6. Übung

Aufgabe 3.2:

Berechnen Sie das Zeitsignal $x[n]$ für folgende Spektren:

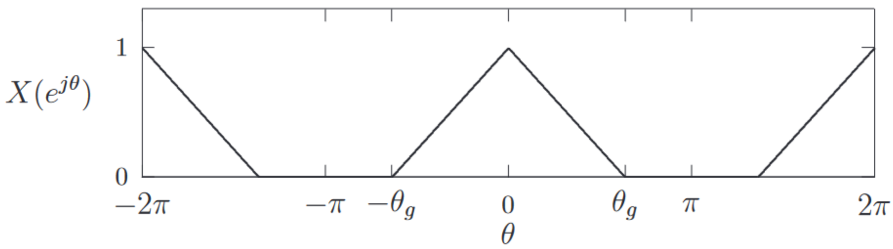
a) $\boldsymbol{X(e^{j\theta}) = \cos^2(\theta)}$

$$X(e^{j\theta}) = \cos^2(\theta)$$
$$= \frac{1}{2} \cdot (1 + \cos(2\theta))$$
$$= \frac{1}{2} \cdot \left(1 + \frac{e^{j2\theta} + e^{-j2\theta}}{2}\right)$$
$$\downarrow$$
$$x[n] = \frac{1}{2} \cdot \left(\mathcal{F}^{-1}(1) + \frac{1}{2} \cdot \left(\mathcal{F}^{-1}(e^{j2\theta}) + \mathcal{F}^{-1}(e^{-j2\theta})\right)\right)$$

Formelsammlung

$$\delta[n] \leftrightarrow 1$$
$$\delta[n - N_0] \leftrightarrow e^{-j\theta N_0}$$
$$\downarrow$$
$$x[n] = \frac{1}{2} \cdot \left(\delta[n] + \frac{1}{2} \cdot (\delta[n+2] + \delta[n-2])\right)$$
$$= \frac{1}{2} \cdot \delta[n] + \frac{1}{4} \cdot \delta[n+2] + \frac{1}{4} \cdot \delta[n-2]$$

b)



$$X(e^{j\theta}) = \Lambda\left(\frac{\theta}{\theta_g}\right)$$
$$= \left(\frac{1}{\sqrt{\theta_g}} \cdot rect\left(\frac{\theta}{\theta_g}\right)\right) * \left(\frac{1}{\sqrt{\theta_g}} \cdot rect\left(\frac{\theta}{\theta_g}\right)\right)$$

Formelsammlung

$$x[n]\cdot y[n] \leftrightarrow \frac{1}{2\pi}\cdot (X * Y)\left(e^{j\theta}\right)$$

$$\frac{\sin(\alpha n)}{\pi n} \leftrightarrow X\left(e^{j\theta}\right) = \begin{cases} 1 & 0 \le |\theta| \le \alpha \\ 0 & \alpha < |\theta| < \pi \end{cases}$$

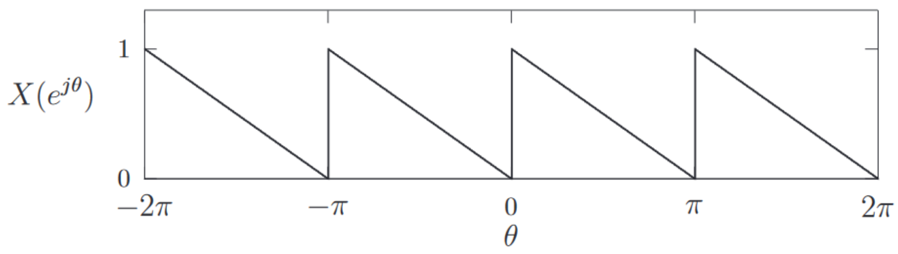
$$\downarrow$$

$$\frac{\sin\left(\frac{\theta_g}{2}n\right)}{\pi n} \leftrightarrow rect\left(\frac{\theta}{\theta_g}\right)$$

$$x[n] = 2\pi \cdot \mathcal{F}^{-1}\left(\frac{1}{\sqrt{\theta_g}}\cdot rect\left(\frac{\theta}{\theta_g}\right)\right)^2$$

$$= \frac{2\pi}{\theta_g}\cdot\left(\frac{\sin\left(\frac{\theta_g}{2}n\right)}{\pi n}\right)^2$$

c)



$$x[n] = \frac{1}{2\pi}\cdot \int_{-\pi}^{\pi} X\left(e^{j\theta}\right)\cdot e^{j\theta n}\, d\theta$$

$$= \frac{1}{2\pi}\cdot\left(\int_{-\pi}^{0} -\frac{\theta}{\pi}\cdot e^{j\theta n}\, d\theta + \int_{0}^{\pi}\left(1-\frac{\theta}{\pi}\right)\cdot e^{j\theta n}\, d\theta\right)$$

$$= \frac{1}{2\pi}\cdot\left(-\frac{1}{\pi}\cdot\int_{-\pi}^{0}\theta\cdot e^{j\theta n}\, d\theta + \int_{0}^{\pi} e^{j\theta n}\, d\theta - \frac{1}{\pi}\cdot\int_{0}^{\pi}\theta\cdot e^{j\theta n}\, d\theta\right)$$

$$= \frac{1}{2\pi}\cdot\left(-\frac{1}{\pi}\cdot\int_{-\pi}^{\pi}\theta\cdot e^{j\theta n}\, d\theta + \int_{0}^{\pi} e^{j\theta n}\, d\theta\right)$$

$$= \frac{1}{2\pi} \cdot \left(-\frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \theta \cdot e^{j\theta n}\, d\theta + \left(\frac{e^{j\theta n}}{jn} \Big|_{0}^{\pi} \right) \right)$$

$$= \frac{1}{2\pi} \cdot \left(-\frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \theta \cdot e^{j\theta n}\, d\theta + \left(\frac{e^{j\pi n}}{jn} - \frac{1}{jn} \right) \right)$$

$$= \frac{1}{2\pi} \cdot \left(-\frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \theta \cdot e^{j\theta n}\, d\theta + \frac{(-1)^n - 1}{jn} \right)$$

$$\downarrow$$

$$\int f'g = fg - \int fg'$$

$$\int e^{j\theta n} \cdot \theta\, d\theta = \frac{e^{j\theta n}}{jn} \cdot \theta - \int \frac{e^{j\theta n}}{jn}\, d\theta$$

$$= \frac{e^{j\theta n}}{jn} \cdot \theta - \frac{1}{jn} \int e^{j\theta n}\, d\theta$$

$$= \frac{e^{j\theta n}}{jn} \cdot \theta - \frac{1}{jn} \cdot \frac{e^{j\theta n}}{jn}$$

$$= \frac{e^{j\theta n}}{jn} \cdot \theta + \frac{e^{j\theta n}}{n^2}$$

$$= \frac{e^{j\theta n}}{n^2} \cdot \left(\frac{n \cdot \theta}{j} + 1 \right)$$

$$= \frac{e^{j\theta n}}{n^2} \cdot (1 - jn\theta)$$

$$\downarrow$$

$$= \frac{1}{2\pi} \cdot \left(-\frac{1}{\pi} \cdot \left(\frac{e^{j\theta n}}{n^2} \cdot (1 - jn\theta) \right) \Big|_{-\pi}^{\pi} + \frac{(-1)^n - 1}{jn} \right)$$

$$= \frac{1}{2\pi} \cdot \left(-\frac{1}{\pi} \cdot \left(\frac{e^{j\pi n}}{n^2} \cdot (1 - jn\pi) - \frac{e^{-j\pi n}}{n^2} \cdot (1 + jn\pi) \right) + \frac{(-1)^n - 1}{jn} \right)$$

$$= \frac{1}{2\pi} \cdot \left(-\frac{1}{\pi} \cdot \left(\frac{(-1)^n}{n^2} \cdot (1 - jn\pi) - \frac{(-1)^{-n}}{n^2} \cdot (1 + jn\pi) \right) + \frac{(-1)^n - 1}{jn} \right)$$

$$\downarrow^{1}$$

$$= \frac{1}{2\pi} \cdot \left(-\frac{1}{\pi} \cdot \frac{(-1)^n}{n^2} \cdot (-j2\pi n) + \frac{(-1)^n - 1}{jn} \right)$$

---

[1] $(-1)^n = (-1)^{-n},\ \forall n \in \mathbb{Z}$

$$= \frac{1}{2\pi n} \cdot \left(-2 \cdot \frac{(-1)^n}{j} + \frac{(-1)^n - 1}{j}\right)$$

$$= \frac{-2 \cdot (-1)^n + (-1)^n - 1}{j2\pi n}$$

$$= -\frac{1 + (-1)^n}{j2\pi n}$$

Wenn $n = 0$

$$= \frac{1}{2\pi} \cdot \left(\int_{-\pi}^{0} -\frac{\theta}{\pi}\, d\theta + \int_{0}^{\pi} 1 - \frac{\theta}{\pi}\, d\theta\right)$$

$$= \frac{1}{2\pi} \cdot \left(-\frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \theta\, d\theta + \int_{0}^{\pi} 1\, d\theta\right)$$

$$= \frac{1}{2\pi} \cdot \left(-\frac{1}{\pi} \cdot \left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) + (\pi)\right)$$

$$= \frac{1}{2}$$

$$x[n] = \begin{cases} -\dfrac{1 + (-1)^n}{j2\pi n} & n \neq 0 \\ \dfrac{1}{2} & n = 0 \end{cases}$$

d) $\boldsymbol{X\left(e^{j\theta}\right) = \frac{e^{-j\theta}}{1 + \frac{1}{6} \cdot e^{-j\theta} - \frac{1}{6} \cdot e^{-j2\theta}}}$

Formelsammlung

$$a^n \cdot \sigma[n] \leftrightarrow \frac{1}{1 - a \cdot e^{-j\theta}}, \qquad |a| < 1$$

$$\downarrow$$ [2]

$$\frac{A}{1 - a \cdot e^{-j\theta}} - \frac{B}{1 - b \cdot e^{-j\theta}} = \frac{A - Abe^{-j\theta} - B + Bae^{-j\theta}}{1 - be^{-j\theta} - ae^{-j\theta} + abe^{-j2\theta}}$$

$$A - B = 0, \qquad ab = -\frac{1}{6}, \qquad -a - b = \frac{1}{6}, \qquad Ba - Ab = 1$$

$$\downarrow$$

[2] Danke @Nico!

$$A = B$$

$$b = -\frac{1}{6a}$$

$$-a + \frac{1}{6a} = \frac{1}{6}$$
$$6a^2 + a - 1 = 0$$
$$a = \frac{1}{3}$$

$$b = -\frac{3}{6}$$
$$= -\frac{1}{2}$$

$$\frac{A}{3} + \frac{A}{2} = 1$$
$$A = B = \frac{6}{5}$$

$$x[n] = \mathcal{F}^{-1}\left(\frac{A}{1 - a \cdot e^{-j\theta}} - \frac{B}{1 - b \cdot e^{-j\theta}}\right)$$
$$= A \cdot \mathcal{F}^{-1}\left(\frac{1}{1 - a \cdot e^{-j\theta}}\right) - B \cdot \mathcal{F}^{-1}\left(\frac{1}{1 - b \cdot e^{-j\theta}}\right)$$
$$= \frac{6}{5}\left(\left(\frac{1}{3}\right)^n - \left(-\frac{1}{2}\right)^n\right) \cdot \sigma[n]$$

Aufgabe 3.4:

Von einem zeitdiskreten System sei die Beziehung zwischen dem Spektrum $\boldsymbol{X(e^{j\theta})}$ des Eingangssignals $\boldsymbol{x[n]}$ und dem Spektrum $\boldsymbol{Y(e^{j\theta})}$ des Ausgangssignals $\boldsymbol{y[n]}$ bekannt:

$$\boldsymbol{Y(e^{j\theta}) = \int_{\theta-\frac{\pi}{4}}^{\theta+\frac{\pi}{4}} X(e^{j\Omega})\, d\Omega}$$

a) Berechnen Sie den Zusammenhang zwischen $\boldsymbol{x[n]}$ und $\boldsymbol{y[n]}$.

$$Y(e^{j\theta}) = \int_{\theta-\frac{\pi}{4}}^{\theta+\frac{\pi}{4}} X(e^{j\Omega})\, d\Omega$$

$$= \int_{\infty}^{\infty} X(e^{j\Omega}) \cdot rect\left(\frac{(\Omega-\theta)}{2} \cdot \frac{4}{\pi}\right) d\Omega$$

$$= \int_{\infty}^{\infty} X(e^{j\Omega}) \cdot rect\left(\frac{(\theta-\Omega)}{2} \cdot \frac{4}{\pi}\right) d\Omega$$

$$= X(e^{j\Omega}) * rect\left(\frac{\theta}{2} \cdot \frac{4}{\pi}\right)$$

Formelsammlung

$$x[n] \cdot y[n] \leftrightarrow \frac{1}{2\pi} \cdot (X * Y)(e^{j\theta})$$

$$\frac{\sin(\alpha n)}{\pi n} \leftrightarrow X(e^{j\theta}) = \begin{cases} 1 & 0 \le |\theta| \le \alpha \\ 0 & \alpha < |\theta| < \pi \end{cases}$$

$$y[n] = 2\pi \cdot x[n] \cdot \mathcal{F}^{-1}\left(rect\left(\frac{\theta}{2} \cdot \frac{4}{\pi}\right)\right)$$

$$\left(\alpha = \frac{\pi}{4}\right)$$

$$= 2\pi \cdot x[n] \cdot \frac{\sin\left(\frac{\pi}{4} \cdot n\right)}{\pi n}$$

$$= 2 \cdot \frac{\sin\left(\frac{\pi}{4} \cdot n\right)}{n} \cdot x[n]$$

b) Prüfen Sie ob das System[3]

a. linear,

$$y[n] = 2 \cdot \frac{\sin\left(\frac{\pi}{4} \cdot n\right)}{n} \cdot x[n]$$

$$\downarrow$$

$$2 \cdot \frac{\sin\left(\frac{\pi}{4} \cdot n\right)}{n} \cdot (a \cdot x_1[n] + b \cdot x_2[n])$$

$$= a \cdot y_1[n] + b \cdot y_2[n]$$

$$\rightarrow \text{linear}$$

b. zeitinvariant,

$$y[n] = 2 \cdot \frac{\sin\left(\frac{\pi}{4} \cdot n\right)}{n} \cdot x[n]$$

$$\downarrow$$

$$y[n-5] \neq 2 \cdot \frac{\sin\left(\frac{\pi}{4} \cdot (n-5)\right)}{(n-5)} \cdot x[n-5]$$

$$\rightarrow \text{zeitvariant}$$

c. kausal,

$\rightarrow$ kausal[4]

d. stabil,

$$y[n] = 2 \cdot \frac{\sin\left(\frac{\pi}{4} \cdot n\right)}{n} \cdot x[n]$$

$$\downarrow$$

$$\lim_{n \to 0} \left(\frac{\sin\left(\frac{\pi}{4} \cdot n\right)}{n} \right)$$

$$(l'Hospital)$$

$$\lim_{n \to 0} \left(\frac{\pi}{4} \cdot \cos\left(\frac{\pi}{4} \cdot n\right) \right)$$

$$= \frac{\pi}{4}$$

$$\downarrow$$

[3] vgl. z.B. Übung 3 → Bsp. 2.1

[4] weil der Ausgang instantan reagiert und nicht von Werten aus der Zukunft abhängt wie z.B. $y[n] = x[n+5]$

$$y[n] \leq 2 \cdot \frac{\sin\left(\frac{\pi}{4} \cdot n\right)}{n} \cdot M$$

$$y[n] \leq C \cdot M$$

$$\rightarrow \text{stabil}$$

e. reell- oder komplexwertig ist.

$$\rightarrow \text{reell}$$

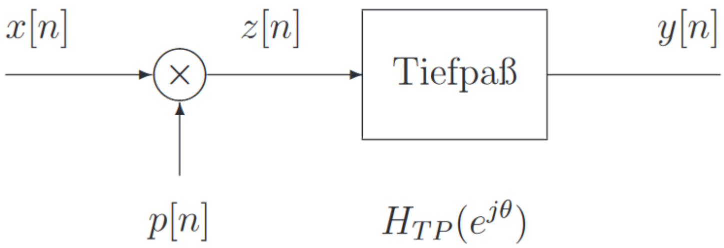
c) Falls das gegebene System durch eine Impulsantwort $h[n]$ beschrieben werden kann, so geben Sie $h[n]$ an.

Das System hat keine eindeutige Impulsantwort, weil es zeitvariant ist

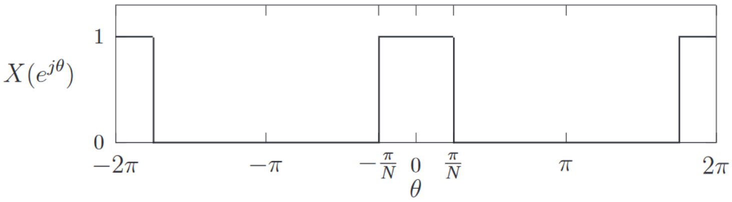
Aufgabe 3.5:

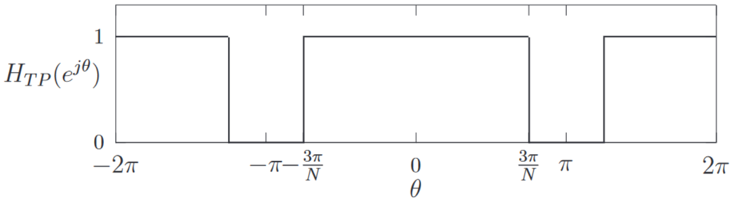
Das Tiefpasssignal $x[n]$ wird mit dem abgebildeten zeitdiskreten System verarbeitet, wobei das Modulationssignal ein Eins-Puls mit der Periode N ist:

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$



Das Spektrum des Eingangssignals und die Übertragungsfunktion des Tiefpassfilters sind in den folgenden Abbildungen dargestellt:





a) Berechnen und skizzieren Sie die Fouriertransformationen $P(e^{j\theta}), Z(e^{j\theta}), Y(e^{j\theta})$ der Signale $p[n], z[n]$ und $y[n]$.

Formelsammlung

$$\sum_{k=-\infty}^{\infty} \delta[n-kN] \leftrightarrow \frac{2\pi}{N} \cdot \sum_{k=-\infty}^{\infty} \delta\left(\theta - \frac{2\pi k}{N}\right)$$

$$\frac{\sin(\alpha n)}{\pi n} \leftrightarrow X(e^{j\theta}) = \begin{cases} 1 & 0 \leq |\theta| \leq \alpha \\ 0 & \alpha < |\theta| < \pi \end{cases}$$

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN], \qquad P(e^{j\theta}) = \frac{2\pi}{N} \cdot \sum_{k=-\infty}^{\infty} \delta\left(\theta - \frac{2\pi k}{N}\right)$$

$$x[n] = \frac{\sin\left(\frac{\pi}{N} \cdot n\right)}{\pi n}, \qquad X(e^{j\theta}) = rect\left(\frac{\theta}{2} \cdot \frac{N}{\pi}\right)$$

$$z[n] = x[n] \cdot p[n]$$

$$= \frac{\sin\left(\frac{\pi}{N} \cdot n\right)}{\pi n} \cdot \sum_{k=-\infty}^{\infty} \delta[n-kN]$$

$$= \frac{\sin\left(\frac{\pi}{N} \cdot n\right)}{\pi n} \cdot \delta_N[n]$$

$$= 0, \qquad \forall n \neq 0$$

$$n = 0:$$

$$\lim_{n \to 0} \left(\frac{\sin\left(\frac{\pi}{N} \cdot n\right)}{\pi n} \right)$$

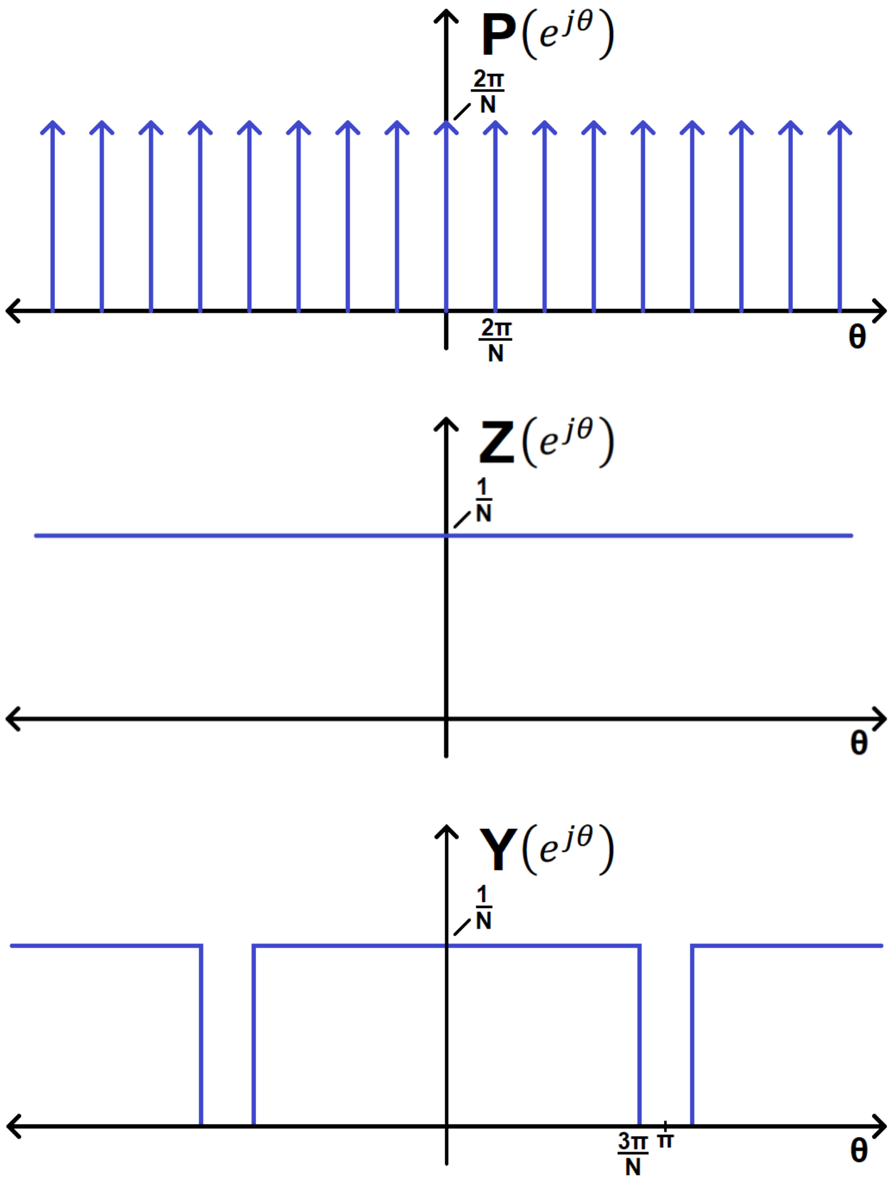
$$(l'Hospital)$$

$$= \lim_{n \to 0} \left(\frac{\frac{\pi}{N} \cdot \cos\left(\frac{\pi}{N} \cdot n\right)}{\pi} \right) = \frac{1}{N}$$

$$z[n] = \frac{\sin\left(\frac{\pi}{N} \cdot n\right)}{\pi n} \cdot \delta_N[n], \qquad Z(e^{j\theta}) = \frac{1}{N}$$

$$Y(e^{j\theta}) = Z(e^{j\theta}) \cdot H_{TP}(e^{j\theta})$$

$$= \frac{1}{N} \cdot rect\left(\frac{\theta}{2} \cdot \frac{N}{3\pi}\right)$$



b) Bestimmen Sie das Ausgangssignal $y[n]$ des Gesamtsystems.

$$Y(e^{j\theta}) = \frac{1}{N} \cdot rect\left(\frac{\theta}{2} \cdot \frac{N}{3\pi}\right)$$

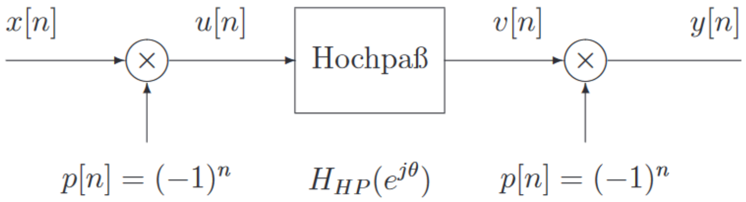
Formelsammlung

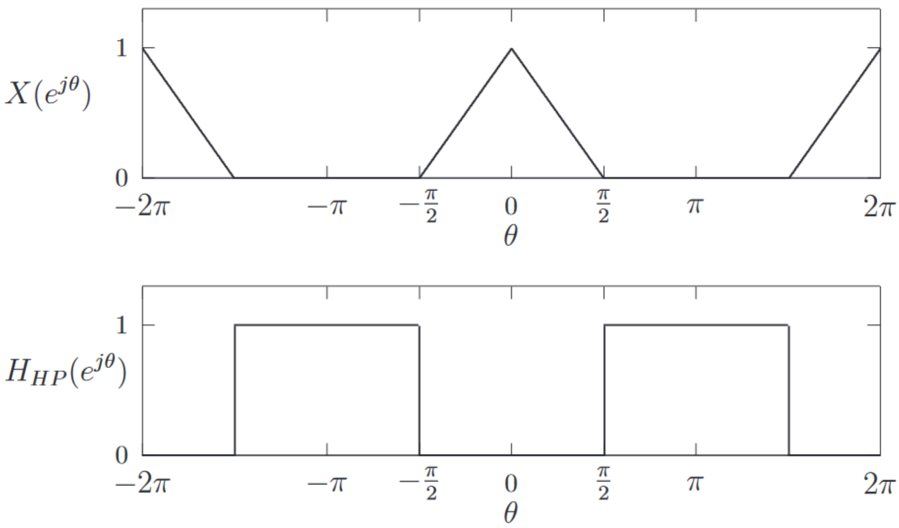
$$\frac{\sin(\alpha n)}{\pi n} \leftrightarrow X(e^{j\theta}) = \begin{cases} 1 & 0 \leq |\theta| \leq \alpha \\ 0 & \alpha < |\theta| < \pi \end{cases}$$

$$y[n] = \frac{1}{N} \cdot \frac{\sin\left(\frac{3\pi}{N} \cdot n\right)}{\pi n}$$

Aufgabe 3.6:

Ein zeitdiskretes System, bestehend aus zwei Modulatoren und einem idealisierten Hochpassfilter, wird mit einem Tiefpasssignal $x[n]$, dessen Spektrum gegeben ist, angeregt.





a) **Berechnen und skizzieren Sie die Fouriertransformationen $P(e^{j\theta}), U(e^{j\theta}), V(e^{j\theta})$, $Y(e^{j\theta})$ der Signale $p[n], u[n], v[n]$ und $u[n]$.**

Formelsammlung

$$e^{j\theta_0 n} \leftrightarrow 2\pi\delta_{2\pi}(\theta - \theta_0)$$

$$x[n] \cdot y[n] \leftrightarrow \frac{1}{2\pi} \cdot (X * Y)\left(e^{j\theta}\right)$$

$$\left(\alpha \cdot \delta(\theta - T) * x(\theta) = \alpha \cdot x(\theta - T)\right)$$

$$p[n] = (-1)^n = e^{j\pi n}$$

$$P\left(e^{j\theta}\right) = 2\pi\delta_{2\pi}(\theta - \pi)$$

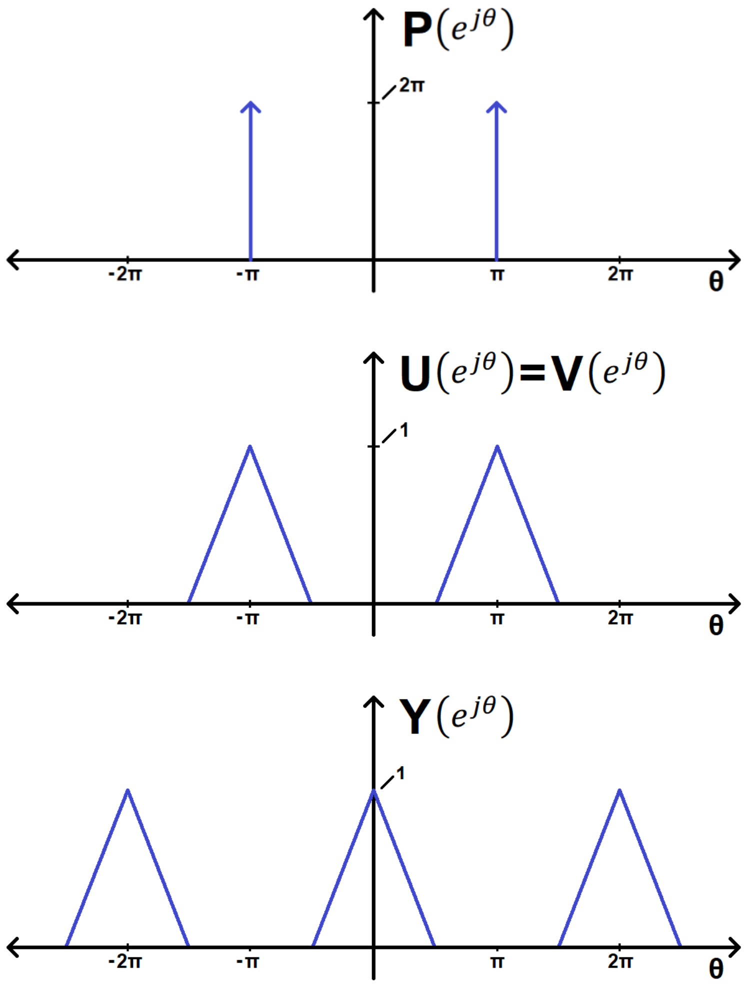
$$U\left(e^{j\theta}\right) = \frac{1}{2\pi} \cdot (X * P)\left(e^{j\theta}\right)$$

$$= X\left(e^{j\theta}\right) * \delta_{2\pi}(\theta - \pi)$$

$$= X\left(e^{j(\theta-\pi)}\right)$$

$$V(e^{j\theta}) = U(e^{j\theta}) \cdot H_{HP}(e^{j\theta})$$
$$= X(e^{j(\theta-\pi)}) \cdot rect\left(\frac{(\theta-\pi)}{2} \cdot \frac{2}{\pi}\right)$$
$$= X(e^{j(\theta-\pi)})$$

$$Y(e^{j\theta}) = \frac{1}{2\pi} \cdot (V * P)(e^{j\theta})$$
$$= X(e^{j(\theta-\pi)}) * \delta_{2\pi}(\theta-\pi)$$
$$= X(e^{j(\theta-2\pi)})$$
$$\downarrow ^{5}$$
$$= X(e^{j\theta})$$



[5] Frequenzbereich ist 2π-periodisch

b) Bestimmen Sie das Ausgangssignal $\boldsymbol{y[n]}$.

Formelsammlung

$$x[n] \cdot y[n] \leftrightarrow \frac{1}{2\pi} \cdot (X * Y)\left(e^{j\theta}\right)$$

$$\frac{\sin(\alpha n)}{\pi n} \leftrightarrow X\left(e^{j\theta}\right) = \begin{cases} 1 & 0 \leq |\theta| \leq \alpha \\ 0 & \alpha < |\theta| < \pi \end{cases}$$

$$\left(rect\left(\frac{t}{k}\right) * rect\left(\frac{t}{k}\right) = k \cdot \Lambda\left(\frac{t}{k}\right)\right)$$

$$Y\left(e^{j\theta}\right) = \Lambda\left(\theta \cdot \frac{2}{\pi}\right)$$

$$= \left(\sqrt{\frac{2}{\pi}} \cdot rect\left(\theta \cdot \frac{2}{\pi}\right)\right) * \left(\sqrt{\frac{2}{\pi}} \cdot rect\left(\theta \cdot \frac{2}{\pi}\right)\right)$$

$$\downarrow$$

$$y[n] = 2\pi \cdot \mathcal{F}^{-1}\left(\sqrt{\frac{2}{\pi}} \cdot rect\left(\theta \cdot \frac{2}{\pi}\right)\right)^2$$

$$= 4 \cdot \mathcal{F}^{-1}\left(rect\left(\theta \cdot \frac{2}{\pi}\right)\right)^2$$

$$= 4 \cdot \left(\frac{\sin\left(\frac{\pi}{4} \cdot n\right)}{\pi n}\right)^2$$