6. Übung Wahrscheinlichkeit und stochastische Prozesse WS 21

1. $X_1, \dots, X_n$ ist eine Stichprobe einer Verteilung mit der Dichte (für $\theta > 0$)

$$f(x, \theta) = \theta x^{\theta - 1}[0 \le x \le 1].$$

Bestimmen Sie den Momentenschätzer. und den Maximum Likelihood Schätzer für θ.

$\theta \neq -1$!

$$E(X) = \int_{-\infty}^{\infty} x f_x(x)\, dx$$

$$\Rightarrow E(X) = \int_0^1 x \cdot \theta x^{\theta-1}\, dx = \theta \cdot \int_0^1 x^{\theta} dx = \theta \cdot \left[\frac{x^{\theta+1}}{\theta+1}\right]_0^1 =$$

$$= \theta \cdot \frac{1}{\theta+1} = \frac{\theta}{\theta+1}$$

$E_\theta(X) = m(\theta)$ & wir setzen $m(\hat\theta_n) = \overline{X}_n$

Wenn Fkt. $m(\theta)$ stetig umkehrbar ist, gilt: $\hat\theta_n = m^{-1}(\overline{X}_n)$ & $\hat\theta_n$ konsistent.

$$E_\theta(X) = \frac{\theta}{\theta+1} \Rightarrow \overline{X}_n = \frac{\hat\theta_n}{\hat\theta_n + 1} \Rightarrow \text{umformen nach } \hat\theta_n$$

$$\overline{X}_n \cdot \hat\theta_n + \overline{X}_n = \hat\theta_n$$

$$\overline{X}_n = \hat\theta_n - \overline{X}_n \cdot \hat\theta_n$$

$$\overline{X}_n = \hat\theta_n (1 - \overline{X}_n) \Rightarrow \hat\theta_n = \frac{\overline{X}_n}{1 - \overline{X}_n} \quad \text{Momentschätzer}$$

Likelihood funktion: $L(X_1, \dots, X_n; \theta) = \prod_{i=1}^{n} f_\theta(X_i)$

$$\Rightarrow L(X_1, \dots, X_n; \theta) = \prod_{i=1}^{n} \theta \cdot X_i^{\theta-1} = \theta^n \cdot \prod_{i=1}^{n} X_i^{\theta-1} \Rightarrow \text{umformen nach } \theta$$

Logarithmieren: $\ln(L) = \ln\left(\theta^n \cdot \left(\prod_{i=1}^{n} X_i^{\theta-1}\right)\right)$

$$= \ln(L) = \ln(\theta^n) + \underbrace{\ln\left(\prod_{i=1}^{n} X_i^{\theta-1}\right)}$$

$$= \ln(X_1^{\theta-1} \cdot X_2^{\theta-1} \cdot \ldots \cdot X_n^{\theta-1}) = \ln(X_1^{\theta-1}) + \ln(X_2^{\theta-1}) + \ldots + \ln(X_n^{\theta-1}) =$$

$$= (\theta-1) \cdot \ln(X_1) + (\theta-1) \cdot \ln(X_2) + \ldots + (\theta-1) \cdot \ln(X_n) =$$

$$= (\theta-1)(\ln(X_1) + \ln(X_2) + \ldots + \ln(X_n)) = (\theta-1) \cdot \sum_{i=1}^{n} \ln(X_i)$$

$$= \ln(L) = n \cdot \ln(\theta) + (\theta-1) \cdot \sum_{i=1}^{n} \ln(X_i)$$

Maximum-Likelihood-Schätzer ist der Wert von θ, der die Likelihoodfunktion maximiert
$\Rightarrow$ nach θ ableiten und $=0$ setzen

$$\frac{\partial}{\partial\theta}\ln(L) = n\cdot\frac{1}{\theta} + 1\cdot\sum_{i=1}^{n}\ln(x_i) \stackrel{!}{=} 0$$

$$\Rightarrow \quad \frac{n}{\theta} + \sum_{i=1}^{n}\ln(x_i) = 0$$

$$\frac{n}{\theta} = -\sum_{i=1}^{n}\ln(x_i)$$

$$n = \theta\cdot\left(-\sum_{i=1}^{n}\ln(x_i)\right) \Rightarrow \hat{\theta} = -\frac{n}{\sum_{i=1}^{n}\ln(x_i)}$$

ML-Schätzer (Max. Likelihood)

2. Bestimmen Sie den Maximum Likelihood Schätzer für den Parameter λ einer Poissonverteilung und zeigen Sie, dass er effizient ist.

$$P_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

k=x in dem Falle

Likelihood funktion: $L(X_1, \dots, X_n; \theta) = \prod_{i=1}^{n} f_\theta(X_i)$

$$\Rightarrow L(X_1,\dots,X_n;\theta) = \prod_{i=1}^{n} \frac{\lambda^{X_i}}{X_i!} e^{-\lambda} = e^{-\lambda n} \cdot \prod_{i=1}^{n} \frac{\lambda^{X_i}}{X_i!} \Rightarrow \text{umformen nach } \lambda$$

Logarithmieren: $\ln(L) = \ln\left(e^{-\lambda n} \cdot \prod_{i=1}^{n} \frac{\lambda^{X_i}}{X_i!}\right)$

$$\Rightarrow \ln(L) = \ln(e^{-\lambda n}) + \underbrace{\ln\left(\prod_{i=1}^{n} \frac{\lambda^{X_i}}{X_i!}\right)}$$

$$= \ln\left(\frac{\lambda^{X_1}}{X_1!} \cdot \frac{\lambda^{X_2}}{X_2!} \cdot \dots \cdot \frac{\lambda^{X_n}}{X_n!}\right) = \ln\left(\frac{\lambda^{X_1}}{X_1!}\right) + \ln\left(\frac{\lambda^{X_2}}{X_2!}\right) + \dots + \ln\left(\frac{\lambda^{X_n}}{X_n!}\right) =$$

$$= \ln(\lambda^{X_1}) - \ln(X_1!) + \ln(\lambda^{X_2}) - \ln(X_2!) + \dots + \ln(\lambda^{X_n}) - \ln(X_n!) =$$

$$= \ln(\lambda^{X_1}) + \ln(\lambda^{X_2}) + \dots + \ln(\lambda^{X_n}) - \sum_{i=1}^{n} \ln(X_i!) =$$

$$= X_1 \cdot \ln(\lambda) + X_2 \cdot \ln(\lambda) + \dots + X_n \cdot \ln(\lambda) - \sum_{i=1}^{n} \ln(X_i!) =$$

$$= \ln(\lambda) \cdot \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \ln(X_i!)$$

$$\Rightarrow \ln(L) = -\lambda n \cdot \overbrace{\ln(e)}^{=1} + \ln(\lambda) \cdot \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \ln(X_i!)$$

$\Rightarrow$ nach λ ableiten und $=0$ setzen

$$\frac{\partial}{\partial\lambda} \ln(L) = -n + \frac{1}{\lambda} \cdot \sum_{i=1}^{n} X_i \overset{!}{=} 0$$

arith. Mittel

$$\Rightarrow \frac{1}{\lambda} \cdot \sum_{i=1}^{n} X_i = n \quad \Rightarrow \quad \hat{\lambda}_n = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}_n$$

damit gilt: $E(\hat{\lambda}_n) = \bar{X}_n = \lambda$ und $V(\hat{\lambda}_n) = \frac{\bar{X}_n}{n} = \frac{\lambda}{n}$

Cramér-Rao: Wenn f_θ 2 mal nach θ diff.bar & Regularitätsvoraussetzungen erfüllt, gilt für jeden erwartungstreuen Schätzer:

$$V(\hat{\theta}_n) \geq \frac{1}{I_n(\theta)} = \frac{1}{nI(\theta)}$$

dabei gilt: $I_n(\theta) = -E_\theta\left(\frac{\partial^2}{\partial\theta^2} \ln(L)\right)$

und

$$I(\theta) = -E_\theta\left(\frac{\partial^2}{\partial\theta^2} \ln(f_\theta(X))\right)$$

$$\frac{\partial^2}{\partial \lambda^2} \ln(L) = -\frac{1}{\lambda^2} \cdot \sum_{i=1}^{n} X_i$$

$$\Rightarrow I_n(\lambda) = -E_\lambda\left(\frac{\partial^2}{\partial \lambda^2} \ln(L)\right) = \frac{1}{\lambda^2} \cdot \sum_{i=1}^{n} E_\lambda(X_i) = \frac{1}{\lambda^2} \cdot n \cdot \lambda = \frac{n}{\lambda}$$

Cramér-Rao: $\frac{\lambda}{n} \geq \frac{1}{\frac{n}{\lambda}} = \frac{\lambda}{n}$ ✓ $\hat{\lambda}_n$ ist effizient

3. Bestimmen Sie den Maximum Likelihood Schätzer für den Parameter einer Exponentialverteilung mit der Dichte

$$f(x, \theta) = \frac{1}{\theta} e^{-x/\theta} [x \geq 0]$$

und zeigen Sie, dass er effizient ist.

Likelihood funktion: $L(X_1, \dots, X_n; \theta) = \prod_{i=1}^{n} f_\theta(X_i)$

$$\Rightarrow L(X_1, \dots, X_n; \theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{X_i}{\theta}} = \frac{1}{\theta^n} \cdot \prod_{i=1}^{n} e^{-\frac{X_i}{\theta}}$$

Logarithmieren: $\ln(L) = \ln\left(\frac{1}{\theta^n} \cdot \prod_{i=1}^{n} e^{-\frac{X_i}{\theta}}\right)$

$$\ln(L) = \ln\left(\frac{1}{\theta^n}\right) + \ln\left(\prod_{i=1}^{n} e^{-\frac{X_i}{\theta}}\right)$$

$$\ln(L) = \ln(1) - n \cdot \ln(\theta) + \sum_{i=1}^{n} \ln\left(e^{-\frac{X_i}{\theta}}\right)$$

$$\ln(L) = \ln(1) - n \cdot \ln(\theta) + \sum_{i=1}^{n} \left(-\frac{X_i}{\theta}\right)$$

$$\ln(L) = \ln(1) - n \cdot \ln(\theta) - \frac{1}{\theta} \cdot \sum_{i=1}^{n} X_i$$

nach θ ableiten und $=0$ setzen:

$$\frac{\partial}{\partial \theta} \ln(L) = -\frac{n}{\theta} + \frac{1}{\theta^2} \cdot \sum_{i=1}^{n} X_i \overset{!}{=} 0$$

arith. Mittel

$$\frac{n}{\theta} = \frac{1}{\theta^2} \cdot \sum_{i=1}^{n} X_i \Rightarrow \theta n = \sum_{i=1}^{n} X_i \Rightarrow \hat{\theta}_n = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}_n$$

$$E(\hat{\theta}_n) = \frac{\sum_{i=1}^{n} E(X_i)}{n} = \frac{\cancel{n} \cdot \theta}{\cancel{n}} = \theta$$

→ erwartungstreu

$$V(\hat{\theta}_n) = \frac{1}{n^2} \sum_{i=1}^{n} \underbrace{Var(X_i)}_{\theta^2} = \frac{\theta^2}{n}$$

Cramér-Rao: Wenn f_θ 2 mal nach θ diffbar & Regularitätsvoraussetzungen erfüllt, gilt für jeden erwartungstreuen Schätzer:

$$V(\hat{\theta}_n) \geq \frac{1}{I_n(\theta)} = \frac{1}{n I(\theta)}$$

dabei gilt: $I_n(\theta) = -E_\theta\left(\frac{\partial^2}{\partial \theta^2} \ln(L)\right)$

und $I(\theta) = -E_\theta\left(\frac{\partial^2}{\partial \theta^2} \ln(f_\theta(x))\right)$

$$I_n(\theta) = -E\left(\frac{n}{\theta^2} - \frac{2}{\theta^3} \cdot \sum_{i=1}^{n} X_i\right) = -\left(\frac{n}{\theta^2} - \frac{2}{\theta^3} \cdot n \cdot \theta\right)$$

$$= -\frac{n}{\theta^2} + \frac{2n}{\theta^2} = \frac{n}{\theta^2}$$

$$V(\hat{\theta}_n) = \frac{\theta^2}{n} \geq \frac{1}{I_n(\theta)} = \frac{\theta^2}{n} \rightarrow \text{effizient}$$

4. $(X_1, \ldots, X_n)$ sei eine Stichprobe einer Gleichverteilung auf $[0, \theta]$. Der Maximum Likelihood Schätzer für θ ist, wie in der Vorlesung gezeigt wurde, $T = \max(X_1, \ldots, X_n)$, und der Schätzer $\hat{\theta}_n = \frac{n+1}{n} T$ ist effizient. Bestimmen Sie seine Varianz und vergleichen Sie sie mit der des Momentenschätzers.

Momentenschätzer:

$$\hat{\theta}_n^m = 2 \cdot \bar{X}_n$$

$$V(\hat{\theta}_n^m) = 4 \cdot \frac{\sum_{i=1}^n V(X_i)}{n^2} = \frac{\frac{4\theta^2}{12}}{n} = \frac{\frac{\theta^2}{3}}{n} = \frac{\theta^2}{3n}$$

$\hat{\theta}_n = \frac{n+1}{n} T$ ist effizient

$$V(\hat{\theta}_n) = \frac{(n+1)^2}{n^2} \cdot \boxed{V(T)}$$

$T = \max(X_1, \ldots, X_n)$

Dichte für $T \rightarrow f_x = \frac{n x^{n-1}}{\theta^n}$ (gegeben; $(0 \leq x \leq \theta)$)

$$E(T) = \int_0^\theta x \frac{n \cdot x^{n-1}}{\theta^n} = \frac{n \cdot x^{n+1}}{\theta^n \cdot (n+1)} \Bigg|_0^\theta = \frac{n \cdot \theta}{n+1}$$

$$E(T^2) = \int_0^\theta x^2 \cdot \frac{n \cdot x^{n-1}}{\theta^n} = \frac{n \cdot x^{n+2}}{\theta^n \cdot (n+2)} \Bigg|_0^\theta = \frac{n \cdot \theta^2}{n+2}$$

$$V(T) = E(T^2) - E(T)^2 = \frac{n \cdot \theta^2}{n+2} - \frac{n^2 \theta^2}{(n+1)^2} = \frac{n \cdot (n+1)^2 \cdot \theta^2 - n^2 \theta^2 (n+2)}{(n+2)(n+1)^2}$$

$$= \frac{(n^3 + 2n^2 + n) \cdot \theta^2 - (n^3 + 2n^2)\theta^2}{(n+2)(n+1)^2} = \frac{n \cdot \theta^2}{(n+2)(n+1)^2} = V(T)$$

$$\rightarrow V(\hat{\theta}_n) = \frac{(n+1)^2}{n^2} \cdot \boxed{V(T)} = \frac{(n+1)^2 \cdot n \cdot \theta^2}{n^2 \cdot (n+2)(n+1)^2} = \frac{\theta^2}{n \cdot (n+2)}$$

5. In einer Stichprobe von 100 Punschtrinkern hatten 20 alkoholfreien Punsch. Bestimmen Sie ein 95%-Konfidenzintervall für den Anteil des alkoholfreien Punsches an der verkauften Menge.

$n = 100$

$\hat{p} = \frac{20}{100} = 0{,}2$

$$\boxed{\hat{p} \pm z_{\frac{1+\gamma}{2}} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

$z_{0{,}975} = 1{,}96$

$0{,}2 \pm 1.96 \cdot \sqrt{\frac{0{,}16}{100}} \rightarrow [0{,}1216, 0{,}2784]$ 95% Int. für μ

Die Verteilungsfunktion der Standardnormalverteilung:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

	0	1	2	3	4	5	6	7	8	9
0.0	.500	.504	.508	.512	.516	.520	.524	.528	.532	.536
0.1	.540	.544	.548	.552	.556	.560	.564	.567	.571	.575
0.2	.579	.583	.587	.591	.595	.599	.603	.606	.610	.614
0.3	.618	.622	.626	.629	.633	.637	.641	.644	.648	.652
0.4	.655	.659	.663	.666	.670	.674	.677	.681	.684	.688
0.5	.691	.695	.698	.702	.705	.709	.712	.716	.719	.722
0.6	.726	.729	.732	.736	.739	.742	.745	.749	.752	.755
0.7	.758	.761	.764	.767	.770	.773	.776	.779	.782	.785
0.8	.788	.791	.794	.797	.800	.802	.805	.808	.811	.813
0.9	.816	.819	.821	.824	.826	.829	.831	.834	.836	.839
1.0	.841	.844	.846	.848	.851	.853	.855	.858	.860	.862
1.1	.864	.867	.869	.871	.873	.875	.877	.879	.881	.883
1.2	.885	.887	.889	.891	.893	.894	.896	.898	.900	.901
1.3	.903	.905	.907	.908	.910	.911	.913	.915	.916	.918
1.4	.919	.921	.922	.924	.925	.926	.928	.929	.931	.932
1.5	.933	.934	.936	.937	.938	.939	.941	.942	.943	.944
1.6	.945	.946	.947	.948	.949	.951	.952	.953	.954	.954
1.7	.955	.956	.957	.958	.959	.960	.961	.962	.962	.963
1.8	.964	.965	.966	.966	.967	.968	.969	.969	.970	.971
1.9	.971	.972	.973	.973	.974	.974	.975	.976	.976	.977
2.0	.977	.978	.978	.979	.979	.980	.980	.981	.981	.982
2.1	.982	.983	.983	.983	.984	.984	.985	.985	.985	.986
2.2	.986	.986	.987	.987	.987	.988	.988	.988	.989	.989
2.3	.989	.990	.990	.990	.990	.991	.991	.991	.991	.992
2.4	.992	.992	.992	.992	.993	.993	.993	.993	.993	.994
2.5	.994	.994	.994	.994	.994	.995	.995	.995	.995	.995
2.6	.995	.995	.996	.996	.996	.996	.996	.996	.996	.996
2.7	.997	.997	.997	.997	.997	.997	.997	.997	.997	.997
2.8	.997	.998	.998	.998	.998	.998	.998	.998	.998	.998
2.9	.998	.998	.998	.998	.998	.998	.998	.999	.999	.999

6. Bestimmen Sie mithilfe des zentralen Grenzwertsatzes ein approximatives Konfidenzintervall für den Parameter einer Exponentialverteilung mit der Dichte

$$f(x,\theta) = \frac{1}{\theta}e^{-x/\theta}[x \geq 0].$$

Wie in 3) gezeigt ist der Schätzer $\boxed{\hat\theta_n = \bar X_n}$ effizient, laut dem zentralen Grenzwertsatz ist die Stichprobenverteilung normalverteilt

$$\underbrace{\bar X_n}_{\hat\theta_n} \pm z_{\frac{1+\gamma}{2}} \cdot \sqrt{V(\hat\theta_n)} \overset{*}{=} \underline{\bar X_n \pm z_{\frac{1+\gamma}{2}} \cdot \frac{\bar X_n}{\sqrt n}}$$

$$* \quad \sqrt{V(\hat\theta_n)} = \sqrt{\frac{\theta^2}{n}} = \frac{\theta}{\sqrt n} \leftrightarrow \frac{\bar X_n}{\sqrt n}$$

7. Bestimmen Sie für die folgende Stichprobe einer Normalverteilung
0.7 1.3 1.2 1.5 1.8 0.9 1.1 1.4 1.9 1.7
95%-Konfidenzintervalle für μ und σ^2.

95 %

$$n = 10 \qquad \bar X_n = \frac{\sum_{i=1}^{n} x_i}{n} = 1.35 \qquad S_n^2 = \frac{\sum_{i=1}^{n}(x_i - \bar X_n)^2}{n-1} = 0{,}151\dot 6$$

$$t_{0,975} = 2.262$$

Quantile $t_{n;p}$ der t-Verteilung mit n Freiheitsgraden:

n	.9	.95	.975	.99	.995	n	.9	.95	.975	.99	.995
1	3.078	6.314	12.706	31.821	63.675	26	1.316	1.706	2.056	2.479	2.779
2	1.886	2.920	4.303	6.965	9.725	27	1.314	1.703	2.052	2.473	2.467
3	1.638	2.353	3.183	4.541	5.841	28	1.313	1.701	2.048	2.467	2.763
4	1.533	2.132	2.776	3.747	4.604	29	1.311	1.699	2.045	2.462	2.756
5	1.476	2.015	2.571	3.365	4.032	30	1.310	1.697	2.042	2.457	2.750
6	1.440	1.943	2.447	3.143	3.707	31	1.309	1.696	2.040	2.453	2.744
7	1.415	1.895	2.365	2.998	3.499	32	1.309	1.694	2.037	2.449	2.738
8	1.397	1.860	2.306	2.896	3.355	33	1.308	1.692	2.035	2.445	2.733
9	1.383	1.833	2.262	2.821	3.250	34	1.307	1.691	2.032	2.441	2.728
10	1.372	1.812	2.228	2.764	3.169	35	1.306	1.690	2.030	2.438	2.724
11	1.363	1.796	2.201	2.718	3.106	40	1.303	1.684	2.021	2.423	2.704
12	1.356	1.782	2.179	2.681	3.055	45	1.301	1.679	2.014	2.412	2.690
13	1.350	1.771	2.160	2.650	3.012	50	1.299	1.676	2.009	2.403	2.678
14	1.345	1.761	2.145	2.624	2.977	55	1.297	1.673	2.004	2.396	2.668
15	1.341	1.753	2.131	2.602	2.947	60	1.296	1.671	2.000	2.390	2.660
16	1.337	1.746	2.120	2.583	2.921	65	1.295	1.669	1.997	2.385	2.654
17	1.333	1.740	2.110	2.567	2.898	70	1.294	1.667	1.994	2.381	2.648
18	1.330	1.734	2.101	2.552	2.878	75	1.293	1.665	1.992	2.377	2.643
19	1.328	1.729	2.093	2.539	2.861	80	1.292	1.664	1.990	2.374	2.639
20	1.325	1.725	2.086	2.528	2.845	85	1.292	1.663	1.988	2.371	2.635
21	1.323	1.721	2.080	2.518	2.831	90	1.291	1.662	1.987	2.368	2.632
22	1.321	1.717	2.074	2.508	2.819	95	1.291	1.661	1.985	2.366	2.629
23	1.319	1.714	2.069	2.500	2.807	100	1.290	1.660	1.984	2.364	2.626
24	1.318	1.711	2.064	2.492	2.797	105	1.290	1.659	1.983	2.362	2.623
25	1.316	1.708	2.060	2.485	2.787	∞	1.282	1.645	1.960	2.326	2.576

Intervall für μ

$$\bar X_n \pm t_{9;0,975} \cdot \sqrt{\frac{S_n^2}{n}} =$$

$$= 1.35 \pm 2.262 \cdot \sqrt{\frac{0{,}1516}{10}} =$$

$$= [1{,}0714 , 1{,}6286]$$

Intervall für σ^2

$$\left[\frac{(n-1)S_n^2}{\chi^2_{n-1;\frac{1+\gamma}{2}}}, \frac{(n-1)S_n^2}{\chi^2_{n-1;\frac{1-\gamma}{2}}}\right].$$

$$= [0{,}071552 ; 0{,}5055]$$

Quantile $\chi^2_{n;p}$ der χ^2-Verteilung mit n Freiheitsgraden:

n	.005	.01	.02	.025	.05	.1	.5	.9	.95	.975	.98	.99	.995
1	.000	.000	.001	.001	.004	.016	.455	2.706	3.841	5.024	5.412	6.635	7.879
2	.010	.020	.040	.051	.103	.211	1.386	4.605	5.991	7.378	7.824	9.210	10.597
3	.072	.115	.185	.216	.352	.584	2.366	6.251	7.815	9.348	9.837	11.345	12.838
4	.207	.297	.429	.484	.711	1.064	3.357	7.779	9.488	11.143	11.668	13.277	14.860
5	.412	.554	.752	.831	1.145	1.610	4.351	9.236	11.070	12.832	13.308	15.086	16.750
6	.676	.872	1.134	1.237	1.635	2.204	5.348	10.645	12.592	14.449	15.033	16.812	18.548
7	.989	1.239	1.564	1.690	2.167	2.833	6.346	12.017	14.067	16.013	16.622	18.475	20.278
8	1.344	1.646	2.032	2.180	2.733	3.490	7.344	13.362	15.507	17.535	18.168	20.090	21.955
9	1.735	2.088	2.532	2.700	3.325	4.168	8.343	14.684	16.919	19.023	19.679	21.666	23.589
10	2.156	2.558	3.059	3.247	3.940	4.865	9.342	15.987	18.307	20.483	21.161	23.209	25.188
11	2.603	3.053	3.609	3.816	4.575	5.578	10.341	17.275	19.675	21.920	22.618	24.725	26.757
12	3.074	3.571	4.178	4.404	5.226	6.304	11.340	18.549	21.026	23.336	24.054	26.217	28.300
13	3.565	4.107	4.765	5.009	5.892	7.042	12.340	19.812	22.362	24.736	25.472	27.688	29.819
14	4.075	4.660	5.368	5.629	6.571	7.790	13.339	21.064	23.685	26.119	26.873	29.141	31.319
15	4.601	5.229	5.985	6.262	7.261	8.547	14.339	22.307	24.996	27.488	28.259	30.578	32.801
16	5.142	5.812	6.614	6.908	7.962	9.312	15.338	23.542	26.269	28.845	29.633	32.000	34.267
17	5.697	6.408	7.255	7.564	8.672	10.085	16.338	24.769	27.587	30.191	30.995	33.409	35.718
18	6.265	7.015	7.906	8.231	9,390	10.835	17.338	25.909	28.869	31.526	32.346	34.805	37.156
19	6.844	7.633	8.567	8.907	10.117	11.651	18.338	27.204	30.144	32.852	33.687	36.191	38.582
20	7.434	8.260	9.237	9.591	10.851	12.443	19.337	28.412	31.410	34.170	35.020	37.566	39.997
21	8.034	8.897	9.915	10.283	11.591	13.240	20.337	29.615	32.671	35.479	36.343	38.932	41.401
22	8.643	9.542	10.600	10.982	12.338	14.041	21.337	30.813	33.924	36.781	37.659	40.289	42.796
23	9.260	10.196	11.293	11.689	13.091	14.848	22.337	32.007	35.172	38.076	38.968	41.638	44.181
24	9.886	10.856	11.992	12.401	13.848	15.659	23.337	33.196	36.415	39.364	40.270	42.980	45.559
25	10.520	11.524	12.697	13.120	14.611	16.473	24.337	34.382	37.652	40.646	41.566	44.324	46.928
26	11.160	12.198	13.409	13.844	15.379	17.292	25.336	35.563	38.885	41.923	42.856	45.642	48.290
27	11.808	12.879	14.125	14.573	16.151	18.114	26.336	36.741	40.113	43.194	44.140	46.963	49.645
28	12.461	13.565	14.847	15.308	16.928	18.939	27.336	37.916	41.337	44.461	45.419	48.278	50.993
29	13.121	14.256	15.574	16.047	17.708	19.768	28.336	39.087	42.557	45.722	46.693	49.588	52.336
30	13.787	14.953	16.306	16.791	18.493	20.599	29.336	40.256	43.773	46.979	47.962	50.892	53.672
31	14.458	15.655	17.042	17.539	19.281	21.434	30.336	41.422	44.985	48.232	49.226	52.191	55.003
32	15.134	16.362	17.783	18.291	20.072	22.271	31.336	42.585	46.194	49.480	50.487	53.486	56.328
33	15.815	17.074	18.527	19.047	20.867	23.110	32.336	43.745	47.400	50.725	51.743	54.776	57.648
34	16.501	17.789	19.275	19.806	21.664	23.952	33.336	44.903	48.602	51.966	52.995	56.061	58.964
35	17.192	18.509	20.027	20.569	22.465	24.797	34.336	46.059	49.802	53.203	54.244	57.342	60.275
40	20.707	22.164	23.838	24.433	26.509	29.051	39.335	51.805	55.758	59.342	60.436	63.691	66.766
45	24.311	25.901	27.720	28.366	30.612	33.350	44.335	57.505	61.656	65.410	66.555	69.957	73.166
50	27.991	29.707	31.664	32.357	34.764	37.689	49.335	63.167	67.505	71.420	72.613	76.154	79.490
55	31.735	33.570	35.659	36.398	38.958	42.060	54.335	68.796	73.311	77.380	78.619	82.292	85.749
60	35.534	37.485	39.699	40.482	43.188	46.459	59.335	74.397	79.082	83.298	84.580	88.397	91.952
65	39.383	41.444	43.779	44.603	47.450	50.883	64.335	79.973	84.821	89.177	90.501	94.422	98.105
70	43.275	45.442	47.893	48.758	51.739	55.329	69.334	85.527	90.531	95.023	96.388	100.425	104.215
75	47.206	49.475	52.039	52.942	56.054	59.795	74.334	91.061	96.217	100.839	102.243	106.393	110.286
80	51.172	53.540	56.213	57.153	60.391	64.278	79.334	96.578	101.879	106.629	108.069	112.329	116.321
85	55.170	57.634	60.412	61.389	64.749	68.777	84.334	102.079	107.522	112.393	113.871	118.236	122.325
90	59.196	61.754	64.635	65.647	69.126	73.291	89.334	107.565	113.145	118.136	119.649	124.116	128.299
95	63.250	65.898	68.879	69.925	73.520	77.818	94.334	113.038	118.752	123.858	125.405	129.973	134.247
100	67.328	70.065	73.142	74.222	77.929	82.358	99.334	118.498	124.342	129.561	131.142	135.806	140.169

$\frac{1-\gamma}{2}$ $\frac{1+\gamma}{2}$

8. $(X_1, \ldots, X_n)$ ist eine Stichprobe einer Normalverteilung mit $\mu = \sigma^2 = \theta$. Bestimmen Sie den Maximum-Likelihood-Schätzer für θ.

$\mu = \sigma^2 = \theta$

$$L(x_1, \ldots, x_n, \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} \cdot e^{\frac{-(x_i-\theta)^2}{2\theta}} = \frac{1}{\sqrt{2\pi\theta}^n} \cdot e^{\frac{-\sum_{i=1}^{n}(x_i-\theta)^2}{2\theta}}$$

Logarithmieren und ableiten

$$\ln(L(x_1, \ldots, x_n, \theta)) = -\frac{n}{2} \cdot \ln(2\pi) - \frac{n}{2} \cdot \ln(\theta) - \frac{\sum_{i=1}^{n}(x_i-\theta)^2}{2\theta}$$

$$\frac{\partial \ln(L(x_1, \ldots, x_n, \theta))}{\partial \theta} = -\frac{n}{2\theta} - \frac{1}{2}\left(\frac{\sum_{i=1}^{n}(x_i^2 - 2x_i\theta + \theta^2)}{\theta}\right)' = 0$$

$$\boxed{\begin{aligned} &[(x_i^2 - 2x_i\theta + \theta^2) \cdot \theta^{-1}]' = \\ &(x_i^2 \cdot \theta^{-1} - 2x_i + \theta)' = \\ &= (-x_i^2\theta^{-2} + 1) = \end{aligned}}$$

$$= -\frac{n}{2\theta} + \frac{\sum_{i=1}^{n} x_i^2}{2\theta^2} - \frac{n}{2} = 0$$

$$\frac{\sum_{i=1}^{n} x_i^2}{\theta^2} = n + \frac{n}{\theta}$$

$$= \theta^2 n + \theta n - \sum_{i=1}^{n} x_i^2$$

$$= \theta^2 + \theta - \frac{\sum_{i=1}^{n} x_i^2}{n}$$

$$= \theta_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{\sum_{i=1}^{n} x_i^2}{n}}$$

Da gelten muss $\theta > 0 \rightarrow \theta = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\sum_{i=1}^{n} x_i^2}{n}}$