SigSys2-UE8

Aufgabe 4.4:

Ein kausales LTI–System sei durch die folgende Differenzengleichung charakterisiert:

$$y[n] - \frac{1}{2}y[n-1] + \frac{1}{4}y[n-2] = x[n].$$

$H = \frac{Y}{X}$

a) Berechnen Sie die Übertragungsfunktion $H(z)$ dieses Systems.

FS: $x[n+n_0] \circ\!\!-\!\!\bullet\ z^{n_0}X(z)$

$$Y(z) - \frac{1}{2z}Y(z) + \frac{1}{4z^2}Y(z) = X(z)$$

$$Y(z)\left(1 - \frac{1}{2z} + \frac{1}{4z^2}\right) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2z} + \frac{1}{4z^2}} = \frac{z^2}{z^2 - \frac{1}{2}z + \frac{1}{4}}$$

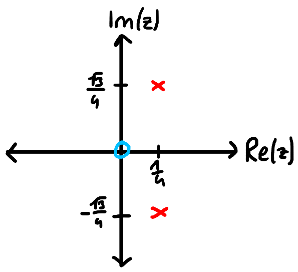
b) Skizzieren Sie das Pol/Nullstellendiagramm von $H(z)$.

doppelte Nullstelle bei $z=0$

$$z_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

$$= \frac{1}{4} \pm \sqrt{\frac{1}{16} - \frac{4}{16}}$$

$$= \frac{1}{4} \pm j\frac{\sqrt{3}}{4} \quad \text{Pole}$$



c) Bestimmen Sie mit Hilfe der $\mathcal{Z}$–Transformation die Systemantwort $y[n]$ für das Eingangssignal

$$x[n] = \left(\frac{1}{2}\right)^n \sigma[n].$$

FS: $\alpha^n\sigma[n] \circ\!\!-\!\!\bullet\ \frac{z}{z-\alpha}$

$$X(z) = \frac{z}{z - \frac{1}{2}}$$

$$Y(z) = H(z)\cdot X(z)$$

$$Y(z) = \frac{z^2}{z^2 - \frac{1}{2}z + \frac{1}{4}} \cdot \frac{z}{z - \frac{1}{2}} = z\overbrace{\left(\frac{z^2}{\left(z^2 - \frac{1}{2}z + \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}\right)}^{F(z)}$$

PBZ: $F(z) = \frac{A + Bz}{z^2 - \frac{1}{2}z + \frac{1}{4}} + \frac{C}{z - \frac{1}{2}} \quad \Big| \cdot \left(z^2 - \frac{1}{2}z + \frac{1}{4}\right)\left(z - \frac{1}{2}\right)$

$$z^2 = (A + Bz)\left(z - \frac{1}{2}\right) + C\left(z^2 - \frac{1}{2}z + \frac{1}{4}\right)$$

$$z^2 = Az - \frac{1}{2}A + Bz^2 - \frac{1}{2}Bz + Cz^2 - \frac{1}{2}Cz + \frac{1}{4}C$$

$$z^2 = z^2(B + C) + z\left(A - \frac{1}{2}B - \frac{1}{2}C\right) - \frac{A}{2} + \frac{C}{4}$$

Koeffvergleich: I: $B + C = 1$

II: $A - \frac{B}{2} - \frac{C}{2} = 0$

III: $-\frac{A}{2} + \frac{C}{4} = 0 \Rightarrow A = \frac{C}{2}$

$\hookrightarrow$ in II: $\frac{C}{2} - \frac{B}{2} - \frac{C}{2} = 0 \Longrightarrow B = 0,\ C = 1,\ A = \frac{1}{2}$

$$Y(z) = z\left(\frac{\frac{1}{2}}{z^2-\frac{1}{2}z+\frac{1}{4}} + \frac{1}{z-\frac{1}{2}}\right)$$

FS: $\rho^n \sin(\alpha n)\,\sigma[n] \;\circ\!\!-\!\!\bullet\; \dfrac{\rho z \sin\alpha}{z^2-2\rho z\cos\alpha+\rho^2} \qquad |z|>\rho$

FS: $\alpha^n \sigma[n] \;\circ\!\!-\!\!\bullet\; \dfrac{z}{z-\alpha}$

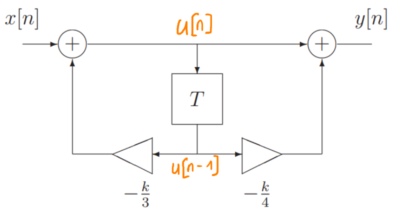
$\rho = \frac{1}{2}, \; \alpha = \frac{\pi}{3}, \; \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$$\Rightarrow \frac{\frac{1}{2}z \cdot \frac{\sqrt{3}}{2}}{z^2 - \cancel{\frac{2\pi}{3}}\, z \cdot \cancel{\frac{1}{2}} + \frac{1}{4}}$$

$$\Rightarrow Y(z) \;\bullet\!\!-\!\!\circ\; \frac{2}{\sqrt{3}}\left(\frac{1}{2}\right)^n \sin\left(\frac{\pi}{3}n\right)\sigma[n] + \left(\frac{1}{2}\right)^n \sigma[n]$$

Aufgabe 4.5:

Von einem digitalen Filter ist das Schaltbild gegeben:



FS: $x[n+n_0] \;\circ\!\!-\!\!\bullet\; z^{n_0}X(z)$

a) Berechnen Sie die Übertragungsfunktion $H(z)$ dieses Filters.

$u[n] = x[n] - \frac{k}{3}u[n-1]$

$y[n] = u[n] - \frac{k}{4}u[n-1]$

I: $U(z) = X(z) - \frac{k}{3z}U(z) \quad \Rightarrow \quad U(z) = \dfrac{X(z)}{1+\frac{k}{3z}}$

einsetzen in II

II: $Y(z) = U(z) - \frac{k}{4z}U(z)$

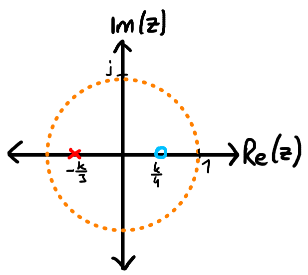
$$= \frac{X(z)}{1+\frac{k}{3z}} - \frac{k}{4z}\frac{X(z)}{1+\frac{k}{3z}} = X(z)\left(\frac{1}{1+\frac{k}{3z}} - \frac{k}{4z}\cdot\frac{1}{1+\frac{k}{3z}}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+\frac{k}{3z}}\left(1-\frac{k}{4z}\right) = \frac{1-\frac{k}{4z}}{1+\frac{k}{3z}} = \frac{z-\frac{k}{4}}{z+\frac{k}{3}}$$

b) Wählen Sie den Parameter k so, dass das Filter stabil ist. Skizzieren Sie das Pol/Nullstellendiagramm für dieses stabile Filter.

Nullstelle bei $z = \frac{k}{4}$, Polstelle bei $z = -\frac{k}{3}$

$\left|\frac{k}{3}\right| < 1 \Rightarrow |k| < 3$, weil es im Konvergenzbereich vom Einheitskreis liegen muss



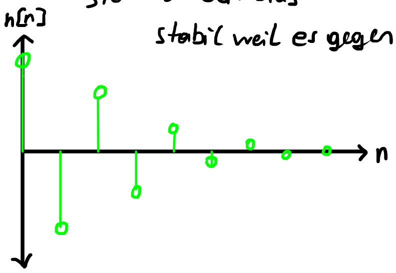
c) Berechnen und skizzieren Sie die Impulsantwort $h[n]$ und die Sprungantwort $a[n]$ des stabilen Filters.

Impulsantwort:

$$H(z) = \frac{z - \frac{k}{4}}{z + \frac{k}{3}} = \frac{z}{z + \frac{k}{3}} - \frac{k}{4} z^{-1} \frac{z}{z + \frac{k}{3}}$$ (von vorhin)

sieht so ca. aus

stabil weil es gegen 0 abklingt



FS: $\alpha^n \sigma[n] \circ\!\!-\!\!\bullet \frac{z}{z-\alpha}$

FS: $x[n+n_0] \circ\!\!-\!\!\bullet z^{n_0} X(z)$

$$H(z) \bullet\!\!-\!\!\circ h[n] = \left(-\tfrac{k}{3}\right)^n \sigma[n] \underbrace{- \tfrac{k}{4}\left(-\tfrac{k}{3}\right)^{n-1}}_{= \frac{k}{4}\cdot\left(-\frac{k}{3}\right)^n \cdot \left(-\frac{k}{3}\right)^{-1} = \left(-\frac{k}{3}\right)^n \cdot \frac{\cancel{k}}{4} \cdot \left(-\frac{3}{\cancel{k}}\right) = \left(-\frac{k}{3}\right)^n \cdot \left(-\frac{3}{4}\right)} \sigma[n-1] = \tfrac{7}{4}\left(-\tfrac{k}{3}\right)^n \sigma[n] - \tfrac{3}{4}\delta[n]$$ (ka wie) ???

Sprungantwort $a[n]$:

$x[n] = \sigma[n]$

FS: $\sigma[n] \circ\!\!-\!\!\bullet \frac{z}{z-1}$

$X(z) = \frac{z}{z-1}$

$$a[n] = h[n] * x[n] \quad \circ\!\!-\!\!\bullet \quad A(z) = H(z)\cdot X(z)$$

$$A(z) = z\left(\frac{z - \frac{k}{4}}{\left(z + \frac{k}{3}\right)(z-1)}\right) = z\underbrace{\left(\frac{A}{z + \frac{k}{3}} + \frac{B}{z-1}\right)}_{F(z)}$$

PBZ:

$$z - \tfrac{k}{4} = A(z-1) + B\left(z + \tfrac{k}{3}\right)$$

$z=1$: $1 - \frac{k}{4} = B\left(1 + \frac{k}{3}\right) \Rightarrow \boxed{B = \frac{1 - \frac{k}{4}}{1 + \frac{k}{3}}}$

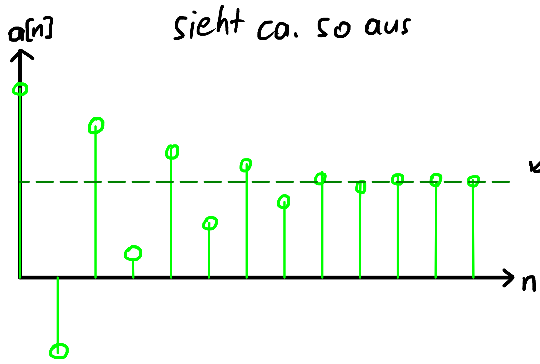
$z = -\frac{k}{3}$: $-\frac{k}{3} - \frac{k}{4} = A\left(-\frac{k}{3} - 1\right) \Rightarrow A = \frac{-\frac{k}{3} - \frac{k}{4}}{-\frac{k}{3} - 1} = \frac{\frac{4k+3k}{12}}{\frac{k+3}{3}} = \frac{3\cdot 7k}{12(k+3)} \Big| :\frac{k}{k} = \frac{\overset{7}{\cancel{21}}}{\underset{4}{\cancel{12}}\left(1+\frac{3}{k}\right)} = \boxed{\frac{7}{4}\,\frac{1}{1+\frac{3}{k}}}$

$$A(z) = z\left(\frac{7}{4}\,\frac{1}{1+\frac{3}{k}} \cdot \frac{1}{z + \frac{k}{3}} + \frac{1 - \frac{k}{4}}{1 + \frac{k}{3}} \cdot \frac{1}{z-1}\right)$$

$$= \frac{7}{4}\,\frac{1}{1+\frac{3}{k}} \cdot \frac{z}{z + \frac{k}{3}} + \frac{1 - \frac{k}{4}}{1 + \frac{k}{3}} \cdot \frac{z}{z-1}$$

$$a[n] = \left(\frac{7}{4}\,\frac{1}{1+\frac{3}{k}}\left(-\frac{k}{3}\right)^n + \frac{1 - \frac{k}{4}}{1 + \frac{k}{3}}\right)\sigma[n]$$

sieht ca. so aus



konv. gegen diesen Wert

Aufgabe 4.6:

Für jede der angegebenen Differenzengleichungen mit zugehöriger Anfangsbedingung für $y[-1]$ und dem Anregungssignal $x[n]$ berechne man mit Hilfe der $\mathcal{Z}$–Transformation das Signal $y[n]$.

Häufig vorkommende Spezialfälle bei DZGs

$$x[n-1] \overset{\mathcal{ZT}_1}{\Longleftrightarrow} z^{-1}X(z) + x[-1]$$
$$x[n-2] \overset{\mathcal{ZT}_1}{\Longleftrightarrow} z^{-2}X(z) + z^{-1}x[-1] + x[-2]$$
$$x[n+1] \overset{\mathcal{ZT}_1}{\Longleftrightarrow} zX(z) - zx[0]$$
$$x[n+2] \overset{\mathcal{ZT}_1}{\Longleftrightarrow} z^2X(z) - z^2x[0] - zx[1].$$

FS: $x[n+n_0] \circ\!\!-\!\!\bullet\; z^{n_0}X(z)$

FS: $\alpha^n \sigma[n] \circ\!\!-\!\!\bullet\; \frac{z}{z-\alpha}$

a) $y[n] + 3y[n-1] = x[n], \quad y[-1] = 1, \quad x[n] = \left(\frac{1}{2}\right)^n \sigma[n]$

$$X(z) = \frac{z}{z-\frac{1}{2}}$$

$$Y(z) + 3\left(\frac{1}{z}Y(z) + \underbrace{y[-1]}_{=1}\right) = X(z)$$
$$Y(z) + 3\frac{1}{z}Y(z) + 3 = X(z)$$
$$Y(z)\left(1 + \frac{3}{z}\right) + 3 = X(z)$$
$$Y(z)\left(1 + \frac{3}{z}\right) = \frac{z}{z-\frac{1}{2}} - 3$$
$$Y(z) = \left(\frac{z}{z-\frac{1}{2}} - 3\right) \cdot \frac{1}{1+\frac{3}{z}}$$
$$Y(z) = \left(\frac{z}{z-\frac{1}{2}} - 3\right) \cdot \frac{z}{z+3}$$
$$Y(z) = z\Big(\underbrace{\frac{z}{(z-\frac{1}{2})(z+3)}}_{F(z)}\Big) - 3 \cdot \frac{z}{z+3}$$

1 z herausgehoben damit man später auf Form $\frac{z}{z-\alpha}$ kommt und rücktransformieren kann

PBZ: $F(z) = \frac{z}{(z-\frac{1}{2})(z+3)} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z+3} \quad \Big/ \cdot (z-\frac{1}{2})(z+3)$

$$z = A(z+3) + B(z-\tfrac{1}{2})$$

$z = \frac{1}{2}$: $\quad \frac{1}{2} = \frac{7}{2}A \Rightarrow A = \frac{1}{7}$

$z = -3$: $\quad -3 = -\frac{7}{2}B \Rightarrow B = \frac{6}{7}$

$$F(z) = \frac{1}{7}\frac{1}{z-\frac{1}{2}} + \frac{6}{7}\frac{1}{z+3}$$
$$Y(z) = z\left(\frac{1}{7}\frac{1}{z-\frac{1}{2}} + \frac{6}{7}\frac{1}{z+3}\right) - 3 \cdot \frac{z}{z+3}$$
$$Y(z) = \frac{1}{7}\frac{z}{z-\frac{1}{2}} + \frac{6}{7}\frac{z}{z+3} - 3 \cdot \frac{z}{z+3}$$
$$\bullet\!\!-\!\!\circ$$
$$y[n] = \frac{1}{7}\left(\frac{1}{2}\right)^n\sigma[n] + \frac{6}{7}(-3)^n\sigma[n] \underbrace{-3}_{=-\frac{21}{7}} \cdot (-3)^n\sigma[n]$$
$$y[n] = \left(\frac{1}{7}\left(\frac{1}{2}\right)^n - \frac{15}{7}(-3)^n\right)\sigma[n]$$

b) $y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{2}x[n-1], \quad y[-1] = 0, \quad x[n] = \sigma[n]$

FS: $\sigma[n] \circ\!\!-\!\!\bullet\; \frac{z}{z-1}$

$$X(z) = \frac{z}{z-1}$$

$$Y(z) - \frac{1}{2}\left(\frac{1}{z}Y(z) + \underbrace{y[-1]}_{=0}\right) = X(z) - \frac{1}{2}\left(\frac{1}{z}X(z) + \underbrace{x[-1]}_{=0}\right)$$
$$Y(z)\left(1 - \frac{1}{2z}\right) = X(z)\left(1 - \frac{1}{2z}\right)$$
$$Y(z)\left(1 - \frac{1}{2z}\right) = \frac{z}{z-1}\left(1 - \frac{1}{2z}\right)$$
$$Y(z) = \frac{z}{z-1}\cancel{\left(1 - \frac{1}{2z}\right)} \cdot \frac{1}{\cancel{1 - \frac{1}{2z}}}$$
$$\bullet\!\!-\!\!\circ$$
$$y[n] = \sigma[n]$$

c) $y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{2}x[n-1], \quad y[-1] = 1, \quad x[n] = \sigma[n]$

FS: $\sigma[n] \circ\!\!-\!\!\bullet \frac{z}{z-1}$

$$Y(z) - \frac{1}{2}\left(\frac{1}{z}Y(z) + \underbrace{y[-1]}_{=1}\right) = X(z) - \frac{1}{2}\left(\frac{1}{z}X(z) + \underbrace{x[-1]}_{=0}\right)$$

$$Y(z) - \frac{1}{2z}Y(z) - \frac{1}{2} = X(z)\left(1 - \frac{1}{2z}\right)$$

$X(z) = \frac{z}{z-1}$

$$Y(z)\left(1 - \frac{1}{2z}\right) - \frac{1}{2} = \frac{z}{z-1}\left(1 - \frac{1}{2z}\right)$$

$$Y(z) = \left(\frac{z}{z-1}\left(1 - \frac{1}{2z}\right) + \frac{1}{2}\right) \cdot \frac{1}{1 - \frac{1}{2z}}$$

$$Y(z) = \left(\frac{\cancel{z}}{z-1} \cdot \frac{2z-1}{2\cancel{z}} + \frac{1}{2}\right) \cdot \frac{z}{z-\frac{1}{2}}$$

$$Y(z) = \left(\frac{2z-1}{2z-2} + \frac{1}{2}\right) \cdot \frac{z}{z-\frac{1}{2}}$$

$$Y(z) = \left(\frac{2z-1}{2z-2} + \frac{z-1}{2(z-1)}\right) \cdot \frac{z}{z-\frac{1}{2}}$$

$$Y(z) = \frac{3z-2}{2z-2} \cdot \frac{z}{z-\frac{1}{2}} = z\left(\frac{3z-2}{2(z-1)(z-\frac{1}{2})}\right) = z\frac{1}{2}\underbrace{\left(\frac{3z-2}{(z-1)(z-\frac{1}{2})}\right)}_{F(z)}$$

PBZ: $\frac{3z-2}{(z-1)(z-\frac{1}{2})} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}} \quad \Big| \cdot (z-1)(z-\frac{1}{2})$

$$3z-2 = A(z-\tfrac{1}{2}) + B(z-1)$$

$z = \frac{1}{2}$: $\quad -\frac{1}{2} = -\frac{1}{2}B \Rightarrow B = 1$

$z = 1$: $\quad 1 = \frac{1}{2}A \Rightarrow A = 2$

$$\Rightarrow F(z) = \frac{2}{z-1} + \frac{1}{z-\frac{1}{2}}$$

$$Y(z) = z\frac{1}{2}\left(\frac{2}{z-1} + \frac{1}{z-\frac{1}{2}}\right) = \frac{z}{z-1} + \frac{1}{2}\frac{z}{z-\frac{1}{2}}$$

FS: $\alpha^n\sigma[n] \circ\!\!-\!\!\bullet \frac{z}{z-\alpha}$

FS: $\sigma[n] \circ\!\!-\!\!\bullet \frac{z}{z-1}$

$$Y(z) \bullet\!\!-\!\!\circ\, y[n] = \sigma[n] + \frac{1}{2}\cdot\left(\frac{1}{2}\right)^n\sigma[n] = \left(1 + \left(\frac{1}{2}\right)^{n+1}\right)\sigma[n]$$