Name: ______________________________

Registration number: ______________________________

186.835 Mathematical Programming
Final Exam
June 29, 2020

Instructors: Georg Brandstätter, Mario Ruthmair
Duration: 120 min

Question	max. Points	Points
1	7	
2	6	
3	6	
4	8	
5	7	
6	6	
	40	

- This is an open book exam which means you can use any resources available, but you have to do it alone.
- Ensure that your solutions and **all necessary intermediate steps** are described / shown in an understandable and readable manner.
- You can do the exercises either electronically (LaTeX, Word, etc.) or on paper.
- Compile all your solutions (electronic documents, scans, photos) into a **single PDF** and **upload it to TUWEL** until the end of the exam.
- If you have any questions during the exam, we have Cisco

Good Luck!

1. *(7 points) Modeling: Building Charging Stations*

The city of Vienna wants to build charging stations for electric vehicles. There are five potential locations $L = \{a, b, c, d, e\}$ and associated costs for building a charging station, and each location is near to a subset of the city districts $D = \{1, 2, 3, 4, 5\}$, see the table below. The goal is to minimize the total costs for building charging stations such that for at least 80% of the districts there is at least one station nearby.

location	costs	near districts
a	20	1, 5
b	40	1, 3, 4
c	30	2, 3
d	10	5
e	50	3, 4

(a) Formulate this problem as a (mixed) integer linear program. Describe all variables and constraints.

(b) Preprocessing: Are there any locations which cannot be in an optimal solution? Use only logical arguments, without solving the problem. How would you incorporate this knowledge in your formulation?

(c) State **all** optimal solutions!

2. *(6 points) Lagrangian Relaxation*

Consider the following binary integer program:

$$
\begin{aligned}
z = \max \quad & 2x_1 + 4x_2 + 3x_3 & (1)\\
& x_1 + x_2 + x_3 \leq 2 & (2)\\
& x_1 + 3x_2 + 2x_3 \leq 4 & (3)\\
& x_1, x_2, x_3 \in \{0, 1\} & (4)
\end{aligned}
$$

(a) Identify the optimal value z and state an optimal solution.

(b) Relax constraint (3) in the usual Lagrangian way and write down the Lagrangian subproblem $z(u)$ depending on the Lagrangian multiplier u associated to the relaxed constraint.

(c) Run 2 iterations of the subgradient algorithm, starting with $u^0 = 0$ and using step sizes $\mu_0 = 1$ and $\mu_1 = 0.5$. State in each iteration the optimal value of the Lagrangian subproblem and an according optimal solution. If the solution is feasible for the original problem, compute the corresponding original objective value.

(d) When is an optimal solution to the Lagrangian subproblem a dual bound and when is it a primal bound?

3. *(6 points) Traveling Salesman By Branch-and-Cut*

A feasible solution to the traveling salesman problem in an undirected graph $G = (V, E)$ can be characterized as follows:

- Each node has exactly two incident edges.
- There are no disconnected subtours.

Assume that we run a branch-and-cut approach for the undirected graph below based on a formulation using only binary variables $x_e \in \{0,1\}$ for all **undirected edges** $e \in E$. We start only with a subset of all the constraints and in some branch-and-bound node we obtain the LP relaxation solution shown in the graph below: The value next to each edge corresponds to its LP solution value (variable names are mostly omitted for better readability).

(a) Find at least one **violated** degree constraint, if one exists.

(b) Find at least one **violated** cycle elimination constraint for cycles of length up to 4, if one exists.

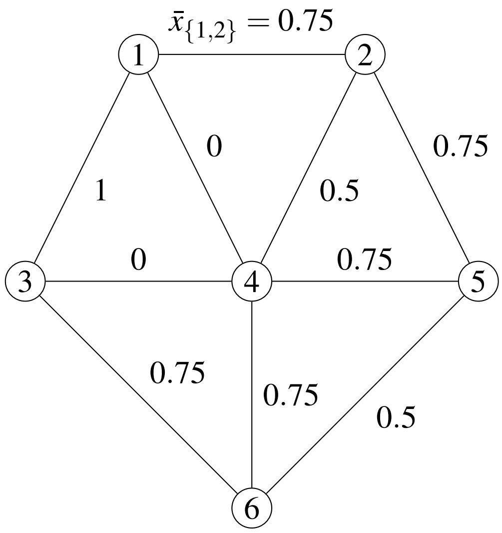
(c) Find at least one **violated** subtour elimination constraint, if one exists.

(d) Find at least one **violated** undirected cutset constraint, if one exists.

(e) The LP solution gives us a dual bound in the current branch-and-bound node. For which parts of the branch-and-bound tree is this bound a valid dual bound?

(f) For which parts of the branch-and-bound tree are the violated constraints identified above valid?

(g) We try to construct a feasible solution by rounding the LP solution: Fractional values ≥ 0.75 are rounded up, all others are set to zero. Do we get a feasible traveling salesman tour? If yes, for which parts of the branch-and-bound tree is the corresponding primal bound valid?



4. *(8 points) Modeling: Efficient Bar Hopping*

   To make best use of the currently limited bar opening hours, Edmund wants to plan his Friday night out as efficiently as possible. There is a set of bars $L = \{1, 2, \ldots, n\}$ that he considers visiting. Drinking his favorite drink at each bar $i \in L$ takes t_i minutes and costs b_i Euro. To travel **from** any bar $i \in L$ **to** another bar $j \in L$, he can use two modes of transportation: (a) he can walk, which takes w_{ij} minutes and is free, or (b) he can take a taxi, which takes u_{ij} minutes and costs c_{ij} Euro.

   Edmund starts the evening with a drink at his favorite bar 1 near his workplace. He wants do visit as many different bars as possible, having one drink at each bar he visits. From the time he arrives at bar 1 until the (simultaneous) closing time of all bars, he has T minutes. In total, he wants to spend at most B Euro. He does **not** return to bar 1 at the end of the night.

   (a) Formulate this problem as a (mixed) integer linear program. Describe all variables and constraints you used. You may use a compact formulation (polynomially many variables and constraints) or one solved by branch-and-cut.

   (b) What would an optimal solution look like if both time budget T and money budget B were unlimited?

5. *(7 points) Chvátal-Gomory Cutting Planes*

Consider the following integer linear program

$$\max\ x_1 + x_2 \tag{5}$$
$$\text{s.t.}\ -2x_1 + x_2 \leq 1 \tag{6}$$
$$3x_1 + 2x_2 \leq 12 \tag{7}$$
$$x_1 \leq 4 \tag{8}$$
$$x_1 + x_2 \geq 0 \tag{9}$$
$$x_1 \geq 0 \tag{10}$$
$$x_2 \geq 0 \tag{11}$$
$$\mathbf{x} \in \mathbb{Z}^2 \tag{12}$$

whose integer feasible set (6)–(12) we denote by X.

(a) State **all** redundant constraints and remove them from the formulation.

(b) Use the Chvátal-Gomory procedure to find $\text{conv}(X)$. You may use the Gomory Cutting Plane procedure **or** find the multiplier vector $\mathbf{u}$ in any other way you want. (**Hint:** Try drawing the feasible set X by hand or with, e.g., `https://www.desmos.com/calculator`)

(c) State the optimal solution for the LP

$$\max\ x_1 + x_2 \tag{13}$$
$$\text{s.t.}\ \mathbf{x} \in \text{conv}(X) \tag{14}$$

Does it differ from the optimal solution to the original LP relaxation (5)–(11)?

6. *(6 points) Cover Inequalities*

Consider the knapsack set

$$X = \{\{0,1\}^5 : 11x_1 + 8x_2 + 6x_3 + 5x_4 + 3x_5 \leq 15\}$$

(a) State a minimum cover C with $|C| \geq 3$ and its corresponding valid cover inequality.

(b) State the **extended** cover inequality for C.

(c) **Lift** the cover inequality for C to obtain a strong valid inequality, i.e., compute all lifting coefficients.

(d) State an optimal solution for the following ILP

$$\max\ x_1 + x_2 + x_3 + x_4 + x_5 \tag{15}$$

$$\text{s.t. } \mathbf{x} \in X \tag{16}$$