The program equivalence example again

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This document describes the steps to show the validity of the *EUF*-formula associated to the program equivalence example presented in the lecture. The proof is based on semantics. The original formula is

$$\psi\colon \quad \varphi \wedge \varphi_n \to o2a \doteq o0b.$$

We translate ψ into the EUF-formulas

$$\psi^{EUF}\colon \quad \varphi^{EUF} \wedge \varphi_n^{EUF} \to o2a \doteq o0b$$

where

$$\begin{aligned} \varphi^{EUF}\colon &\quad o0a \doteq in \wedge o1a \doteq G(o0a, in) \wedge o2a \doteq G(o1a, in) \\ \varphi_n^{EUF}\colon &\quad o0b \doteq G(G(in, in), in). \end{aligned}$$

We show that ψ^{EUF} is E-valid which immediately implies the E-validity of ψ.

Basic observations

1. Our theory here is equality with uninterpreted functions. The theory axioms (or more precisely axiom schemes) are functional consistency and the equality axioms. We have to show ψ^{EUF} E-valid, i.e., we have to check that all interpretations satisfying all instances of the theory axioms also satisfy ψ^{EUF}.

2. We can therefore restrict our models considered in the following steps to those ones which satisfy all instances of the functional consistency scheme. The same holds for the instances of the equality axioms, i.e., instances of the schemes

$$\begin{aligned} \text{reflexivity:} &\quad x \doteq x \\ \text{symmetry:} &\quad x \doteq y \to y \doteq x \\ \text{transitivity:} &\quad x \doteq y \wedge y \doteq z \to x \doteq z \end{aligned}$$

3. The formula ψ^{EUF} above is satisfied by an E-interpretation iff its left conjunction $\varphi^{EUF} \wedge \varphi_n^{EUF}$ is *not* satisfied or its right equality $o2a \doteq o0b$ is. It is therefore sufficient to consider only E-interpretation structures M which satisfy the left conjunction (since the other E-interpretation structures trivially satisfy the implication). We have to show then that $o2a \doteq o0b$ is satisfied by M.

Performing the reasoning

1. Take an *arbitrary* model M of $\varphi^{EUF} \wedge \varphi_n^{EUF}$ (obeying the restrictions as discussed above). By the semantics of $\wedge$, the equalities

$$o0a \doteq in,\ o1a \doteq G(o0a, in),\ o2a \doteq G(o1a, in),\ o0b \doteq G(G(in, in), in)$$

are all satisfied by M. Moreover, $in \doteq in$ is satisfied in M by reflexivity.

2. Take the following instance

$$o0a \doteq in \wedge in \doteq in \rightarrow G(o0a, in) \doteq G(in, in)$$

of the functional consistency scheme which is satisfied by M. Since the left-hand side of the implication is satisfied by M, so is the right equality.

3. Take the following instance

$$o1a \doteq G(o0a, in) \wedge G(o0a, in) \doteq G(in, in) \rightarrow o1a \doteq G(in, in)$$

of the transitivity scheme which is satisfied by M. Since the left-hand side of the implication is satisfied by M, so is the right equality.

4. Take the following instance

$$o1a \doteq G(in, in) \wedge in \doteq in \rightarrow G(o1a, in) \doteq G(G(in, in), in)$$

of the functional consistency scheme which is satisfied by M. Since the left-hand side of the implication is satisfied by M, so is the right equality.

5. Take the following instance

$$o2a \doteq G(o1a, in) \wedge G(o1a, in) \doteq G(G(in, in), in) \rightarrow o2a \doteq G(G(in, in), in)$$

of the transitivity scheme which is satisfied by M. Since the left-hand side of the implication is satisfied by M, so is the right equality.

6. Take the following instance

$$o0b \doteq G(G(in, in), in) \rightarrow G(G(in, in), in) \doteq o0b$$

of the symmetry scheme which is satisfied by M. Since the left-hand side of the implication is satisfied by M, so is the right equality.

7. Take the following instance

$$o2a \doteq G(G(in.in), in) \wedge G(G(in, in), in) \doteq o0b \rightarrow o2a \doteq o0b$$

of the transitivity scheme which is satisfied by M. Since the left-hand side of the implication is satisfied by M, so is the right equality.

Since we have chosen our E-interpretation structure M arbitrarily, it satisfies the eigenvariable condition. Therefore, ψ^{EUF} is satisfied by all those structures and consequently ψ^{EUF} is E-valid. This implies the E-validity of ψ.