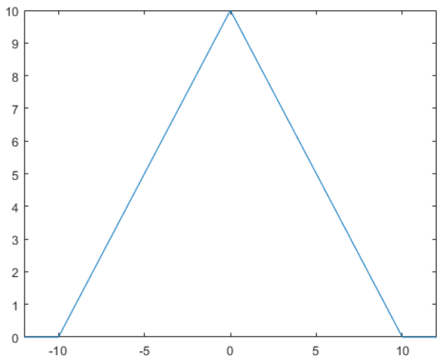
$$e^{j\theta_0 n} \leftrightarrow 2\pi \cdot \delta_{2\pi} \cdot (\theta - \theta_0)$$
$$\downarrow$$
$$X(e^{j\theta}) = 2\pi \cdot \delta_{2\pi} \cdot (\theta - \pi)$$

g)





$$\Lambda(\mathrm{t}) = rect(t) * rect(t)$$



$$k \cdot \Lambda\left(\frac{t}{k}\right) = rect\left(\frac{t}{k}\right) * rect\left(\frac{t}{k}\right)$$
$$10 \cdot \Lambda\left(\frac{t}{10}\right) = rect\left(\frac{t}{10}\right) * rect\left(\frac{t}{10}\right)$$

$$x[n] = \Lambda\left(\frac{t}{10}\right)$$

$$= \frac{1}{10} \cdot \left(rect\left(\frac{t}{10}\right) * rect\left(\frac{t}{10}\right)\right)$$

$$= \left(\frac{1}{\sqrt{10}} \cdot rect\left(\frac{t}{10}\right)\right) * \left(\frac{1}{\sqrt{10}} \cdot rect\left(\frac{t}{10}\right)\right)$$

Formelsammlung

$$\delta[n - N_0] \leftrightarrow e^{-j\theta N_0}$$

$$x[n - N_0] \leftrightarrow e^{-j\theta N_0} \cdot X(e^{j\theta})$$

$$\downarrow$$

$$rect\left(\frac{t}{10}\right) = e^{j\theta \cdot \frac{9}{2}} \cdot \mathcal{F}\left(\sum_{n=0}^{9} \delta[n - N_0]\right)$$

$$= e^{j\theta \cdot 4.5} \cdot \sum_{n=0}^{9} e^{-j\theta n}$$

Formelsammlung

$$(x * y)[n] \leftrightarrow X(e^{j\theta}) \cdot Y(e^{j\theta})$$

$$x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases} \leftrightarrow \frac{\sin\left((2N_1 + 1) \cdot \frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

$$\downarrow$$

$$(N_1 = 4.5)$$

$$X(e^{j\theta}) = \left(\frac{1}{\sqrt{10}} \cdot \frac{\sin\left((2 \cdot 4.5 + 1) \cdot \frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\right)^2$$

$$= \frac{1}{10} \cdot \left(\frac{\sin(5\theta)}{\sin\left(\frac{\theta}{2}\right)}\right)^2$$