Übungstest – Discrete Mathematics - WS 2014 (**Gruppe alle**)

1. State Euler's formula relating the number of vertices, edges and faces in a graph. For which graphs is it valid? Prove the formula by using induction on the number of faces. You may use the formula relating the number of edges and the number of vertices in a tree without proof.

   Deduce that for a connected simple planar graph the inequality $\#E \leq 3\#V - 6$ is satisfied, and that K_5 is therefore non-planar.

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2. - State the axioms required that a set I forms the independent sets of a matroid.
   - Let $G = (V, E)$ be a connected graph and let $\mathcal{F}$ be the set of spanning forests of G. Show that $\mathcal{F}$ is the set of independent sets of a matroid having the edges of G as elements.

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3. Let $G = (V, E, w)$ a weighted directed graph with source s and sink t.
   (a) What is a flow Φ on G?
   (b) What is an augmenting path for Φ?
   (c) What is a cut of G, and what is its capacity?
   (d) Suppose that there is no augmenting path for Φ. Define a cut whose capacity equals the value of Φ.

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4. (a) What does the Ramsey number $R(r, s)$ specify?
   (b) Compute $R(2, 4)$.
   (c) Prove that $R(3, 3) = 6$.

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5. (a) State the axioms required that a set T is a topology.
   (b) Let X and Y be topological spaces. What does it mean that a mapping $f : X \to Y$ is continuous?
   (c) Show that K_4 can be embedded on a torus and that K_3 cannot be embedded on the torus. Relate this to the Euler characteristic of the torus.

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6. Finish the following argument showing that a connected simple planar graph G with minimal degree five can be properly coloured with at most five colours.

   - let v be a vertex of degree five with neighbours $v_1, \ldots, v_5$.
   - by induction, we can suppose that $G \setminus \{v\}$ can be five-coloured, we pick such a colouring.
   - suppose that the colour of v_i is i.
   - consider the subgraph $P_{i,j}$ of $G \setminus \{v\}$ consisting of the vertices of colour i and j only.

   Now show that there are two vertices v_i and v_j which are in different components of $P_{i,j}$.
