# VU Einführung in Wissensbasierte Systeme 

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## Solutions Exercise Sheet 2

## Exercise 2.1

Consider the theory

$$
\begin{aligned}
T:= & \{\forall x \forall y((P(x) \vee Q(y)) \rightarrow R(x, y)), \\
& \forall x((S(x) \wedge Q(x)) \rightarrow P(x)), \\
& Q(b), P(a), S(b)\}
\end{aligned}
$$

containing only the constants $a$ and $b$, along with the extended theories

$$
\begin{aligned}
& T_{1}:=T \cup\{\forall x(S(x) \rightarrow \neg P(x))\} \quad \text { and } \\
& T_{2}:=T \cup\{\forall x(\neg S(x) \rightarrow \neg R(x, x))\} .
\end{aligned}
$$

For each $T_{i}, i \in\{1,2\}$, answer the following questions:
(a) Determine CWA $\left(T_{i}\right)$ and $\mathrm{CWA}^{\{S, P\}}\left(T_{i}\right)$.
(b) Prove or refute whether $\operatorname{CWA}\left(T_{i}\right)$ and $\mathrm{CWA}{ }^{\{S, P\}}\left(T_{i}\right)$ are consistent.

## Solution to Exercise 2.1

>By definition,

- $\operatorname{CWA}\left(T_{i}\right)=\operatorname{Cn}\left(T_{i} \cup T_{i a s m}\right)$, where

$$
T_{i a s m}=\left\{\neg P \mid T_{i} \not \vDash P, P \text { is a ground atom }\right\},
$$

and

- $\mathrm{CWA}^{\{S, P\}}\left(T_{i}\right)=\operatorname{Cn}\left(T_{i} \cup T_{i a s m}^{\{S, P\}}\right)$, where

$$
\begin{gathered}
T_{i \text { asm }}^{\{S, P\}}=\left\{\neg P \mid T_{i} \not \vDash P, P\right. \text { is a ground atom with } \\
\text { predicate symbol } S \text { or } P\},
\end{gathered}
$$

for $i=1,2$.
$\Leftrightarrow$ We need to determine $T_{i \text { asm }}$ for $i=1,2$.

## Solution to Exercise 2.1 (ctd.)

> First of all, we observe that $T_{1}$ is inconsistent:

- Since

$$
\begin{aligned}
& T_{1} \models S(b) \wedge Q(b) \wedge((S(b) \wedge Q(b)) \rightarrow P(b)) \wedge(S(b) \rightarrow \neg P(b)) \\
& \text { we get } T_{1} \models P(b) \text { and } T_{1} \models \neg P(b) . \\
& \Longrightarrow T_{1 \text { asm }}=T_{1 \text { asm }}\{S, P\} \\
& \operatorname{CWA}\left(T_{1}\right)=\mathrm{CWA}^{\{S, P\}}\left(T_{1}\right)=\operatorname{Cn}\left(T_{1}\right) .
\end{aligned}
$$

$\Longrightarrow \operatorname{CWA}\left(T_{1}\right)$ and $\mathrm{CWA}^{\{S, P\}}\left(T_{1}\right)$ are inconsistent too.
$>$ As for $T_{2}$, this theory is consistent:

- Consider the $\Sigma$-structure $\mathcal{I}=\langle\mathcal{U}, I, \emptyset\rangle$ such that

$$
\begin{aligned}
-\mathcal{U}=\left\{d_{a}, d_{b}\right\}, I(a)=d_{a}, I(b)=d_{b}, I(P)=I(S)=\mathcal{U} \\
I(Q)=\left\{d_{b}\right\}, I(R)=\mathcal{U} \times \mathcal{U}
\end{aligned}
$$

- Then, $\mathcal{I}$ is a model of $T_{2}$. Moreover, $\mathcal{I} \not \vDash Q(a)$ while $\mathcal{I} \models A$ for each ground atom $A \neq Q(a)$.


## Solution to Exercise 2.1 (ctd.)

$>$ In view of model $\mathcal{I}$, we have $T_{2} \notin Q(a) . \Longrightarrow \neg Q(a) \in T_{2 \text { asm }}$.
> Moreover, it holds that $T_{2} \models A$ for each ground atom $A \neq Q(a)$ :

- $T_{2} \models P(a), Q(b), S(b)$ since $\{P(a), Q(b), S(b)\} \subseteq T$ and $T \subseteq T_{2}$.
- $T_{2} \models P(b)$ since $T \models P(b)$ and $T \subseteq T_{2}$.
- $T_{2} \models R(a, a), R(a, b), R(b, a), R(b, b)$ since

$$
\begin{aligned}
- & T_{2} \models(P(a) \vee Q(a)),(P(a) \vee Q(b)),(P(b) \vee \\
& Q(a)),(P(b) \vee Q(b)) \text { and } \\
- & T_{2} \models \forall x \forall y((P(x) \vee Q(y)) \rightarrow R(x, y)) .
\end{aligned}
$$

- $T_{2} \models S(a)$ since $T_{2} \models R(a, a) \wedge(\neg S(a) \rightarrow \neg R(a, a))$.
$\Rightarrow T_{2 a s m}=\{\neg Q(a)\}$ and $T_{2 \text { asm }}{ }^{\{S, P\}}=\emptyset$. Hence,
$\operatorname{CWA}\left(T_{2}\right)=\operatorname{Cn}\left(T_{2} \cup\{\neg Q(a)\}\right)$ and $\operatorname{CWA}^{\{S, P\}}\left(T_{2}\right)=\operatorname{Cn}\left(T_{2}\right)$.
- As $\mathcal{I}$ is clearly also a model of $\operatorname{CWA}\left(T_{2}\right)$ as well as of $\operatorname{CWA}^{\{S, P\}}\left(T_{2}\right)$, both $\operatorname{CWA}\left(T_{2}\right)$ and $\operatorname{CWA}^{\{S, P\}}\left(T_{2}\right)$ are consistent.


## Exercise 2.2

Consider the following theorem from the lecture:
Let $T$ be a consistent theory. Then, $\operatorname{CWA}(T)$ is inconsistent iff there are ground atoms $A_{1}, \ldots, A_{n}($ for $n>1$ ) such that $T \models A_{1} \vee \cdots \vee A_{n}$ but $T \not \vDash A_{i}$, for all $i=1, \ldots, n$.
(a) Prove the only-if direction.
(b) Prove the if direction.

## Solution to Exercise 2.2 (a)

> Assume that $\operatorname{CWA}(T)$ is inconsistent. We first show that $T_{\text {asm }} \neq \emptyset$.

- Assume that $T_{\text {asm }}=\emptyset$. Then,

$$
\operatorname{CWA}(T)=C n\left(T \cup T_{a s m}\right)=C n(T)
$$

and hence $T$ would be inconsistent in view of our hypothesis CWA( $T$ ) is inconsistent.

- This is a contradiction to our assumption that $T$ is consistent. $\Longrightarrow T_{\text {asm }} \neq \emptyset$.
> We now use the compactness property:
- $T$ is satisfiable $\Longleftrightarrow$ each finite subset of $T$ is satisfiable.
- Hence:
- $T$ is unsatisfiable $\Longleftrightarrow$ not every finite subset of $T$ is satisfiable $\Longleftrightarrow$ there is a finite subset of $T$ which is unsatisfiable.


## Solution to Exercise 2.2 (a) (ctd.)

> We know that $\operatorname{CWA}(T)$ is unsatisfiable and

$$
\mathrm{CWA}(T)=C n\left(T \cup T_{\text {asm }}\right) .
$$

Hence, $T \cup T_{a s m}$ is unsatisfiable too.
$\Rightarrow$ By compactness, there is a finite subset $T^{0} \subseteq T \cup T_{\text {asm }}$ such that $T^{0}$ is unsatisfiable.

- Let $T_{\text {asm }}^{0}:=T^{0} \cap T_{\text {asm }}$.
- Then, $T_{a s m}^{0}$ is a finite set of ground atoms and $T \cup T_{\text {asm }}^{0}$ is unsatisfiable.
- Furthermore, $T_{\text {asm }}^{0} \neq \emptyset$, otherwise $T^{0} \subseteq T$ would hold and that would mean that $T$ is unsatisfiable, which contradicts the fact that $T$ is consistent.


## Solution to Exercise 2.2 (a) (ctd.)

$>$ Let now $T_{\text {asm }}^{0}$ be $\left\{\neg A_{1}, \ldots, \neg A_{n}\right\}$.
Claim: $n>1$.
Proof of Claim:

- $n$ must be greater than 0 because $T_{\text {asm }}^{0} \neq \emptyset$.
- Assume that $n=1$.
- Then, $T \cup\left\{\neg A_{1}\right\}$ is unsatisfiable, and hence $T \models A_{1}$ follows.
- But $\neg A_{1} \in T_{\text {asm }}$ and hence $T \not \vDash A_{1}$ by definition of $T_{\text {asm }}$. $\Longrightarrow$ Contradiction.
$\Rightarrow n>1$.
$\Rightarrow T \cup\left\{\neg A_{1}, \ldots, \neg A_{n}\right\}$ is unsatisfiable.
$\Rightarrow T \cup\left\{\neg A_{1} \wedge \cdots \wedge \neg A_{n}\right\}$ is unsatisfiable.
$\Rightarrow T \cup\left\{\neg\left(A_{1} \vee \cdots \vee A_{n}\right)\right\}$ is unsatisfiable.
$\Leftrightarrow T \models A_{1} \vee \cdots \vee A_{n}$. This proves the only-if direction and thus Part (a).


## Solution to Exercise 2.2 (b)

$>$ Assume there are ground atoms $A_{1}, \ldots, A_{n}$, for $n>1$, such that $T \models A_{1} \vee \cdots \vee A_{n}$ but $T \not \vDash A_{i}$ for each $i=1, \ldots, n$.
$\Longrightarrow\left\{\neg A_{1}, \ldots, \neg A_{n}\right\} \subseteq T_{\text {asm }}$ (since $T \not \vDash A_{i}$ for $i=1, \ldots, n$ ).
$\Longrightarrow\left\{\neg A_{1}, \ldots, \neg A_{n}\right\} \subseteq C n\left(T \cup T_{\text {asm }}\right)$.
$\Longrightarrow T \cup T_{\text {asm }} \models \neg A_{1} \wedge \cdots \wedge \neg A_{n}$.
$\Longrightarrow T \cup T_{\text {asm }}=\neg\left(A_{1} \vee \cdots \vee A_{n}\right)$.
$\Rightarrow$ Since $T \models A_{1} \vee \cdots \vee A_{n}$ by hypothesis, it follows also that $T \cup T_{\text {asm }} \models A_{1} \vee \cdots \vee A_{n}$ and hence $T \cup T_{\text {asm }}$ is inconsistent.
$\Leftrightarrow \operatorname{CWA}(T)=\operatorname{Cn}\left(T \cup T_{\text {asm }}\right)$ is inconsistent.

## Exercise 2.3

$>$ Consider the open default theory $T=(W, \Delta)$, where

$$
\begin{aligned}
W & =\{\forall x(P(x) \vee Q(x)), \exists x(R(x) \vee P(x)), \forall x(\neg Q(x) \rightarrow \neg R(x))\}, \\
\Delta & =\left\{\frac{P(x): \neg Q(x)}{\neg Q(x)}, \frac{Q(x): \neg P(x)}{\neg P(x)}, \frac{\top: \neg R(x)}{\neg R(x)}\right\} .
\end{aligned}
$$

> Compute the closure of $T$ and determine the possible candidates for being an extension.
> For each candidate $E$, compute the classical reduct $\Delta_{E}$ and determine all extensions of $T$.

## Solution to Exercise 2.3

$>$ A closure of $T$ is given by $\bar{T}=(\bar{W}, \bar{\Delta})$, where

$$
\begin{aligned}
\bar{W} & =\{\forall x(P(x) \vee Q(x)), R(a) \vee P(a), \forall x(\neg Q(x) \rightarrow \neg R(x))\}, \\
\bar{\Delta} & =\left\{\frac{P(a): \neg Q(a)}{\neg Q(a)}, \frac{Q(a): \neg P(a)}{\neg P(a)}, \frac{\top: \neg R(a)}{\neg R(a)}\right\} .
\end{aligned}
$$

> The possible candidates for being extensions are

- $E_{1}=\operatorname{Cn}(\bar{W})$,
- $E_{2}=C n(\bar{W} \cup\{\neg Q(a)\})$,
- $E_{3}=C n(\bar{W} \cup\{\neg P(a)\})$,
- $E_{4}=C n(\bar{W} \cup\{\neg R(a)\})$,
- $E_{5}=C n(\bar{W} \cup\{\neg Q(a), \neg P(a)\})$ (inconsistent),
- $E_{6}=C n(\bar{W} \cup\{\neg Q(a), \neg R(a)\})$,
- $E_{7}=C n(\bar{W} \cup\{\neg P(a), \neg R(a)\})$ (inconsistent),
- $E_{8}=C n(\bar{W} \cup\{\neg Q(a), \neg P(a), \neg R(a)\})$ (inconsistent).


## Solution to Exercise 2.3 (ctd.)

$$
\begin{aligned}
\bar{W} & =\{\forall x(P(x) \vee Q(x)), R(a) \vee P(a), \forall x(\neg Q(x) \rightarrow \neg R(x))\}, \\
\bar{\Delta} & =\left\{\frac{P(a): \neg Q(a)}{\neg Q(a)}, \frac{Q(a): \neg P(a)}{\neg P(a)}, \frac{\top: \neg R(a)}{\neg R(a)}\right\} .
\end{aligned}
$$

$>$ We determine the classical reducts $\Delta_{E_{i}}$ and $\Gamma_{\bar{T}}\left(E_{i}\right)=C n^{\Delta_{E_{i}}}(\bar{W})$ for $i=1, \ldots, 8$ :

- $E_{1}=\operatorname{Cn}(\bar{W})$ :

$$
\begin{aligned}
& -\Delta_{E_{1}}=\{P(a) / \neg Q(a), Q(a) / \neg P(a), \top / \neg R(a)\} . \\
& -\Gamma_{\bar{T}}\left(E_{1}\right)=\operatorname{Cn}(\bar{W} \cup\{\neg R(a), \neg Q(a)\})=E_{6} \neq E_{1} .
\end{aligned}
$$

- $E_{2}=C n(\bar{W} \cup\{\neg Q(a)\}):$

$$
\begin{aligned}
& -\Delta_{E_{2}}=\{P(a) / \neg Q(a), \top / \neg R(a)\} . \\
& -\Gamma_{\bar{T}}\left(E_{2}\right)=\operatorname{Cn}(\bar{W} \cup\{\neg R(a), \neg Q(a)\})=E_{6} \neq E_{2} .
\end{aligned}
$$

- $E_{3}=\operatorname{Cn}(\bar{W} \cup\{\neg P(a)\})$ :

$$
\begin{aligned}
& -\Delta_{E_{3}}=\{Q(a) / \neg P(a)\} \\
& -\Gamma_{\bar{T}}\left(E_{3}\right)=C n(\bar{W})=E_{1} \neq E_{3} .
\end{aligned}
$$

## Solution to Exercise 2.3 (ctd.)

$$
\begin{aligned}
\bar{W} & =\{\forall x(P(x) \vee Q(x)), R(a) \vee P(a), \forall x(\neg Q(x) \rightarrow \neg R(x))\}, \\
\bar{\Delta} & =\left\{\frac{P(a): \neg Q(a)}{\neg Q(a)}, \frac{Q(a): \neg P(a)}{\neg P(a)}, \frac{\top: \neg R(a)}{\neg R(a)}\right\} .
\end{aligned}
$$

- $E_{4}=C n(\bar{W} \cup\{\neg R(a)\}):$

$$
\begin{aligned}
& -\Delta_{E_{4}}=\{P(a) / \neg Q(a), \top / \neg R(a)\} . \\
& -\Gamma_{\bar{T}}\left(E_{4}\right)=\operatorname{Cn}(\bar{W} \cup\{\neg R(a), \neg Q(a)\})=E_{6} \neq E_{4} .
\end{aligned}
$$

- $E_{5}=\operatorname{Cn}(\bar{W} \cup\{\neg Q(a), \neg P(a)\})$ (inconsistent):

$$
\begin{aligned}
& -\Delta_{E_{5}}=\emptyset \\
& -\Gamma_{\bar{T}}\left(E_{5}\right)=\operatorname{Cn}(\bar{W})=E_{1} \neq E_{5}
\end{aligned}
$$

- $E_{6}=C n(\bar{W} \cup\{\neg Q(a), \neg R(a)\}):$
$-\Delta_{E_{6}}=\{P(a) / \neg Q(a), T / \neg R(a)\}=\Delta_{E_{2}}$.
$-\Gamma_{\bar{T}}\left(E_{6}\right)=\Gamma_{\bar{T}}\left(E_{2}\right)=E_{6} . \Longrightarrow$ Extension!


## Solution to Exercise 2.3 (ctd.)

$$
\begin{aligned}
\bar{W} & =\{\forall x(P(x) \vee Q(x)), R(a) \vee P(a), \forall x(\neg Q(x) \rightarrow \neg R(x))\}, \\
\bar{\Delta} & =\left\{\frac{P(a): \neg Q(a)}{\neg Q(a)}, \frac{Q(a): \neg P(a)}{\neg P(a)}, \frac{\top: \neg R(a)}{\neg R(a)}\right\} .
\end{aligned}
$$

- $E_{7}=C n(\bar{W} \cup\{\neg P(a), \neg R(a)\})$ (inconsistent):

$$
\begin{aligned}
& -\Delta_{E_{7}}=\emptyset \\
& -\Gamma_{\bar{T}}\left(E_{7}\right)=\operatorname{Cn}(\bar{W})=E_{1} \neq E_{7}
\end{aligned}
$$

- $E_{8}=C n(\bar{W} \cup\{\neg Q(a), \neg P(a), \neg R(a)\})$ (inconsistent):

$$
\begin{aligned}
& -\Delta_{E_{8}}=\emptyset \\
& -\Gamma_{\bar{T}}\left(E_{8}\right)=C n(\bar{W})=E_{1} \neq E_{8}
\end{aligned}
$$

$E_{6} \cap \mathcal{L}_{T}$ is the single extension of $T$ !

## Exercise 2.4

$>$ Show that any closed normal default theory $(W, \Delta)$ has an extension.

## Solution:

> Let $T=(W, \Delta)$ be a closed normal default theory.

- If $W$ is inconsistent, then $\operatorname{Cn}(W)$ is an extension as it satisfies all properties of an extension.
- Assume now that $W$ is consistent. We define a sequence $\left(E_{i}\right)_{i \geq 0}$ as follows:

$$
\begin{aligned}
E_{0} & =W \\
E_{i+1} & =\operatorname{Cn}\left(E_{i}\right) \cup T_{i},
\end{aligned}
$$

where $T_{i}$ is a maximal set of closed formulas satisfying the following conditions:

1. $E_{i} \cup T_{i}$ is consistent;
2. if $B \in T_{i}$, then there is a default $(A: B / B) \in \Delta$ such that $E_{i} \models A$.

## Solution to Exercise 2.4 (ctd.)

$>$ Let $E=\bigcup_{i \geq 0} E_{i}$.
$>$ Claim: $E$ is an extension of $T$.
> Proof of claim: Let $H_{i}=\left\{B \mid(A: B / B) \in \Delta, E_{i} \models A, \neg B \notin E\right\}$. We show that $T_{i}=H_{i}$, for all $i$. This proves the claim by the semirecursive characterisation of extensions (cf. the hint in Ex. Sheet 2).

1. We first show that $T_{i} \subseteq H_{i}$.

- Let $B \in T_{i}$. Then, by definition of $T_{i}$, there is some default $(A: B / B) \in \Delta$ such that $E_{i} \models A$.
- Suppose $\neg B \in E$. Since $E=\bigcup_{i \geq 0} E_{i}$, there is some $k \geq 0$ such that $\neg B \in E_{k}$.
- Since $B \in T_{i}$ and $T_{i} \subseteq E_{i+1}$, we have that $B \in E_{i+1}$.
- Let $m=\max (i+1, k)$. Since $E_{I} \subseteq E_{I+1}$, for all $I \geq 0$, it follows that $B \in E_{m}$ and $\neg B \in E_{m}$. $\Longrightarrow E_{m}$ is inconsistent. Contradiction.
- Therefore, $\neg B \notin E$ must hold. We obtain $B \in H_{i}$.
$\Longrightarrow$ This proves that $T_{i} \subseteq H_{i}$.


## Solution to Exercise 2.4 (ctd.)

2. Suppose that $T_{i} \subset H_{i}$ holds. Then, there is some $B \in H_{i}$ such that $B \notin T_{i}$.

- By the maximality of $T_{i}$, we have that $E_{i} \cup T_{i} \cup\{B\}$ is inconsistent. Since $E_{i} \cup T_{i} \subseteq E_{i+1} \subseteq E$ holds, $E \cup\{B\}$ is also inconsistent.
- It follows that $E \models \neg B$. Furthermore, it is easy to see that $E=C n(E)$. Hence, $\neg B \in E$.
- But $B \in H_{i}$, and so $\neg B \notin E$ must hold, by definition of $H_{i}$. Contradiction.
$\Longrightarrow T_{i} \subset H_{i}$ cannot hold and we obtain $T_{i}=H_{i}$.


## Exercise 2.5

> Consider the theory

$$
\begin{aligned}
W:=\{ & \forall x \forall y((P(x) \vee Q(y)) \rightarrow R(x, y)), \\
& \forall x((S(x) \wedge Q(x)) \rightarrow P(x)), \\
& Q(b), P(a), S(b), \\
& \forall x(\neg S(x) \rightarrow \neg R(x, x))\} .
\end{aligned}
$$

> Construct a closed, normal default theory $T:=(W, \Delta)$ such that there exists an extension $E$ of $T$ with $\operatorname{CWA}(W)=E$.

## Solution to Exercise 2.5

$>$ We define the closed normal default theory $T=(W, \Delta)$ with

$$
\begin{aligned}
\Delta:= & \left\{\frac{\top: \neg P(a)}{\neg P(a)}, \frac{\top: \neg Q(a)}{\neg Q(a)}, \frac{\top: \neg S(a)}{\neg S(a)}, \frac{\top: \neg R(a, a)}{\neg R(a, a)}\right\} \cup \\
& \left\{\frac{\top: \neg P(b)}{\neg P(b)}, \frac{\top: \neg Q(b)}{\neg Q(b)}, \frac{\top: \neg S(b)}{\neg S(b)}, \frac{\top: \neg R(b, b)}{\neg R(b, b)}\right\} \cup \\
& \left\{\frac{\top: \neg R(a, b)}{\neg R(a, b)}, \frac{\top: \neg R(b, a)}{\neg R(b, a)}\right\} .
\end{aligned}
$$

$>$ Now, theory $W$ is actually theory $T_{2}$ from Exercise 2.1. $\Longrightarrow \operatorname{CWA}(W)=C n(W \cup\{\neg Q(a)\})$.
> To show that CWA $(W)$ is an extension of $T$, we only need to construct the reduct $\Delta_{\mathrm{CWA}(W)}$ and show that $\Gamma_{T}(\operatorname{CWA}(W))=\operatorname{CWA}(W)$.
> From what we know from Exercise 2.1, it follows that $\Delta_{\mathrm{CWA}(W)}=\{\top / \neg Q(a)\}$, so $\Gamma_{T}(\mathrm{CWA}(W))=\operatorname{CWA}(W)$ indeed holds.

## Exercise 2.6

$>$ Extend the default theory $T=(W, \Delta)$, where

$$
\begin{aligned}
W & =\{\forall x(P(x) \vee Q(x)), R(a), \forall x(R(x) \rightarrow P(x))\} \text { and } \\
\Delta & =\left\{\frac{P(x): \neg Q(x)}{\neg Q(x)}, \frac{Q(x): \neg P(x)}{\neg P(x)}\right\},
\end{aligned}
$$

such that it has no extensions anymore.
> For achieving this, use only predicates occurring in $T$.
> Show then that your enlarged theory has no extensions, i.e., consider possible extensions, compute the classical reducts and fixed points, and show that no candidate set is an extension.

## Solution to Exercise 2.6

- Since $T$ is normal, we have to add a non-normal default.
> We define $T^{\prime}=\left(W, \Delta^{\prime}\right)$, where

$$
\Delta^{\prime}=\left\{\frac{P(x): \neg Q(x)}{\neg Q(x)}, \frac{Q(x): \neg P(x)}{\neg P(x)}, \frac{T: \neg Q(x)}{\neg P(x)}\right\} .
$$

> To determine the extensions of $T^{\prime}$, we first built its closure $\overline{T^{\prime}}=\left(\bar{W}, \overline{\Delta^{\prime}}\right)$, where $\bar{W}=W$ and

$$
\overline{\Delta^{\prime}}=\left\{\frac{P(a): \neg Q(a)}{\neg Q(a)}, \frac{Q(a): \neg P(a)}{\neg P(a)}, \frac{T: \neg Q(a)}{\neg P(a)}\right\} .
$$

> Candidates for being extensions of $\overline{T^{\prime}}$ :

- $E_{1}=C n(W)$;
- $E_{2}=C n(W \cup\{\neg Q(a)\}) ;$
- $E_{3}=C n(W \cup\{\neg P(a)\})$ (inconsistent);
- $E_{4}=C n(W \cup\{\neg Q(a), \neg P(a)\})$ (inconsistent). $\Longrightarrow E_{3}=E_{4}$.


## Solution to Exercise 2.6 (ctd.)

$$
\begin{aligned}
\bar{W}=W & =\{\forall x(P(x) \vee Q(x)), R(a), \forall x(R(x) \rightarrow P(x))\}, \\
\bar{\Delta} & =\left\{\frac{P(a): \neg Q(a)}{\neg Q(a)}, \frac{Q(a): \neg P(a)}{\neg P(a)}, \frac{T: \neg Q(a)}{\neg P(a)}\right\} .
\end{aligned}
$$

$>$ We determine the classical reducts $\Delta_{E_{i}}$ and $\Gamma_{\bar{T}}\left(E_{i}\right)=C n^{\Delta_{E_{i}}}(\bar{W})$ for $i=1, \ldots, 4$ :

- $E_{1}=C n(W)$ :

$$
\begin{aligned}
& -\Delta_{E_{1}}=\{P(a) / \neg Q(a), \top / \neg P(a)\} . \\
& -\Gamma_{\bar{T}}\left(E_{1}\right)=\operatorname{Cn}(W \cup\{\neg P(a), \neg Q(a)\})=E_{4} \neq E_{1} .
\end{aligned}
$$

- $E_{2}=C n(W \cup\{\neg Q(a)\}):$

$$
\begin{aligned}
& -\Delta_{E_{2}}=\{P(a) / \neg Q(a), \top / \neg P(a)\}=\Delta_{E_{1}} . \\
& -\Gamma_{\bar{T}}\left(E_{2}\right)=\Gamma_{\bar{T}}\left(E_{1}\right)=E_{4} \neq E_{2} .
\end{aligned}
$$

## Solution to Exercise 2.6 (ctd.)

$$
\begin{aligned}
\bar{W}=W & =\{\forall x(P(x) \vee Q(x)), R(a), \forall x(R(x) \rightarrow P(x))\}, \\
\bar{\Delta} & =\left\{\frac{P(a): \neg Q(a)}{\neg Q(a)}, \frac{Q(a): \neg P(a)}{\neg P(a)}, \frac{T: \neg Q(a)}{\neg P(a)}\right\} .
\end{aligned}
$$

- $E_{3}=C n(W \cup\{\neg P(a)\})$ (inconsistent):

$$
\begin{aligned}
& -\Delta_{E_{3}}=\emptyset \\
& -\Gamma_{\bar{T}}\left(E_{3}\right)=C n(W)=E_{1} \neq E_{3} .
\end{aligned}
$$

- $E_{4}=C n(W \cup\{\neg Q(a), \neg P(a)\})=E_{3}$ (inconsistent):

$$
-\Delta_{E_{4}}=\Delta_{E_{3}}=\emptyset
$$

$$
-\Gamma_{\bar{T}}\left(E_{4}\right)=\Gamma_{\bar{T}}\left(E_{3}\right)=E_{1} \neq E_{4}
$$

No candidate set satisfies the fixed-point condition.
$\Longrightarrow \bar{T}$, and hence also $T$, has no extension.

## Exercise 2.7

> Consider the following information:

1. British People usually like tea.
2. British People who drink tea usually do not drink coffee.
3. Scientists usually prefer to drink coffee.
4. Lisa is a British scientist.

- Formalise the given information in terms of an open default theory.
$>$ To this end, use
- $B(x)$ for " $x$ is British",
- $T(x)$ for " $x$ drinks tea",
- $C(x)$ for " $x$ drinks coffee",
- $S(x)$ for " $x$ is a scientist",
- the constant symbol / for "Lisa".


## Exercise 2.7 (ctd.)

- Furthermore, determine the closure $\bar{T}=\left(W^{\prime}, \Delta^{\prime}\right)$ of $T$ and decide which of the following extension candidates of $\bar{T}$ are actually extensions of $\bar{T}$ :
- $E_{2}=C n\left(W^{\prime} \cup\{T(I)\}\right)$,
- $E_{3}=C n\left(W^{\prime} \cup\{\neg C(I)\}\right)$,
- $E_{5}=C n\left(W^{\prime} \cup\{T(I), \neg C(I)\}\right)$,
- $E_{6}=C n\left(W^{\prime} \cup\{T(I), C(I)\}\right)$.
> Lastly, determine how many unique extension candidates of $\bar{T}$ there would have been.


## Solution to Exercise 2.7

$>$ We formalise the information as the theory $T=(W, \Delta)$, where

$$
\begin{aligned}
W & =\{B(I), S(I)\} \\
\Delta & =\left\{\frac{B(x): T(x)}{T(x)}, \frac{B(x) \wedge T(x): \neg C(x)}{\neg C(x)}, \frac{S(x): C(x)}{C(x)}\right\} .
\end{aligned}
$$

> Next, we determine the closure $\bar{T}=(\bar{W}, \bar{\Delta})=\left(W^{\prime}, \Delta^{\prime}\right)$ of $T$ :

$$
\begin{aligned}
W^{\prime} & =\{B(I), S(I)\}, \\
\Delta^{\prime} & =\left\{\frac{B(I): T(I)}{T(I)}, \frac{B(I) \wedge T(I): \neg C(I)}{\neg C(I)}, \frac{S(I): C(I)}{C(I)}\right\} .
\end{aligned}
$$

## Solution to Exercise 2.7 (ctd.)

$$
\begin{aligned}
& \bar{W}=W^{\prime}=W \\
& \bar{\Delta}=\{B(I), S(I)\}, \\
&=\Delta^{\prime}=\left\{\frac{B(I): T(I)}{T(I)}, \frac{B(I) \wedge T(I): \neg C(I)}{\neg C(I)}, \frac{S(I): C(I)}{C(I)}\right\} .
\end{aligned}
$$

> Candidates for being extensions of $\bar{T}$ are:

- $E_{1}=C n\left(W^{\prime}\right)$;
- $E_{2}=\operatorname{Cn}\left(W^{\prime} \cup\{T(I)\}\right)$;
- $E_{3}=C n\left(W^{\prime} \cup\{\neg C(I)\}\right) ;$
- $E_{4}=C n\left(W^{\prime} \cup\{C(I)\}\right)$;
- $E_{5}=C n\left(W^{\prime} \cup\{T(I), \neg C(I)\}\right)$;
- $E_{6}=C n\left(W^{\prime} \cup\{T(I), C(I)\}\right) ;$
- $E_{7}=C n\left(W^{\prime} \cup\{\neg C(I), C(I)\}\right)$;
- $E_{8}=C n\left(W^{\prime} \cup\{T(I), \neg C(I), C(I)\}\right)$.
$>E_{7}$ and $E_{8}$ are inconsistent $\Longrightarrow E_{7}=E_{8}$.
$\Longrightarrow$ We have only 7 unique candidates.


## Solution to Exercise 2.7 (ctd.)

$$
\begin{aligned}
\bar{W}=W^{\prime} & =W \\
\bar{\Delta} & =\{B(I), S(I)\}, \\
\Delta^{\prime} & =\left\{\frac{B(I): T(I)}{T(I)}, \frac{B(I) \wedge T(I): \neg C(I)}{\neg C(I)}, \frac{S(I): C(I)}{C(I)}\right\} .
\end{aligned}
$$

$>$ Our task is to check which of $E_{2}, E_{3}, E_{5}$, and $E_{6}$ are extensions.
$>$ To this end, we determine the classical reducts $\Delta_{E_{i}}$ and $\Gamma_{\bar{T}}\left(E_{i}\right)=C n^{\Delta_{E_{i}}}(\bar{W})$ for $i=2,3,5,6$ :

- $E_{2}=C n\left(W^{\prime} \cup\{T(I)\}\right):$

$$
\begin{aligned}
& -\Delta_{E_{2}}=\{B(I) / T(I), B(I) \wedge T(I) / \neg C(I), S(I) / C(I)\} . \\
& -\Gamma_{\bar{T}}\left(E_{2}\right)=C n\left(W^{\prime} \cup\{T(I), \neg C(I), C(I)\}\right)=E_{8} \neq E_{2} .
\end{aligned}
$$

- $E_{3}=C n\left(W^{\prime} \cup\{\neg C(I)\}\right):$

$$
\begin{aligned}
& -\Delta_{E_{3}}=\{B(I) / T(I), B(I) \wedge T(I) / \neg C(I)\} . \\
& -\Gamma_{\bar{T}}\left(E_{3}\right)=C n\left(W^{\prime} \cup\{T(I), \neg C(I)\}\right)=E_{5} \neq E_{3} .
\end{aligned}
$$

## Solution to Exercise 2.7 (ctd.)

$$
\begin{aligned}
\bar{W}=W^{\prime} & =W \\
\bar{\Delta} & =\{B(I), S(I)\}, \\
\Delta^{\prime} & =\left\{\frac{B(I): T(I)}{T(I)}, \frac{B(I) \wedge T(I): \neg C(I)}{\neg C(I)}, \frac{S(I): C(I)}{C(I)}\right\} .
\end{aligned}
$$

- $E_{5}=C n\left(W^{\prime} \cup\{T(I), \neg C(I)\}\right)$ :

$$
\begin{aligned}
& -\Delta_{E_{5}}=\{B(I) / T(I), B(I) \wedge T(I) / \neg C(I)\}=\Delta_{E_{3}} . \\
& -\Gamma_{\bar{T}}\left(E_{5}\right)=\Gamma_{\bar{T}}\left(E_{3}\right)=E_{5} \Longrightarrow \text { Extension! }
\end{aligned}
$$

- $E_{6}=C n\left(W^{\prime} \cup\{T(I), C(I)\}\right):$

$$
\begin{aligned}
& -\Delta_{E_{6}}=\{B(I) / T(I), S(I) / C(I)\} \\
& -\Gamma_{\bar{T}}\left(E_{6}\right)=\operatorname{Cn}\left(W^{\prime} \cup\{T(I), C(I)\}\right)=E_{6} \Longrightarrow \text { Extension! }
\end{aligned}
$$

$\Leftrightarrow E_{2}$ and $E_{3}$ are not extensions of $T$; $E_{5}$ and $E_{6}$ are extensions of $T$.

