Sigsys 2 - UE5

Aufgabe 3.1:

Berechnen Sie die Fouriertransformation $X(e^{j\theta})$ der folgenden Signale:

a) $x[n] = \alpha^n \underbrace{\sin\theta_0 n}_{= \frac{e^{j\theta_0}-e^{-j\theta_0}}{2j}}\, \sigma[n]$ für $|\alpha| < 1$

$$x[n] = \alpha^n \left(\frac{e^{j\theta_0}-e^{-j\theta_0}}{2j}\right)\sigma[n] = \frac{1}{2j}\left(\alpha^n e^{j\theta_0}\sigma[n] - \alpha^n e^{-j\theta_0}\sigma[n]\right)$$

FS: $a^n\sigma[n] \circ\!\!-\!\!\bullet \frac{1}{1-ae^{-j\theta}}$ für $|a|<1$

FS: $e^{j\theta_0 n}x[n] \circ\!\!-\!\!\bullet X(e^{j(\theta-\theta_0)})$

$$X(e^{j\theta}) = \frac{1}{2j}\left(\frac{1}{1-\alpha e^{-j(\theta-\theta_0)}} - \frac{1}{1-\alpha e^{-j(\theta+\theta_0)}}\right) = \frac{1}{2j}\left(\frac{(\cancel{1}-\alpha e^{-j(\theta+\theta_0)}) - (\cancel{1}-\alpha e^{-j(\theta-\theta_0)})}{(1-\alpha e^{-j(\theta+\theta_0)})\cdot(1-\alpha e^{-j(\theta-\theta_0)})}\right)$$

$$= \frac{1}{2j}\left(\frac{\alpha e^{-j(\theta-\theta_0)} - \alpha e^{-j(\theta+\theta_0)}}{1-\alpha e^{-j(\theta-\theta_0)} - \alpha e^{-j(\theta+\theta_0)} + \underbrace{\alpha^2 e^{-j(\theta-\cancel{\theta_0})-j(\theta+\cancel{\theta_0})}}_{=\alpha^2 e^{-j2\theta}}}\right)$$

$$= \frac{1}{2j}\left(\frac{\alpha e^{-j\theta}(e^{j\theta_0}-e^{-j\theta_0})}{(1-\alpha^2 e^{-j2\theta} - \alpha e^{-j\theta}(e^{j\theta}+e^{-j\theta}))}\right) \cdot \frac{2}{2} \text{ erweitern}$$

$$\sin(\theta_0) = \frac{e^{j\theta_0}-e^{-j\theta_0}}{2j} \qquad \cos(\theta_0) = \frac{e^{j\theta_0}+e^{-j\theta_0}}{2}$$

$$= \frac{\alpha e^{-j\theta}\sin(\theta_0)}{1-\alpha^2 e^{-j2\theta} - 2\alpha e^{-j\theta}\cos(\theta_0)}$$

b) $x[n] = 2^n\sigma[-n]$ Substituieren $\tilde{n} = -n$

$$\Rightarrow x[-\tilde{n}] = 2^{-\tilde{n}}\sigma[\tilde{n}] \qquad 2^n = 2^{-\tilde{n}} = \frac{1}{2^{\tilde{n}}} = a \qquad |a|<1 \checkmark$$

FS: $a^n\sigma[n] \circ\!\!-\!\!\bullet \frac{1}{1-ae^{-j\theta}}$ für $|a|<1$

$$X(e^{j\theta}) = X(e^{-j\tilde{\theta}}) = \frac{1}{1-\frac{1}{2}e^{j\tilde{\theta}}} = \frac{1}{1-\frac{1}{2}e^{j\theta}}$$

oder so:

$$\sum_{\tilde{n}=0}^{\infty} 2^{-\tilde{n}} \cdot e^{j\theta\tilde{n}} = \sum_{\tilde{n}=0}^{\infty}\left(\frac{1}{2}e^{j\theta}\right)^{\tilde{n}} = \frac{1}{1-\frac{1}{2}e^{j\theta}}$$

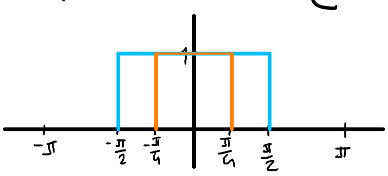
c) $x[n] = \overbrace{\frac{\sin\frac{\pi}{2}n}{\pi n}}^{y[n]}\ \overbrace{\frac{\sin\frac{\pi}{4}n}{\pi n}}^{z[n]}$

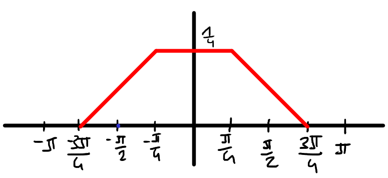
FS: $\frac{\sin(\alpha n)}{\pi n} \circ\!\!-\!\!\bullet X(e^{j\theta}) = \begin{cases} 1 & \text{für } 0 \le |\theta| \le \alpha \\ 0 & \text{für } \alpha < |\theta| < \pi \end{cases}$

$(0 < \alpha < \pi)$

FS: $x[n]y[n] \circ\!\!-\!\!\bullet \frac{1}{2\pi}(X * Y)(e^{j\theta})$

$$y[n] \circ\!\!-\!\!\bullet Y(e^{j\theta}) = \begin{cases} 1 & \text{für } 0 \le |\theta| \le \frac{\pi}{2} \\ 0 & \text{für } \frac{\pi}{2} < |\theta| < \pi \end{cases} \qquad z[n] \circ\!\!-\!\!\bullet Z(e^{j\theta}) = \begin{cases} 1 & \text{für } 0 \le |\theta| \le \frac{\pi}{4} \\ 0 & \text{für } \frac{\pi}{4} < |\theta| < \pi \end{cases}$$





$$X(e^{j\theta}) = \begin{cases} \frac{\frac{1}{4}\theta}{\frac{\pi}{2}} + \frac{3}{8} = \frac{1}{2\pi}\left(\theta + \frac{3\pi}{4}\right) & \text{für } -\frac{3\pi}{4} < \theta < -\frac{\pi}{4} \\ \frac{1}{4} & \text{für } -\frac{\pi}{4} < \theta < \frac{\pi}{4} \\ -\frac{1}{2\pi}\left(\theta - \frac{3\pi}{4}\right) & \text{für } \frac{\pi}{4} < \theta < \frac{3\pi}{4} \\ 0 & \text{sonst in } [-\pi, \pi] \end{cases}$$

d) $x[n] = \left(\frac{1}{2}\right)^{|n|} = \left(\frac{1}{2}\right)^{-n}\cdot\sigma[-n] + \left(\frac{1}{2}\right)^{n}\cdot\sigma[n] - \delta[n]$

FS: $a^n\sigma[n] \circ\!\!-\!\!\bullet \frac{1}{1-ae^{-j\theta}}$ $\quad |a|<1$

$$X(e^{j\theta}) = \frac{1}{1-\frac{1}{2}e^{j\theta}} + \frac{1}{1-\frac{1}{2}e^{-j\theta}} - \overbrace{e^{-j\theta\cdot 0}}^{=1}$$

$$= \frac{\left(1-\frac{1}{2}e^{-j\theta}\right)+\left(1-\frac{1}{2}e^{j\theta}\right)}{\left(1-\frac{1}{2}e^{j\theta}\right)\left(1-\frac{1}{2}e^{-j\theta}\right)} - 1 = \frac{2-\frac{1}{2}\overbrace{\left(e^{j\theta}+e^{-j\theta}\right)}^{\cos(\theta)}}{1-\frac{1}{2}e^{j\theta}-\frac{1}{2}e^{-j\theta}+\frac{1}{4}\underbrace{e^{j\theta-j\theta}}_{=1}} - 1 =$$

$$= \frac{2-\cos(\theta)}{1-\frac{1}{2}\left(e^{j\theta}+e^{-j\theta}\right)+\frac{1}{4}} - 1 = \frac{2-\cos(\theta)}{1-\cos(\theta)+\frac{1}{4}} - 1 = \frac{2-\cos(\theta)}{\frac{5}{4}-\cos(\theta)} - 1 =$$

$$= \frac{(2-\cos(\theta)) - \left(\frac{5}{4}-\cos(\theta)\right)}{\frac{5}{4}-\cos(\theta)} = \frac{\frac{3}{4}}{\frac{5}{4}-\cos(\theta)} = \frac{\frac{3}{\cancel{4}}}{\frac{5-4\cos(\theta)}{\cancel{4}}} = \frac{3}{5-4\cos(\theta)}$$

e) $x[n] = n\left(\frac{1}{2}\right)^{|n|}$

FS: $n\,x[n] \circ\!\!-\!\!\bullet\; j\frac{dX(e^{j\theta})}{d\theta}$

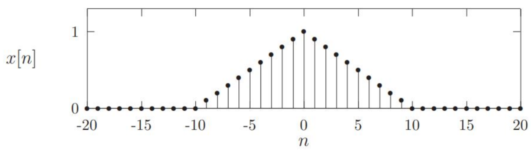
$$X(e^{j\theta}) = j\frac{d}{d\theta}\left(\frac{3}{5-4\cos(\theta)}\right) = 3j\frac{d}{d\theta}\left(\frac{1}{5-4\cos(\theta)}\right) = 3j\cdot\left(\frac{-4\sin(\theta)}{(5-4\cos(\theta))^2}\right) = -12j\,\frac{\sin(\theta)}{(5-4\cos(\theta))^2}$$

f) $x[n] = (-1)^n = e^{j\pi n}$

FS: $e^{j\theta_0 n} \circ\!\!-\!\!\bullet\; 2\pi\delta_{2\pi}(\theta-\theta_0)$

$$X(e^{j\theta}) = 2\pi\delta_{2\pi}(\theta-\pi)$$

g)



als Ergebnis zweier Faltungssignale darstellen

(Alternative: mit $\frac{1}{110}$ arbeiten statt 1 für $0\le n\le 9$)

$$a[n] = \begin{cases} 1 & \text{für } 0\le n\le 9 \\ 0 & \text{sonst} \end{cases} = \sigma[n] - \sigma[n-10]$$

$$b[n] = a[n] * a[n] \circ\!\!-\!\!\bullet\; B(e^{j\theta}) = A(e^{j\theta})\cdot A(e^{j\theta})$$

FS: $\sigma[n] \circ\!\!-\!\!\bullet \frac{1}{1-e^{-j\theta}} + \pi\delta_{2\pi}(\theta)$, $x[n-N_0] \circ\!\!-\!\!\bullet\; e^{-j\theta N_0}X(e^{j\theta})$

$$x[n] = \frac{1}{10}\cdot b[n+9]$$

$$a[n] \circ\!\!-\!\!\bullet A(e^{j\theta}) = \frac{1}{1-e^{-j\theta}} + \pi\delta_{2\pi}(\theta) - e^{-j\theta 10}\cdot\left(\frac{1}{1-e^{-j\theta}} + \pi\delta_{2\pi}(\theta)\right)$$

$$= \left(\frac{1}{1-e^{-j\theta}} + \pi\delta_{2\pi}(\theta)\right)\cdot\left(1-e^{-j\theta 10}\right)$$

$$= \underbrace{\frac{1-e^{-j\theta 10}}{1-e^{-j\theta}}}_{I} + \underbrace{\left(1-e^{-j\theta 10}\right)\left(\pi\delta_{2\pi}(\theta)\right)}_{II}$$

$$\mathrm{I}: \frac{1-e^{-j\theta 10}}{1-e^{-j\theta}} = \frac{e^{-j\theta 5}\left(e^{j\theta 5}-e^{-j\theta 5}\right)}{e^{-j\frac{\theta}{2}}\left(e^{j\frac{\theta}{2}}-e^{-j\frac{\theta}{2}}\right)} = \frac{\cancel{2j}\,e^{-j\theta 5}}{\cancel{2j}\,e^{-j\frac{\theta}{2}}}\cdot\frac{\sin(5\theta)}{\sin\left(\frac{\theta}{2}\right)} = e^{-j\frac{9}{2}\theta}\cdot\frac{\sin(5\theta)}{\sin\left(\frac{\theta}{2}\right)}$$

$$\mathrm{II}: \left(1-\overbrace{e^{-j\theta 10}}^{=1 \text{ wenn } \theta=2\pi k\ (k\in\mathbb{Z})}\right)\left(\pi\delta_{2\pi}(\theta)\right) = \begin{cases}\left(1-e^{-j\theta 10}\right)\pi & \text{wenn } \theta=2\pi k \quad (k\in\mathbb{Z})\\ 0 & \text{sonst}\end{cases}$$

$$=\left(1-\underbrace{e^{-j\cdot 2\pi k\cdot 10}}_{=1 \text{ immer}}\right)\pi = (1-1)\pi = 0$$

$$A(e^{j\theta}) = e^{-j\frac{9}{2}\theta}\cdot\frac{\sin(5\theta)}{\sin\left(\frac{\theta}{2}\right)}$$

$$B(e^{j\theta}) = \left(A(e^{j\theta})\right)^2 = e^{-\frac{18}{2}\theta}\,\frac{\sin^2(5\theta)}{\sin^2\left(\frac{\theta}{2}\right)} = e^{-9\theta}\cdot\frac{\sin^2(5\theta)}{\sin^2\left(\frac{\theta}{2}\right)}$$

$$x[n] = \frac{1}{10}\cdot b[n+9] \quad \circ\!\!-\!\!\bullet \quad X(e^{j\theta}) = \frac{1}{10}\cdot\cancel{e^{-9\theta}}\cdot\frac{\sin^2(5\theta)}{\sin^2\left(\frac{\theta}{2}\right)}\cdot\overbrace{\cancel{e^{j9\theta}}}^{\text{wegen Verschiebung nach links}} = \frac{1}{10}\,\frac{\sin^2(5\theta)}{\sin^2\left(\frac{\theta}{2}\right)}$$

Aufgabe 3.2:

Berechnen Sie das Zeitsignal $x[n]$ für folgende Spektren:

a) $X(e^{j\theta}) = \cos^2\theta$

$$\cos^2(\theta) = \cos(\theta)\cdot\cos(\theta)$$

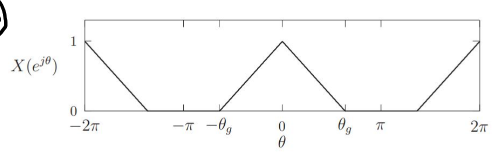
$$\cos(\theta) = \frac{1}{2}\left(e^{j\theta}+e^{-j\theta}\right)$$

$$= \frac{1}{4}\left(e^{j\theta}+e^{-j\theta}\right)^2 = \frac{1}{4}\left(e^{j\theta 2} + 2\overbrace{e^{j\theta}e^{-j\theta}}^{=1} + e^{-j\theta 2}\right) = \frac{1}{4}e^{j\theta 2} + \frac{1}{2} + \frac{1}{4}e^{-j\theta 2}$$

FS: $\delta[n-N_0] \;\circ\!\!-\!\!\bullet\; e^{-j\theta N_0}$

$$X(e^{j\theta}) \;\bullet\!\!-\!\!\circ\; x[n] = \frac{1}{4}\delta[n+2] + \frac{1}{2}\delta[n] + \frac{1}{4}\delta[n-2]$$

b)



Faltung zweier Rechtecke $A(e^{j\theta})$

$$A(e^{j\theta}) = \begin{cases}\frac{1}{\sqrt{\theta_g}} & \text{für } 0\le|\theta|\le\frac{\theta_g}{2}\\ 0 & \text{für } \frac{\theta_g}{2}<|\theta|<\pi\end{cases}$$

$(0<\alpha<\pi)$

FS: $\frac{\sin(\alpha n)}{\pi n} \;\circ\!\!-\!\!\bullet\; \begin{cases}1 & \text{für } 0\le|\theta|\le\alpha\\ 0 & \text{für } \alpha<|\theta|<\pi\end{cases}$

$$A(e^{j\theta}) \;\bullet\!\!-\!\!\circ\; a[n] = \frac{1}{\sqrt{\theta_g}}\;\frac{\sin\left(\frac{\theta_g}{2}n\right)}{\pi n}$$

FS: $x[n]\,y[n] \;\circ\!\!-\!\!\bullet\; \frac{1}{2\pi}(X*Y)(e^{j\theta})$

$$x[n] = a[n]\,a[n]\cdot 2\pi = \frac{2\pi}{\theta_g}\;\frac{\sin^2\left(\frac{\theta_g}{2}n\right)}{\pi^2 n^2}$$