

Einführung in Wissensbasierte Systeme WS 2018/19, 3.0 VU, 184.737

Exercise Sheet 1 – Logic, Part 1

For the presentation part of this exercise, mark your solved exercises in TUWEL until Monday, November 26, 23:55 CET. Be sure that you tick only those exercises that you can solve and explain in detail with the necessary theoretical background. In particular note that ticking exercises which you do not understand can result in a low number of points in the exercise part!

If exercises have subtasks, each subtask counts as one exercise and you will be able to cross them separately.

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✓ **Exercise 1.1:** Check using semantics whether the following statements hold (do not use truth tables). If a statement does not hold, provide a counterexample.

1. $\neg(p \rightarrow q) \models p \rightarrow \neg q$ ✓
2. $p \rightarrow q \models (p \rightarrow \neg q) \rightarrow \neg p$ ✓
3. $p \rightarrow q \models q \rightarrow \neg p$ ✓

✓ **Exercise 1.2:** Let ϕ be a tautology, ψ a contradiction, and χ a contingency (i.e., χ is satisfiable as well as falsifiable). Which of the following formulas are (i) tautological, (ii) contradictory, (iii) contingent, or (iv) logically equivalent to χ ? Justify your answers.

- | | |
|---------------------------|--------------------------------|
| 1. $\phi \wedge \chi$. ✓ | 4. $\psi \vee \chi$. ✓ |
| 2. $\phi \vee \chi$. ✓ | 5. $\phi \vee \psi$. ✓ |
| 3. $\psi \wedge \chi$. ✓ | 6. $\chi \rightarrow \psi$. ✓ |

p ∈ BV

✓ **Exercise 1.3:** Let Γ be a (possibly infinite) set of propositional formulas. Recall that Γ is *satisfiable* if there exists an interpretation I such that I is a model of each $\varphi \in \Gamma$. Similarly, Γ is *unsatisfiable* if there is no interpretation which satisfies each $\varphi \in \Gamma$. The *compactness theorem* states that Γ is satisfiable if every finite subset of it is satisfiable. Prove that the following formulations of the compactness theorem are equivalent:

1. Γ is satisfiable if every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.
2. For any formula φ , if $\Gamma \models \varphi$, then $\Gamma_0 \models \varphi$ for some finite $\Gamma_0 \subseteq \Gamma$.
3. If Γ is unsatisfiable, then there exists some finite $\Gamma_0 \subseteq \Gamma$ such that Γ_0 is unsatisfiable.

In order to prove the equivalence of the three statements, note that it suffices to prove (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1), i.e., you only have to prove three implications. (Hint: Recall that Γ is unsatisfiable iff $\Gamma \models \perp$).

✓ **Exercise 1.4:** Certain inference patterns of propositional logic can be seen as representing proof techniques. E.g., the deduction theorem presented in the lecture says that, in order to prove an implication $\varphi \rightarrow \psi$ from a knowledge base (theory) Γ , we may prove ψ using the assumptions in Γ and, additionally, φ . We now investigate the proof technique used in Exercise 1.3 more closely, i.e., the technique of proving the equivalence of n statements by proving a chain of n implications.

To this end, let $n \geq 1$ and propositional formulas $\varphi_1, \dots, \varphi_n$ be given. We define

$$\Gamma^n := \{\varphi_i \rightarrow \varphi_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\varphi_n \rightarrow \varphi_1\}.$$

Show that for all $n \geq 1$ and all $1 \leq i, j \leq n$ it holds that $\Gamma^n \models \varphi_i \leftrightarrow \varphi_j$.

(Hint: Prove the statement by induction on n . In the induction step, recall that $Cn(\cdot)$ is monotonic and idempotent.)

✓ **Exercise 1.5:** Prove or refute whether the following formulas are tautologies. Establish your claim by purely semantic means (i.e., without applying a calculus like TC1).

- ✓ (a) $\exists x (P(x) \rightarrow P(f(x)))$
- ✓ (b) $\forall x \exists y R(x, y) \rightarrow \exists y \forall x R(x, y)$

Hint: Truth tables do not work.

✓ **Exercise 1.6:** Translate the following arguments into entailment problems of first-order logic and check the validity of each statement. If the argument is valid, provide a proof in TC1. Otherwise, prove that there exists no closed tableau for the (negation of the) corresponding formula in TC1.

- ✓ (a) Influenza is caused by a virus. Pseudomonas is a bacterium and not a virus, therefore it doesn't cause influenza.
- ✓ (b) Every chicken eats corn. Berta eats grass. Therefore Berta is not a chicken.

[EWBS]

- 1.1 ... If $\mathcal{I}(\varphi) = 1$, then \mathcal{I} is called a model of φ .

Entailment: Let W be a set of closed formulas. Then W entails φ , $W \models \varphi$, if and only if $\text{Mod}(W) \subseteq \text{Mod}(\varphi)$.

\Rightarrow every Model of W is a Model of φ

$$1) \neg(p \rightarrow q) \models p \rightarrow \neg q$$

let's take $\mathcal{I} = \langle U, I, \{\}\rangle$ as an arbitrary model of the left side $\neg(p \rightarrow q)$. Therefore, $\mathcal{I} \models \neg(p \rightarrow q)$ and $\mathcal{I} \not\models p \rightarrow q$. So $\mathcal{I} \not\models p \rightarrow q$ iff $\mathcal{I} \models p$ and $\mathcal{I} \not\models q$ must hold.

Now we have to evaluate the right side $p \rightarrow \neg q$ under the model \mathcal{I} . We can follow from $\mathcal{I} \not\models q$ that $\mathcal{I} \models \neg q$ and as a result $\mathcal{I} \models p \rightarrow \neg q$. Therefore the entailment holds.

$$2) p \rightarrow q \models (p \rightarrow \neg q) \rightarrow \neg p$$

We alternatively show the contraposition of the entailment.

$$R \models Q \rightarrow \neg R \models \neg Q$$

Have a look at the contraposition $\neg((p \rightarrow \neg q) \rightarrow \neg p) \models \neg(p \rightarrow q)$. let's take $\mathcal{I} = \langle U, I, \{\}\rangle$ as an arbitrary model of the left side. Therefore, $\mathcal{I} \models \neg((p \rightarrow \neg q) \rightarrow \neg p)$ and $\mathcal{I} \not\models (p \rightarrow \neg q) \rightarrow \neg p$ holds.

$\Rightarrow \mathcal{I} \not\models (p \rightarrow \neg q) \rightarrow \neg p$ iff $\mathcal{I} \models p$ and $\mathcal{I} \not\models q$. Let's evaluate the right side under the model \mathcal{I} . As a result $\mathcal{I} \not\models p \rightarrow q$ and $\mathcal{I} \models \neg(p \rightarrow q)$. Therefore the entailment holds.

$$3) p \rightarrow q \models q \rightarrow \neg p$$

Let's take an arbitrary model $\mathcal{I} = \langle U, I, \{\cdot\} \rangle$ so, that

$$\mathcal{I} \models p \rightarrow q \text{ and } \mathcal{I} \not\models q \rightarrow \neg p.$$

To achieve this and to produce a counterexample we choose

~~the~~ the model \mathcal{I} in a way that $\mathcal{I} \models p$ and $\mathcal{I} \models q$. As a result $\mathcal{I} \models p \rightarrow q$ and $\mathcal{I} \not\models q \rightarrow \neg p$. The entailment doesn't hold because $\text{Mod}(p \rightarrow q) \neq \text{Mod}(q \rightarrow \neg p)$.

1.2

Tautology: all interpretation structures are models

contradiction: no interpretation structure is a model

contingency: satisfiable as well as falsifiable

Φ ... tautology

Ψ ... contradiction

χ ... contingency

1. $\Phi \wedge \chi$

→ look at models

(iv) logically equivalent, because the conjunction of a tautology Φ and χ is logically the same as χ . Tautologies are like the neutral element of conjunction of formulas.

(iii) ~~not~~ contingent, because of the conjunction with a tautology the formula keeps its satisfiability and falsifiability.

2. $\phi \vee X$

(i) tautological: the disjunction of an tautological formula and any other formula is a tautological formula.

3. $\psi \wedge X$

(ii) contradictory: the conjunction of an contradictory formula and any other formula is a contradictory formula, because both sides of the conjunction has to be models, but for contradictions there is no model at all.

4. $\psi \vee X$

(iii) contingent: There is no model for ψ , but every model of X stays a model after the disjunction. Therefore, it keeps its satisfiability and falsifiability.

(iv) logically equivalent to X : a contradiction is the neutral element of disjunction. Every model of X stays a model after disjunction with ψ .

5. $\phi \vee \psi$

(i) tautological: the disjunction of a model with a non model stays a model

6. $X \rightarrow \psi$

(iii) contingent: the implication of a model to a non model is not a model, but the implication of a non model to a model therefore is. The formula is satisfiable and falsifiable. or a non model

1.3]

Ass 1.: In general, classical logic satisfies the monotonicity principle:
If $S \models A$ and $S \subseteq S'$ then $S' \models A$.

Ass 2.: $\{\varphi_1, \dots, \varphi_n\}$ is $\bigwedge_{i=1}^n \varphi_i$, Γ is satisfiable if there exists an interpretation I such that I is a model of each $\varphi \in \Gamma$.

Ass 3.: $I \models \varphi$ for $I(\varphi) = 1$ and $I \not\models \varphi$ for $I(\varphi) = 0$
 $\Rightarrow I(\neg \varphi) = 0 \Leftrightarrow I \models \varphi$

Only retrieves invalid closures from valid ones: $T \models \neg a \Rightarrow T \not\models a$

1. Γ is satisfiable if every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable

2. For any formula φ , if $\Gamma \models \varphi$, then $\Gamma_0 \models \varphi$ for some finite $\Gamma_0 \subseteq \Gamma$.

3. If Γ is unsatisfiable, then there exists some finite $\Gamma_0 \subseteq \Gamma$ such that Γ_0 is unsat.

Show: (1) \Rightarrow (2)

$$\triangleq \Gamma_0 \models \neg \varphi$$

1. Assume (1)

2. Let φ be an arbitrary formula, $\Gamma \models \varphi$ but $\Gamma_0 \not\models \varphi$ for every finite $\Gamma_0 \subseteq \Gamma$

3. $\Gamma_0 \cup \{\neg \varphi\}$ is satisfiable for every finite $\Gamma_0 \subseteq \Gamma$ (1-2)

4. Let $\Psi \subseteq \Gamma \cup \{\neg \varphi\}$ be a finite set.

5a. $\neg \varphi \in \Psi \rightarrow \Psi = \Gamma_0 \cup \{\neg \varphi\}$ for a finite $\Gamma_0 \subseteq \Gamma$ (1-4, Def. 0, Def. c)

6a. Ψ is satisfiable due to (3)

5b. $\neg \varphi \notin \Psi \rightarrow \Psi \subseteq \Gamma$ (1-4, Def. c)

6b. $\Psi \cup \{\neg \varphi\}$ is satisfiable due to (5a) $\rightarrow \Psi$ is satisfiable (5a, Def. 0)

7b. This means that every finite $\Gamma_0 \subseteq \Gamma \cup \{\neg \varphi\}$ is satisfiable (2-6b)

8. Looking at (1) we get $\Gamma \cup \{\neg \varphi\}$ is satisfiable. (1-7) (1-7)

9. $\Gamma \not\models \varphi \rightarrow$ that is, a contradiction and (1-8)

Therefore (1) \Rightarrow (2) holds

Show: $(2) \Rightarrow (3)$ Ass4: Γ is unsatisfiable iff $\Gamma \models \perp$

1. Assume (2)
 2. Let Γ be unsatisfiable, $\Gamma \models \perp$
 3. Looking at (2) we get a finite $\Gamma_0 \subseteq \Gamma$, so that $\Gamma_0 \models \perp$ (1-2)
 4. Γ_0 is unsatisfiable (1-3, Ass4)
 5. Γ_0 is a finite unsatisfiable subset of Γ and satisfies (3) (1-4)
- $\Rightarrow (2) \Rightarrow (3)$ holds.

Show: $(3) \Rightarrow (1)$

1. Assume (3)
2. Let every finite $\Gamma_0 \subseteq \Gamma$ be satisfiable
3. Γ is satisfiable (2, Def. \subseteq)
4. ...

Alternativ: Dieser Beweis entspricht einer Kontraposition.

For example the contrapositive of $P \rightarrow Q$ is thus $\neg Q \rightarrow \neg P$.

(1) is a contraposition of (3)

every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable $\rightarrow \Gamma$ is satisfiable

Γ is unsatisfiable \rightarrow it exists some finite $\Gamma_0 \subseteq \Gamma$ such that Γ_0 is unsat.

1.4 add

Induction step.

1. $\forall n \geq 1, \forall 1 \leq i, j \leq n : \text{Mod}(\Gamma^n) \subseteq \text{Mod}(q_i \leftrightarrow q_j)$
2. $\text{Mod}(\Gamma^n \cup \{q_n \rightarrow q_{n+1}, q_{n+1} \rightarrow q_1\}) \subseteq \text{Mod}(\Gamma^n)$
3. Let I be the general interpretation that isn't a model for $q_n \rightarrow q_1$
 $I \models q_n, I \not\models q_1$
4. Check if I isn't a model for $(q_n \rightarrow q_{n+1} \wedge q_{n+1} \rightarrow q_1)$ either
case: $I \models q_{n+1}, q_n \rightarrow q_{n+1} : 1 \rightarrow 1 \checkmark$
 $q_{n+1} \rightarrow q_1 : 1 \rightarrow 0 \times \Rightarrow \text{no model}$
5. case $I \not\models q_{n+1}, q_n \rightarrow q_{n+1} : 1 \rightarrow 0 \times \Rightarrow \text{no model}$
6. $\text{Mod}(\neg(q_n \rightarrow q_1)) \subseteq \text{Mod}(\neg(q_n \rightarrow q_{n+1} \wedge q_{n+1} \rightarrow q_1))$ (3,4, def. Mod)
7. $\text{Mod}(\Gamma^n \cup \{q_n \rightarrow q_{n+1}, q_{n+1} \rightarrow q_1\} \setminus \{q_n \rightarrow q_1\}) \subseteq \text{Mod}(\Gamma^n)$ (2,5)
8. $C_n(\Gamma^n) \subseteq C_n(\Gamma^{(n+1)})$
9. Let I_1, I_2 the general interpretations that are models for all Γ^n
 $\forall 1 \leq i \leq n : I_1 \models q_i \text{ model of } q_i \quad C_n(\Gamma^{n+1}) \setminus C_n(\Gamma^n)$
 $\forall 1 \leq i \leq n : I_2 \not\models q_i \text{ no model of } q_i \Leftrightarrow \neg q_{n+1} \rightarrow q_i$
10. Check if $I_{1,2}$ is a model for $\forall 1 \leq i \leq n : (q_i \leftrightarrow q_{n+1}) \equiv q_i \rightarrow q_{n+1}$
case $I_1 \models q_{n+1} : q_i \rightarrow q_{n+1} : 1 \rightarrow 1 \checkmark \Rightarrow I_1 \text{ is model}$
 $q_{n+1} \rightarrow q_i : 1 \rightarrow 1 \checkmark \Rightarrow I_1 \text{ is model}$
- case $I_1 \not\models q_{n+1} : q_i \rightarrow q_{n+1} : 1 \rightarrow 0 \times$
 $q_{n+1} \rightarrow q_i : 0 \rightarrow 1 \checkmark \Rightarrow I_1 \text{ is not a model}$

case $I_2 \not\models \varphi_{n+1}$: $\ell_i \rightarrow \ell_{n+1} : 0 \rightarrow 0 \checkmark$
 $\ell_{n+1} \rightarrow \ell_i : 0 \rightarrow 0 \checkmark \Rightarrow I_2 \text{ is a model}$

case $I_2 \models \varphi_{n+1}$: $\ell_i \rightarrow \ell_{n+1} : 0 \rightarrow 1 \checkmark$ $\Rightarrow I_2 \text{ is not a model}$
 $\ell_{n+1} \rightarrow \ell_i : 1 \rightarrow 0 \checkmark$

11. $\text{Mod}(\Gamma^{n+1}) \subseteq \text{Mod}(\forall 1 \leq i \leq n \mid \ell_i \leftrightarrow \ell_{n+1}) \quad (10)$

12. $\forall 1 \leq i \leq n : \Gamma^{n+1} \models \ell_i \leftrightarrow \ell_{n+1}$

13. $\forall 1 \leq i, j \leq n+1 : \Gamma^{n+1} \models \ell_i \leftrightarrow \ell_j \checkmark$

1.5] Prove or refute, formulas are tautologies ?!

$$(a) \exists x (P(x) \rightarrow P(f(x))) \stackrel{\Delta}{=} \exists x (\neg P(x) \vee P(f(x)))$$

Assume $\exists x (P(x) \rightarrow P(f(x)))$ is not a tautology, then $\forall x \neg (P(x) \rightarrow P(f(x)))$, so $\forall x (P(x) \wedge \neg P(f(x)))$ must be satisfiable.

counter example: let $\mathcal{I} = \langle \mathbb{N}, I, \{\}\rangle$, $\alpha = \{\}$ cause formula is closed

Informally, the symbols f, P have the following meanings:

f: $I \in \text{Func}$ and means "successor of"

P: $I \in \text{Pred}$ and means "is even number"

$\mathcal{I} \models \forall x (P(x) \wedge \neg P(f(x)))$ iff for each $c \in \mathbb{N}$, $\mathcal{I}_c = \langle \mathbb{N}, I, \{x \in c\} \rangle$ and
 $\mathcal{I}_c \models P(x) \wedge \neg P(f(x))$

let's choose c as an odd number (3)

⇒ Tautology

$$(b) \forall x \exists y R(x, y) \rightarrow \exists y \forall x R(x, y)$$

To show that given formula is no tautology we have to find an interpretation \mathcal{I} that $\mathcal{I} \models \forall x \exists y R(x, y)$ and $\mathcal{I} \not\models \exists y \forall x R(x, y)$.

Let's have a look at $\mathcal{U} = \{0, 1\}$, $I(R) = \{(0, 0), (1, 1)\} \subset \mathcal{U}^2$ meaning, that x must be the same as y.

$\mathcal{I} \models \forall x \exists y R(x, y)$ is fulfilled, because for every x there is an equal y ($x=0 \rightarrow y=0$, $x=1 \rightarrow y=1$).

$\mathcal{I} \not\models \exists y \forall x R(x, y)$ is not fulfilled, cause for $y=0$, $x=1$ doesn't hold and for $y=1$, $x=0$ doesn't hold.

⇒ no tautology

1.6) a) $\forall x (\text{causeInf}(x) \rightarrow \text{virus}(x)),$
 $\text{bact(pm)}, \neg \text{virus(pm)}$
 $\models \neg \text{causeInf(pm)}$

$A, B, C \models D \rightsquigarrow A \rightarrow (B \rightarrow (C \rightarrow D))$
 $\Leftrightarrow A \vee (\neg B \vee (\neg (C \vee D)))$
 ~~$\neg A \wedge B \wedge C \neg D$~~

$\varphi, F \models \varphi_{\text{gdw}}$: $\varphi \models F \rightarrow \varphi$
 $\varphi \models \varphi_{\text{gdw}} \models \varphi \rightarrow \varphi$
 $\text{gdw. } \neg(\varphi \rightarrow \varphi) \equiv \varphi_1 \neg \varphi_{\text{unsat}}$
 $\text{gdw. } \varphi_1 \neg \varphi \text{ hat closed TC}$

NNF: $\neg (\forall x (\text{causeInf}(x) \rightarrow \text{virus}(x)) \wedge \text{bact(pm)} \wedge \neg \text{virus(pm)}) \rightarrow \neg \text{causeInf(pm)}$
 $\neg \neg (\forall x (\neg \text{causeInf}(x) \vee \text{virus}(x)) \wedge \text{bact(pm)} \wedge \neg \text{virus(pm)}) \vee \neg \text{causeInf(pm)}$
 $\forall x (\neg \text{causeInf}(x) \vee \text{virus}(x)) \wedge \text{bact(pm)} \wedge \neg \text{virus(pm)} \wedge \text{causeInf(pm)}$

$\forall x (\neg \text{causeInf}(x) \vee \text{virus}(x))$

$\begin{array}{c} | \\ \text{bact(pm)} \end{array}$

$\begin{array}{c} | \\ \neg \text{virus(pm)} \end{array}$

$\begin{array}{c} | \\ \text{causeInf(pm)} \end{array}$

$\neg \text{causeInf(pm)} \vee \text{virus(pm)}$

$\neg \text{causeInf(pm)}$

virus(pm)

*

*

The negated formula in NNF is unsat. \Rightarrow formula is valid!

b) $\forall x (\text{chicken}(x) \rightarrow \text{eats}(x, \text{corn})), \text{eats}(\text{bertha}, \text{grass})$
 $\vdash \neg \text{chicken}(\text{bertha})$

Look at the interpretation structure: $\mathcal{I} = \langle U, I, \{\}\rangle$ with

$U = \{0, 1, 2\}$, $I(\text{bertha}) = 0$, $I(\text{corn}) = 1$, $I(\text{grass}) = 2$

$I(\text{eats}) = \{(0, 1), (0, 2)\}$

$I(\text{chicken}) = \{(0)\}$

$\neg \vdash (\forall x (\neg \text{chicken}(x) \vee \text{eats}(x, \text{corn})) \wedge \text{eats}(\text{bertha}, \text{grass})) \vee \neg \text{chicken}(\text{bertha})$
 $\forall x (\neg \text{chicken}(x) \rightarrow \text{eats}(x, \text{corn})) \wedge \text{eats}(\text{bertha}, \text{grass}) \wedge \neg \text{chicken}(\text{bertha})$

2.1] $T = \{\forall x (\neg P(x) \rightarrow Q(x)), \neg P(b), R(a), \forall x (R(x) \rightarrow P(x))\}$

(a) $CWA(T) \& CWA^a(T)$ $T_{\text{consistent}} \equiv T_{\text{rat.}}$

$CWA(T) = \{\varphi \mid T \cup T_{\text{asm}} \models \varphi, \varphi \text{ closed}\} = C_n(T \cup T_{\text{asm}})$

$T_{\text{asm}} = \{\neg P \mid P \text{ is a ground atom and } T \not\models P\}$

T_{asm} : $P(a) : R(a), \forall x (R(x) \rightarrow P(x)) \in T \Rightarrow T \models P(a) \Rightarrow \neg P(a) \notin T_{\text{asm}}$

$P(b) : \neg P(b) \in T, T_{\text{rat.}} \Rightarrow T \not\models P(b) \Rightarrow \neg P(b) \in T_{\text{asm}}$

$Q(a) : T \not\models Q(a) \Rightarrow \neg Q(a) \in T_{\text{asm}}$

$Q(b) : \neg P(b), \forall x (\neg P(x) \rightarrow Q(x)) \in T \Rightarrow T \models Q(b) \Rightarrow \neg Q(b) \notin T_{\text{asm}}$

$R(a) : \neg R(a) \in T, T_{\text{rat.}} \Rightarrow T \models R(a) \Rightarrow \neg R(a) \notin T_{\text{asm}}$

$R(b) : T \not\models R(b) \Rightarrow \neg R(b) \in T_{\text{asm}}$

$\Rightarrow T_{\text{asm}} = \{\neg P(b), \neg Q(a), \neg R(b)\}$

$CWA(T) = C_n(T \cup \{\neg P(b), \neg Q(a), \neg R(b)\})$

$T_{\text{asm}}^a = \{\neg Q(a)\}$ To angeben für consistency

$CWA^a(T) = C_n(T \cup \{\neg Q(a)\})$

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Exercise Sheet 2 – Nonmonotonic Reasoning

For the presentation part of this exercise, mark your solved exercises in TUWEL until Monday, November 26, 23:55 CET. Be sure that you tick only those exercises that you can solve and explain in detail with the necessary theoretical background. In particular note that ticking exercises which you do not understand can result in a low number of points in the exercise part!

If exercises have subtasks, each subtask counts as one exercise and you will be able to cross them separately.

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Exercise 2.1: Consider the following theory:

$$T = \{\forall x(\neg P(x) \rightarrow Q(x)), \neg P(b), R(a), \forall x(R(x) \rightarrow P(x))\}.$$

- Determine $\text{CWA}(T)$ as well as $\text{CWA}^Q(T)$.
- Prove or refute whether $\text{CWA}(T)$ is consistent. What about $\text{CWA}^Q(T)$?

 **Exercise 2.2:** Consider the following theorem from the lecture:

Let T be a consistent theory. Then:

$\text{CWA}(T)$ is inconsistent iff there are ground atoms A_1, \dots, A_n such that $T \models A_1 \vee \dots \vee A_n$, but $T \not\models A_i$ for all $i = 1, \dots, n$.

- Prove the only-if direction.

Hint: First prove that, under the hypothesis of the only-if direction, $T_{\text{asm}} \neq \emptyset$ holds. From this, together with the compactness theorem¹, show that there exists a finite $T_{\text{asm}}^0 \subseteq T_{\text{asm}}$ such that $Cn(T \cup T_{\text{asm}}^0)$ is inconsistent. Assuming that T_{asm}^0 consists of exactly the atomic formulas $\neg A_1, \dots, \neg A_n$, argue that then $T \models A_1 \vee \dots \vee A_n$ must hold.

- Prove the if direction.

 **Exercise 2.3:** Consider the open default theory $T = (W, \Delta)$, where

$$\begin{aligned} W &= \{\exists x(P(x) \vee Q(x)), \forall x(P(x) \vee R(x)), \forall x(R(x) \rightarrow Q(x))\} \\ \Delta &= \left\{ \frac{P(x) : \neg Q(x)}{\neg Q(x)}, \frac{Q(x) : \neg P(x)}{\neg P(x)}, \frac{\top : R(x)}{R(x)} \right\}. \end{aligned}$$

Compute the closure of T and determine the possible candidates for being an extension. Then compute the classical reduct Δ_E for each candidate E , and determine all extensions of T .

¹The compactness theorem states that a set of formulas is satisfiable iff every finite subset of it is satisfiable. Prove first that this is equivalent to the condition that a set of formulas is unsatisfiable iff there is a finite subset of it which is unsatisfiable.

 **Exercise 2.4:** Prove that if a closed normal default theory $T = (W, \Delta)$ has distinct extensions E and E' then $E \cup E'$ is inconsistent.

Note that a default is called *normal* if it is of the form $A : B/B$. A default theory (W, Δ) is called *normal* if all defaults in Δ are normal.

Hint: You may use the *semi-recursive characterisation of extensions*. By this we understand the following property: Let E be a closed set of formulas and $(E_i)_{i \geq 0}$ be a sequence of sets of formulas defined by setting

$$E_0 = W$$

$$E_i = Cn(E_{i-1}) \cup \{C \mid (A : B_1, \dots, B_n/C) \in \Delta, E_{i-1} \models A \text{ and } \neg B_1, \dots, \neg B_n \notin E\} \\ (\text{for } i > 0).$$

Then, E is an extension of $T = (W, \Delta)$ iff

$$E = \bigcup_{i \geq 0} E_i.$$

 **Exercise 2.5:** The property *semi-monotonicity* is a weaker form of monotonicity and is defined as follows:

Let $T = (W, \Delta)$ and $T' = (W, \Delta')$ be default theories with $\Delta \subseteq \Delta'$, and let E be an extension of T . Then T' has an extension E' such that $E \subseteq E'$.

Prove or disprove whether semi-monotonicity holds in default logic.

 **Exercise 2.6:** You are given the propositional default theory $T = (W, \Delta)$, where

$$W = \{a\}, \\ \Delta = \left\{ \frac{a : b}{b}, \frac{b : c, \neg d}{c}, \frac{b : d, \neg c}{d}, \frac{c : e}{\neg e} \right\}.$$

Now consider the following (erroneous) reasoning trying to find an extension E of T :

1. Starting with the certain knowledge $a \in W$, we can apply the default $a : b/b$ to derive b .
2. Having derived b , we can now apply the second default $b : c, \neg d/c$ to derive c .
3. Due to c , the third default is blocked, but we can apply the fourth default $c : e/\neg e$ and derive $\neg e$.
4. No more defaults are applicable, so we arrive at $E = Cn(\{a, b, c, \neg e\})$.

However, E is *not* an extension of T .

Your task is to analyse the above reasoning and explain in detail why this approach does not work. Furthermore, find the actual extensions of T .

 **Exercise 2.7:** Consider the following information:

1. People who live in a city usually rent a flat.
2. People who live in a city and rent a flat are usually not rich.
3. Rich people usually do not rent a flat.
4. Emily lives in a city.
5. Emily is rich.

Formalise the given information in terms of an open default theory and compute all of its extensions. Use

- $\text{city}(x)$ for “ x lives in a city”,
- $\text{flat}(x)$ for “ x rents a flat”,
- $\text{rich}(x)$ for “ x is rich”, and
- the constant symbol e for “Emily”.

2.2) Let T be a consistent theory. Then:

$CWA(T)$ is inconsistent iff there are ground atoms A_1, \dots, A_n such that $T \models A_1 \vee \dots \vee A_n$, but $T \not\models A_i$, $i = 1, \dots, n$

a) Prove the only-if direction

Ass. 1.: $CWA(T) = \{\varphi \mid T \cup T_{asm} \models \varphi, \varphi \text{ closed}\}$

Ass. 2.: Compactness theorem: A set of formulas is satisfiable iff every finite subset of it is satisfiable.

Ass. 3.: T is consistent

Ass. 4.: A theory is consistent iff no contradiction can be derived from it, i.e. iff $T \not\models \perp$

a) Show: If $CWA(T)$ is inconsistent, then $T_{asm} \neq \emptyset$ holds. ✓

1. Assume $CWA(T)$ is inconsistent

2. Let $T_{asm} = \emptyset$

3. $CWA(T) = \{\varphi \mid T \cup T_{asm} \models \varphi, \varphi \text{ closed}\} = \{\varphi \mid T \models \varphi, \varphi \text{ closed}\}$ (2, Def. CWA)

4. $CWA(T) = C_n(T)$ (2-3, Def. C_n)

5. $C_n(T)$ is inconsistent

6. T is inconsistent $\Leftrightarrow T_{asm} \neq \emptyset$ (1-5, Ass. 4)

Ass. 5.: $T_{asm} \neq \emptyset$, Ass. 7: Contradiction theorem: $W \cup \{\varphi\}$ unsat. $\Leftrightarrow W \models \neg \varphi$

Ass. 6.: T is consistent iff T is satisfiable

Show: There is a finite $T_{asm}^0 \subseteq T_{asm}$ such that $C_n(T \cup T_{asm}^0)$ is inconsistent

1. Assume $T \cup T_{asm}$ inconsistent but T consistent (Ass. 3)

2. $T_{asm}^0 \subseteq T_{asm}$ so that $T \cup T_{asm}^0$ is inconsistent (1, Ass. 5, Ass. 2)

3. $T \cup T_{asm}^0$ is unsatisfiable ³⁵ $C_n(T \cup T_{asm})$ consists of all closed formulas (2, Ass. 3, Ass. 6)

4. $T \models \neg T_{asm}^0$ (1-3, Ass. 7)

5. $T \models \neg(\neg A_1 \wedge \dots \wedge \neg A_n)$ (4, T_{asm}^0 of $\neg A_1 \wedge \dots \wedge \neg A_n$)

6. $T \models A_1 \vee \dots \vee A_n$ \Leftrightarrow (1-5)

(b) Prove the if direction: \Leftarrow

Show: \Leftarrow

1. Assume there are ground atoms A_1, \dots, A_n such that $T \models A_1 \vee \dots \vee A_n$, but $T \not\models A_i$ for all $i = 1, \dots, n$
2. T is inconsistent
3. $T \cup T_{\text{asm}}$ is inconsistent because $\{\neg A_1, \dots, \neg A_n\} \in T_{\text{asm}}$ (1-2, Def. U)
4. $\text{CWA}(T)$ is inconsistent, cause it's unsatisfiable (1-3, Def. CWA)
 $\Rightarrow \text{CWA}(T)$ consists of all closed formulas

$C_n(T)$ is a set of all closed formulas if T is inconsistent

2.3] $T = (W, \Delta)$ where

$$W = \{ \exists x (P(x) \vee Q(x)), \forall x (P(x) \vee R(x)), \forall x (R(x) \rightarrow Q(x)) \}$$

$$\Delta = \left\{ \frac{P(a) : \neg Q(a)}{\neg Q(a)}, \frac{Q(a) : \neg P(a)}{\neg P(a)}, \frac{T : R(a)}{R(a)} \right\}$$

Closure \bar{T} of T : $\bar{T} = (\bar{W}, \bar{\Delta})$

$$\bar{W} = \{ P(a) \vee Q(a), \forall x (P(x) \vee R(x)), \forall x (R(x) \rightarrow Q(x)) \}$$

$$\bar{\Delta} = \left\{ \frac{P(a) : \neg Q(a)}{\neg Q(a)}, \frac{Q(a) : \neg P(a)}{\neg P(a)}, \frac{T : R(a)}{R(a)} \right\}$$

Candidates:

$$E_1 = C_n(\bar{W})$$

$$E_2 = C_n(\bar{W} \cup \{\neg Q(a)\})$$

$$E_3 = C_n(\bar{W} \cup \{\neg P(a)\})$$

$$E_4 = C_n(\bar{W} \cup \{R(a)\})$$

$$E_5 = C_n(\bar{W} \cup \{\neg Q(a), \neg P(a)\}) \rightarrow \text{inconsistent}$$

$$E_6 = C_n(\bar{W} \cup \{\neg Q(a), R(a)\}) \rightarrow \text{inconsistent}$$

$$E_7 = C_n(\bar{W} \cup \{\neg P(a), R(a)\})$$

$$E_8 = C_n(\bar{W} \cup \{\neg Q(a), \neg P(a), R(a)\}) \rightarrow \text{inconsistent}$$

$$\Delta_{E_1} = \left\{ \frac{P(a)}{\neg Q(a)}, \frac{Q(a)}{\neg P(a)}, \frac{T}{R(a)} \right\}$$

$$\Delta_{E_2} = \left\{ \frac{P(a)}{\neg Q(a)} \right\}$$

$$\Delta_{E_3} = \left\{ \frac{Q(a)}{\neg P(a)}, \frac{T}{R(a)} \right\}$$

$$\Delta_{E_4} = \left\{ \frac{Q(a)}{\neg P(a)}, \frac{T}{R(a)} \right\}$$

$$\Delta_{E_5} = \Delta_{E_6} = \Delta_{E_8} = \emptyset$$

$$\Delta_{E_7} = \left\{ \frac{Q(a)}{\neg P(a)}, \frac{T}{R(a)} \right\}$$

$$\Gamma_T(E_1) = C_n^{\Delta_{E_1}}(\bar{W}) = C_n(\bar{W} \cup \{R(a), Q(a), \neg P(a)\}) \neq E_1 \text{ no ext.}$$

$$\Gamma_T(E_2) = C_n^{\Delta_{E_2}}(\bar{W}) = C_n(\bar{W}) \neq E_2 \text{ no ext.}$$

$$\Gamma_T(E_3) = C_n^{\Delta_{E_3}}(\bar{W}) = C_n(\bar{W} \cup \{R(a), Q(a), \neg P(a)\}) = E_3 \text{ ext.}$$

$$\Gamma_T(E_4) = C_n^{\Delta_{E_4}}(\bar{W}) = C_n(\bar{W} \cup \{R(a), Q(a)\})$$

$$\Gamma_T(E_5) = \Gamma_T(E_6) = \Gamma_T(E_8) = C_n(\bar{W})$$

E_7 ext.

2.4] Prove if a closed normal default theory $T = (W, \Delta)$ has distinct extensions E & E' then $E \cup E'$ is inconsistent.

Note that a default is called normal if it is of the form $\frac{A : B}{C}$. A default theory (W, Δ) is called normal if all defaults in Δ are normal.

Hint: You may use the semi-recursive characterisation of extensions.

By this we understand the following property: Let E be a closed set of formulas and $(E_i)_{i \geq 0}$ be a sequence of sets of formulas defined by setting

Normal default theories always possesses extensions!

$$\text{Ass.1. } E_0 = W$$

(for $i \geq 0$)

$$E_i = C_n(E_{i-1}) \cup \{C \mid \frac{A : B_1, \dots, B_n}{C} \in \Delta, E_{i-1} \models A \text{ and } \neg B_1, \dots, \neg B_n \notin E\}$$

Then, E is an extension of $T = (W, \Delta)$ iff $E = \bigcup_{i \geq 0} E_i$

Consistency: A theory T is consistent iff no contradiction can be derived from it. $T \not\models \perp$

Show: If a closed normal default theory $T = (W, \Delta)$ has distinct extensions E & E' then $E \cup E'$ is inconsistent.

1. Assume a closed normal default theory $T = (W, \Delta)$ has distinct extensions E & E' , $E \neq E'$

2. $E = \bigcup_{i \geq 0} E_i$ and $E' = \bigcup_{i \geq 0} E'_i$ (Ass.1.)

3. $E_0 = W$ and $E_i = C_n(E_{i-1}) \cup \{B_1, \dots, B_n \mid \frac{A : B_1, \dots, B_n}{B_1, \dots, B_n} \in \Delta, E_{i-1} \models A \text{ and } \neg B_1, \dots, \neg B_n \notin E\}$ (Ass.)

4. $E'_0 = W$ and $E'_i = C_n(E'_{i-1}) \cup \{B_1, \dots, B_n \mid \frac{A : B_1, \dots, B_n}{B_1, \dots, B_n} \in \Delta, E'_{i-1} \models A \text{ and } \neg B_1, \dots, \neg B_n \notin E'\}$ (Ass.)

5. \exists some smallest $i : E_i \neq E'_i$ (1-2, Ass.1)

6. There is a default $\frac{A : B_1, \dots, B_n}{B_1, \dots, B_n} \in \Delta$ with $A \in E_{i-1}$ and $A \in E'_i$ (1-5)

7. further $B_1, \dots, B_n \in E_i$ and $\neg B_1, \dots, \neg B_n \notin E$ so $\neg B_1, \dots, \neg B_n \in E'_i$ (1-6)

8. $A \in E_{i-1}$ and $B_1, \dots, B_n \notin E_i$ so $\neg B_1, \dots, \neg B_n \in E'$ (6-7, Del. Default)

9. $B_1, \dots, B_n \in E$ and $\neg B_1, \dots, \neg B_n \in E'$ (1-8)

(1-9, Del. Consistency)

10. $E \cup E'$ is inconsistent

2.5] The property semi-monotonicity is a weaker form of monotonicity and is defined as follows:

Show: Let $T = (W, \Delta)$ and $T' = (W, \Delta')$ be default theories with $\Delta \subseteq \Delta'$, and let E be an extension of T . Then T' has an extension E' such that $E \subseteq E'$.

Prove or disprove whether semi-monotonicity holds in default logic.

Knowledge:

E is an extension of $T = (W, \Delta)$ iff $\Gamma_T(E) = E$

$$\Delta_E = \{ \varphi / \chi \mid (\varphi : \psi_1, \dots, \psi_n / \chi) \in \Delta \text{ and } \{\neg \psi_1, \dots, \neg \psi_n\} \cap E = \emptyset \}$$

$$C_n^{\Delta_E}(W) = C_n(W \cup \bigcup_{i \geq 0} E_i) \text{ with}$$

$$E_0 = \{ \chi \mid \varphi / \chi \in \Delta_E \text{ and } W \models \varphi \}$$

$$E_i = \{ \chi \mid \varphi / \chi \in \Delta_E \text{ and } W \cup E_{i-1} \models \varphi \}$$

$$\Rightarrow \Gamma_T(E) = C_n^{\Delta_E}(W) = E$$

Show: default logic is not semi-monotonic

1. Consider default theories $T = (A, \Delta = \{ \frac{T:A}{A} \})$ and $T' = (A, \Delta' = \{ \frac{T:A}{A}, \frac{T:B}{\neg B} \})$

2. $\Delta \subseteq \Delta'$ holds

3. T has extension $E = C_n(\{A\})$

$$\hookrightarrow \Delta_E = \{ \frac{T:A}{A} \} \quad \Gamma_T(E) = C_n^{\Delta_E}(A) = C_n(\{A\}) = E \quad \checkmark$$

4. T' has no extension due to the default $\frac{T:B}{\neg B}$

\hookrightarrow if B can be consistently assumed, then infer $\neg B$ contradicts the assumption

5. $E \neq E'$ due to non existence of E'

\Rightarrow semi-monotonicity doesn't hold in default logic

$$2.6) W = \{a\}, \Delta = \left\{ \frac{a=b}{b}, \frac{b=c \neg d}{c}, \frac{b=d \neg c}{d}, \frac{c=e}{\neg e} \right\}$$

- There are multiple errors in this reasoning. First of all, defaults aren't applied in a certain order, they fire concurrently. Furthermore, there is an inconsistent default applied in step 3, because if c is known and we can consistently derive e , then we can infer $\neg e$ is a contradiction and inconsistent (like killing clauses).

Knowledge: S is applicable to E iff $\forall E$ and $\neg \Psi_1 \notin E, \dots, \neg \Psi_n \notin E$

$$\Delta_E = \{ \Psi / X \mid \frac{\Psi: \Psi_1, \dots, \Psi_n}{X} \in \Delta \text{ and } \{ \neg \Psi_1, \dots, \neg \Psi_n \} \cap E = \emptyset \}$$

$$C_n^{\Delta_E}(W) = C_n(W \cup \bigcup_{i=0}^{n-1} E_i)$$

$$E_0 = \{ X \mid \Psi / X \in \Delta_E \text{ and } W \models \Psi \}$$

$$E_1 = \{ X \mid \Psi / X \in \Delta_E \text{ and } W \cup E_{0-1} \models \Psi \}$$

$$\Rightarrow \Gamma_T(E) = C_n^{\Delta_E}(W) = E$$

Show: $E = C_n(\{a, b, c, \neg e\})$ is no extension

$$\Gamma_T(E) = C_n^{\Delta_E}(W) = C_n^{\Delta_E}(W) = C_n(W \cup \{b, c\}) \neq E \quad \text{no extension}$$

$$\Delta_E = \left\{ \frac{a=b}{b}, \frac{b=c \neg d}{c} \right\}$$

$$E_{12} = C_n(\{a, b, c, d\})$$

$$\Delta_{E12} = \left\{ \frac{a=b}{b}, \frac{c=e}{\neg e} \right\}$$

$$E_{13} = C_n(\{a, b, c, \neg e\})$$

$$\Delta_{E13} = \left\{ \frac{a=b}{b}, \frac{b=d \neg c}{c} \right\}$$

$$E_{14} = C_n(\{a, b, d, \neg e\})$$

$$\Delta_{E14} = \left\{ \frac{a=b}{b}, \frac{b=d \neg c}{d} \right\}$$

$$E_{15} = C_n(\{a, c, d, \neg e\})$$

$$\Delta_{E15} = \left\{ \frac{a=b}{b} \right\}$$

$$E_{16} = C_n(\{a, b, c, d, \neg e\})$$

$$\Delta_{E16} = \left\{ \frac{a=b}{b}, \frac{b=c \neg d}{c} \right\}$$

$$E_1 = C_n(\{a\}) \quad \Delta_{E1} = \Delta$$

$$E_2 = C_n(\{a, b\}) \quad \Delta_{E2} = \Delta$$

$$E_3 = C_n(\{a, c\}) \quad \Delta_{E3} = \left\{ \frac{a=b}{b}, \frac{b=c \neg d}{c}, \frac{c=e}{\neg e} \right\}$$

$$E_4 = C_n(\{a, d\}) \quad \Delta_{E4} = \left\{ \frac{a=b}{b}, \frac{b=d \neg c}{d}, \frac{c=e}{\neg e} \right\}$$

$$E_5 = C_n(\{a, \neg e\}) \quad \Delta_{E5} = \left\{ \frac{a=b}{b}, \frac{b=c \neg d}{c}, \frac{b=d \neg c}{d} \right\}$$

$$E_6 = C_n(\{a, b, c\}) \quad \Delta_{E6} = \left\{ \frac{a=b}{b}, \frac{b=c \neg d}{c}, \frac{c=e}{\neg e} \right\}$$

$$E_7 = C_n(\{a, b, d\}) \quad \Delta_{E7} = \left\{ \frac{a=b}{b}, \frac{b=d \neg c}{d}, \frac{c=e}{\neg e} \right\}$$

$$E_8 = C_n(\{a, b, \neg e\}) \quad \Delta_{E8} = \left\{ \frac{a=b}{b}, \frac{b=c \neg d}{c}, \frac{b=d \neg c}{d} \right\}$$

$$E_9 = C_n(\{a, c, d\}) \quad \Delta_{E9} = \left\{ \frac{a=b}{b}, \frac{c=e}{\neg e} \right\}$$

$$E_{10} = C_n(\{a, c, \neg e\}) \quad \Delta_{E10} = \left\{ \frac{a=b}{b}, \frac{b=c \neg d}{c} \right\}$$

$$E_{11} = C_n(\{a, d, \neg e\}) \quad \Delta_{E11} = \left\{ \frac{a=b}{b}, \frac{b=d \neg c}{d} \right\}$$

$$\Gamma_T(E_1) = C_n^{\Delta E_1}(W) = C_n(\{a, b, c, d, \neg e\}) \neq E_1$$

$$\Gamma_T(E_2) = C_n^{\Delta E_2}(W) = C_n(\{a, b, c, d, \neg e\}) \neq E_2$$

$$\Gamma_T(E_3) = C_n^{\Delta E_3}(W) = C_n(\{a, b, c, \neg e\}) \neq E_3$$

$$\Gamma_T(E_4) = C_n^{\Delta E_4}(W) = C_n(\{a, b, d\}) \neq E_4$$

$$\Gamma_T(E_5) = C_n^{\Delta E_5}(W) = C_n(\{a, b, c, d\}) \neq E_5$$

$$\Gamma_T(E_6) = C_n^{\Delta E_6}(W) = C_n(\{a, b, c, \neg e\}) \neq E_6$$

$$\Gamma_T(E_7) = C_n^{\Delta E_7}(W) = C_n(\{a, b, d\}) = E_7 \quad \dots \text{extension}$$

$$\Gamma_T(E_8) = C_n^{\Delta E_8}(W) = C_n(\{a, b, c, d\}) \neq E_8$$

$$\Gamma_T(E_9) = C_n^{\Delta E_9}(W) = C_n(\{a, b, c, \neg e\}) \neq E_9$$

$$\Gamma_T(E_{10}) = C_n^{\Delta E_{10}}(W) = C_n(\{a, b, c\}) \neq E_{10}$$

$$\Gamma_T(E_{11}) = C_n^{\Delta E_{11}}(W) = C_n(\{a, b, d\}) \neq E_{11}$$

$$\Gamma_T(E_{12}) = C_n^{\Delta E_{12}}(W) = C_n(\{a, b, c, \neg e\}) \neq E_{12}$$

$$\Gamma_T(E_{13}) = C_n^{\Delta E_{13}}(W) = C_n(\{a, b, c\}) \neq E_{13}$$

$$\Gamma_T(E_{14}) = C_n^{\Delta E_{14}}(W) = C_n(\{a, b, d\}) \neq E_{14}$$

$$\Gamma_T(E_{15}) = C_n^{\Delta E_{15}}(W) = C_n(\{a, b\}) \neq E_{15}$$

$$\Gamma_T(E_{16}) = C_n^{\Delta E_{16}}(W) = C_n(\{a, b, c\}) \neq E_{16}$$

The actual extension is $E = C_n(\{a, b, d\})$.

2.7] Consider the following information :

- 1. People who live in a city usually rent a flat.
- 2. People who live in a city and rent a flat are usually not rich.
- 3. Rich people usually do not rent a flat.
- 4. Emily lives in a city.
- 5. Emily is rich.

Formalise the given information in terms of an open default theory and compute all of its extensions.

- 1. $S_1 = \frac{\text{city}(x) \rightarrow \text{flat}(x)}{\text{flat}(x)}$
- 2. $S_2 = \frac{\text{city}(x), \text{flat}(x) \rightarrow \neg \text{rich}(x)}{\neg \text{rich}(x)}$
- 3. $S_3 = \frac{\text{rich}(x) \rightarrow \neg \text{flat}(x)}{\neg \text{flat}(x)}$
- 4. $\text{city}(e)$
- 5. $\text{rich}(e)$

$$T = (\underbrace{\{\text{city}(e), \text{rich}(e)\}}_W, \underbrace{\{S_1, S_2, S_3\}}_\Delta)$$

Def.: E is an extension of T iff $\Gamma_T(E) = E$

$$\Delta_E = \{\varphi/x \mid \frac{\varphi: \varphi_1, \dots, \varphi_n}{x} \in \Delta \text{ and } \{\neg \varphi_1, \dots, \neg \varphi_n\} \cap E = \emptyset\}$$

$$C_n^{\Delta_E}(W) = C_n(W \cup \bigcup_{i>0} E_i)$$

$$E_0 = \{x \mid \varphi/x \in \Delta_E \text{ and } W \models \varphi\}$$

$$E_1 = \{x \mid \varphi/x \in \Delta_E \text{ and } W \cup E_{i-1} \models \varphi\}$$

Extensions: $E_1 = C_n(\{\text{city}(e), \text{rich}(e), \text{flat}(e)\})$

$$E_2 = C_n(\{\text{city}(e), \text{rich}(e), \neg \text{flat}(e)\})$$

$$\Delta_{E_1} = \{S_1\} \rightarrow \Gamma_T(E_1) = C_n^{\Delta_{E_1}}(W) = C_n(\{\text{city}(e), \text{rich}(e), \text{flat}(e)\}) = E_1 \checkmark$$

$$\Delta_{E_2} = \{S_3\} \rightarrow \Gamma_T(E_2) = C_n^{\Delta_{E_2}}(W) = C_n(\{\text{city}(e), \text{rich}(e), \neg \text{flat}(e)\}) = E_2 \checkmark$$

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Exercise Sheet 3 – Answer-Set Programming and Probabilistic Reasoning

For the presentation part of this exercise, mark your solved exercises in **TUWEL** until Monday, November 26, 23:55 CET. Be sure that you tick only those exercises that you can solve and explain in detail with the necessary theoretical background. In particular note that ticking exercises which you do not understand can result in a low number of points in the exercise part!

If exercises have subtasks, each subtask counts as one exercise and you will be able to cross them separately.

Please ask questions in the **TISS** Forum or visit our tutors during the tutor hours (see **TUWEL**).

 **Exercise 3.1:** Let P be a Horn logic program and $M_1, M_2 \subseteq HB(P)$ classical models of P . Prove that $M_1 \cap M_2$ is also a classical model of P . What about $M_1 \cup M_2$?

 **Exercise 3.2:** Consider the following disjunctive logic program P :

$$P = \left\{ \begin{array}{l} a \vee b \leftarrow \text{not } c. \\ c \leftarrow b, \text{not } d. \\ d \leftarrow \text{not } c. \\ a \leftarrow d. \end{array} \right\}.$$

-  (i) Determine all answer sets of P . For each proposed answer set S , argue formally that it is indeed an answer set (using the Gelfond-Lifschitz reduct P^S).
-  (ii) Is it possible to add new rules Q to P such that $S' = \{b, c\}$ is an answer set of $P \cup Q$? If yes, give the new rules Q , if not, argue why.
-  (iii) The same as in (??), but for $S' = \{a, b\}$.

 **Exercise 3.3:** For a program P , we denote by $AS(P)$ the set of all answer sets of P . Let P, Q be programs. We say that P, Q are

- (i) *equivalent* if $AS(P) = AS(Q)$ and
- (ii) *strongly equivalent* if $AS(P \cup R) = AS(Q \cup R)$ for every program R .

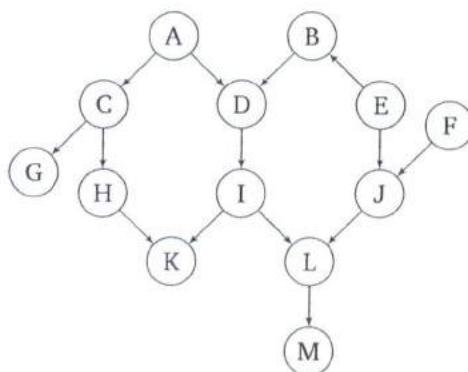
Prove or refute whether

-  (a) (i) implies (ii),
-  (b) (ii) implies (i).

✓ **Exercise 3.4:** Let S_1, S_2 be answer sets of a program P . Show that $S_1 \subseteq S_2$ implies $S_1 = S_2$.

✓ **Exercise 3.5:** A factory produces an item on two machines simultaneously. The produced items from machine 1 contain 4 % rejects; those from machine 2 contain 13 % rejects. All the items from the factory are mixed in a ratio of 3 : 1 and from this mixture one item is drawn randomly. What is the probability that a reject is drawn? What is the (conditional) probability that the item was produced by the first machine, if it is a reject?

✓ **Exercise 3.6:** Consider the following graph of a Bayesian network:



Answer the following questions and give an explanation of your answer.

- ✓ (i) Is B conditionally independent of G ?
- ✓ (ii) Is B conditionally independent of G given evidence M ?
- ✓ (iii) Which evidences are needed such that G is conditionally independent of M ?
- ✓ (iv) Which evidences are needed such that A is conditionally independent of L ?

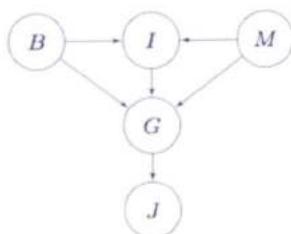
Exercise 3.7: Given the following Bayesian network with the Boolean variables

- $B = \text{BrokeElectionLaw}$,
- $I = \text{Indicted}$,
- $M = \text{PoliticallyMotivatedProsecutor}$,
- $G = \text{FoundGuilty}$, and
- $J = \text{Jailed}$,

where the probabilities for B and M are $P(B) = 0.9$ and $P(M) = 0.1$, respectively. The conditional probabilities for $P(G|B, I, M)$, $P(J|G)$, and $P(I|B, M)$ are as follows:

B	M	$P(I B, M)$
0	0	0.1
0	1	0.5
1	0	0.5
1	1	0.9

G	$P(J G)$
1	0.9
0	0.0



B	I	M	$P(G B, I, M)$
0	0	0	0.0
0	0	1	0.0
0	1	0	0.1
0	1	1	0.2
1	0	0	0.0
1	0	1	0.0
1	1	0	0.8
1	1	1	0.9

- ✓ (i) Which, if any, of the following relations hold in view of the *network structure alone* (ignoring the conditional probability tables for now)?
- ✓ (a) $P(B, I, M) = P(B)P(I)P(M)$.
 - ✓ (b) $P(J|G) = P(J|G, I)$.
 - ✓ (c) $P(M|G, B, I) = P(M|G, B, I, J)$.
- ✓ (ii) Calculate $P(b, i, \neg m, g, j)$.
- ✓ (iii) Calculate the probability that someone goes to jail given that he or she broke the law, has been indicted, and faces a politically motivated prosecutor.

- 3.1] Let P be a Horn logic program and $M_1, M_2 \subseteq HB(P)$ classical models of P . Prove that $M_1 \cap M_2$ is also a classical model of P . What about $M_1 \cup M_2$?

Knowledge: $a_1 \vee \dots \vee a_m :- b_1, \dots, b_k, \text{not } b_{k+1}, \dots, \text{not } b_n$

A logic program is Horn if it is ~~is~~ normal and basic.

- ↳ basic if $n = k$ and $m \geq 1$
- ↳ normal if non-disjunctive and contains no strong negation \neg .

Herbrand base $HB(P)$: all ground atoms constructible from the predicate symbols and the constants appearing in P .

classical model of $a_1 \vee \dots \vee a_m :- b_1, \dots, b_k, \text{not } b_{k+1}, \dots, \text{not } b_n$ iff:

- ↳ $\{b_1, \dots, b_k\} \subseteq M$ and $\{\text{not } b_{k+1}, \dots, \text{not } b_n\} \cap M = \emptyset$, then $\{a_1, \dots, a_m\} \cap M \neq \emptyset$.
- ↳ whenever the body is true, the head must also be true.

An atom is an expression of form $p(t_1, \dots, t_n)$. p .. predicate, t_1, \dots, t_n terms.

$M_1 \cap M_2$:

Assumption: $M_1, M_2 \subseteq HB(P)$ are classical models of Horn program P

Show: $M_1 \cap M_2$ is a classical model of Horn program P

1. Rules of P are of form: $a :- b_1, \dots, b_k$ (Def. Horn Syntax)
 2. Assume $M_1 \cap M_2$ is no classical model of Horn program P
 3. There exists some $\{b_1, \dots, b_k\} \subseteq M_1 \cap M_2$ where $a \notin M_1 \cap M_2$ (Def. class. models)
 4. Due to \cap it must hold that: $\{b_1, \dots, b_k\} \subseteq M_1, M_2$ but $a \notin M_1, M_2$ (1-3, Def. \cap)
 5. Contradiction to the fact that M_1, M_2 are classical models of P .
- ↳ $M_1 \cap M_2$ is a classical model of P

$M_1 \cup M_2$:

Assumption: $M_1, M_2 \subseteq HB(P)$ are classical models of horn program P.

Show: $M_1 \cup M_2$ is a classical model of P

1. Rules of P are of form: $a :- b_1, \dots, b_k$. (Def. Horn Syntax)
2. Assume $M_1 \cup M_2$ is no classical model of horn program P.
3. There exists some $\{b_1, \dots, b_k\} \subseteq M_1 \cup M_2$ where $a \notin M_1 \cup M_2$ (1-2, Def. class. model)
4. Due to \cup it must hold that: $\{b_1, \dots, b_k\} \subseteq M_1$ where $a \notin M_1$ or $\{b_1, \dots, b_k\} \subseteq M_2$ where $a \notin M_2$ or both. (1-3, Def. \cup)
5. To fulfill at least one of these cases from (4), M_1 or M_2 is no classical model of P or both aren't. (1-4, Def. class. mod.)
6. Contradiction, because M_1 and M_2 are classical models of P.
 $\hookrightarrow M_1 \cup M_2$ is a classical model of P

$M_1 \cup M_2$: 

Ass1: $P = \{a :- b, c, d :- b, e :- c\}$

Ass2: $M_1 = \{a, b, c, d\}$... a model of P

Ass3: $M_2 = \{b, d\}$... a model of P

Ass4: $M_3 = \{c, e\}$... a model of P

$M_1 \cup M_2 = \{a, b, c, d, e\}$ is a model of P, but

$M_2 \cup M_3 = \{b, c, d, e\}$ is not a model of P.

\Rightarrow The union of 2 models need not be a model, it can be.

3.2] Consider the following disjunctive logic program P:

$$P = \left\{ \begin{array}{l} a \vee b \leftarrow \text{not } c. \\ c \leftarrow b, \text{not } d. \\ d \leftarrow \text{not } c. \\ a \leftarrow d. \end{array} \right\}$$

- (i) Determine all answer sets of P. For each proposed answer set S, argue formally that it is indeed an answer set (using the Gelfond-Lifschitz reduct P^S).
- (ii) Is it possible to add new rules Q to P such that $S^1 = \{b, c\}$ is an answer set of $P \cup Q$? If yes, give the new rules Q, if not, argue why.
- (iii) The same as in (ii) but for $S^1 = \{a, b\}$.

Knowledge: An interpretation M is an answer set of a ground program P iff it is a minimal model of P^M , i.e., there is no $N \subset M$ which is also a model of P^M . P^M is an adaption of the operator Γ_T of a default theory T.

Let P be a program and M be an interpretation. Then, the reduct P^M is defined as follows:

$$P^M = \{ a_1 \vee \dots \vee a_m \leftarrow \neg b_1, \dots, \neg b_k \mid a_1 \vee \dots \vee a_m \leftarrow b_1, \dots, b_k, \neg b_{k+1}, \dots, \neg b_n \in P \}$$

$$\{ b_{k+1}, \dots, b_n \} \cap M = \emptyset \}$$

- (i) The only (minimal) answer set is $S = \{a, d\}$. The corresponding reduct is $P^S = \{a \vee b \leftarrow \neg c, d \leftarrow \neg c, a \leftarrow d\}$. The minimal model of P^S is $\{a, d\} = S$. Therefore $S = \{a, d\}$ is an answer set of P.

(ii) $S' = \{b, c\}$ should be an answer set of $P \cup Q$. Therefore we have to add rules Q to P . This can be achieved by choosing

$$Q = \{b \leftarrow \text{not } d\}.$$

Let's prove this by investigating the reduct $(P \cup Q)^{S'}$, where

$$(P \cup Q)^{S'} = \{b \leftarrow \text{not } d, c \leftarrow b, \text{not } d, a \leftarrow d\}$$

The minimal model of $(P \cup Q)^{S'}$ is $\{b, c\} = S'$.

$\Rightarrow S'$ is an answer set of $(P \cup Q)$, as well as $S = \{a, d\}$.

(iii) $S' = \{a, b\}$ should be an answer set of $P \cup Q$. There is no set of rules Q that can be added to P so that $S' = \{a, b\}$ becomes an answer set of $P \cup Q$.

The problem is the disjunctive rule $a \vee b \leftarrow \text{not } c$ and the fact that answer sets need to be minimal. Disjunction is (subset) minimal and therefore a and b cannot be both part of the same answer set. Furthermore, if we would add some rule to infer b , a minimal subset would only contain a or b , due to this disjunction.

3.3) For a program P , we denote by $AS(P)$ the set of all answer sets of P . Let P, Q be programs. We say P, Q are

- (i) equivalent if $AS(P) = AS(Q)$ and
- (ii) strongly equivalent if $AS(P \cup R) = AS(Q \cup R)$ for every program R

Prove or refute:

- (a) (i) implies (ii)
- (b) (ii) implies (i)

(a) Show: $AS(P) = AS(Q) \Rightarrow AS(P \cup R) = AS(Q \cup R)$

1. Let's have a look at the programs $P = \{a :- b\}$ and $Q = \{a :- c\}$
2. Both have the same answer sets: $M = \{\emptyset\}$ (1, Def. ans. set)
3. $AS(P) = \{\{\emptyset\}\} = AS(Q)$ (1-2, Def. AS)
4. Define a program $R = \{b :-\}$ containing one fact
5. $AS(P \cup R) = \{\{\emptyset, b\}\}$ and $AS(Q \cup R) = \{\{\emptyset\}\}$ (1-4, Def. AS, ans. set)
6. $AS(P \cup R) \neq AS(Q \cup R)$ (1-5, pr. +)
↳ counterexample, and $AS(P) = AS(Q) \Rightarrow AS(P \cup R) = AS(Q \cup R)$

b) Show: $AS(P \cup R) = AS(Q \cup R) \Rightarrow AS(P) = AS(Q)$

1. Assume $AS(P \cup R) = AS(Q \cup R)$ for every program R
2. Assume $AS(P) \neq AS(Q)$
3. Let R_0 be an empty program $\{\}$
4. $AS(P \cup R) \neq AS(Q \cup R)$ (2-3, Def. AS, Def. \cup)
5. R_0 contradicts $AS(P \cup R) = AS(Q \cup R)$ (4, contrad. 1)
6. $AS(P) = AS(Q)$ must hold (2-5, Def. AS)
7. $AS(P \cup R) = AS(Q \cup R) \Rightarrow AS(P) = AS(Q)$ (1-6, pr. \Rightarrow)

3.4] Let S_1, S_2 be answer sets of a program P. Show that $S_1 \subseteq S_2$ implies $S_1 = S_2$.

Knowledge:

- 1) An interpretation M is an answer set of a ground program P iff it is a minimal model of P^M , i.e., there is no $N \subset M$ which is also a model of P^M .
- 2) $P^M = \{a_1, v_1..v_m \models b_1, \dots, b_k \mid a_1, v_1..v_m \models b_1, \dots, b_k, \text{not } b_{k+1}, \dots, \text{not } b_n \in P\}$
 $\{b_{k+1}, \dots, b_n\} \cap M = \emptyset\}$

Assumptions:

- 1) S_1, S_2 are answer sets of a program P

Show: $S_1 \subseteq S_2 \Rightarrow S_1 = S_2$

1. Assume that $S_1 \subseteq S_2$

2. $S_1 \subset S_2$ or $S_1 = S_2$ must hold

(1, Def. \subseteq)

3. Assume that $S_1 \subset S_2$ holds

4. Let P^{S_2} be the reduct of P to answer set S_2

(Def. reduct)

5. S_2 is minimal, because it's an answer set, so

there is no $S_1 \subset S_2$ which is also a model of P^{S_2} (3-4, ass 1, Def. reduct)

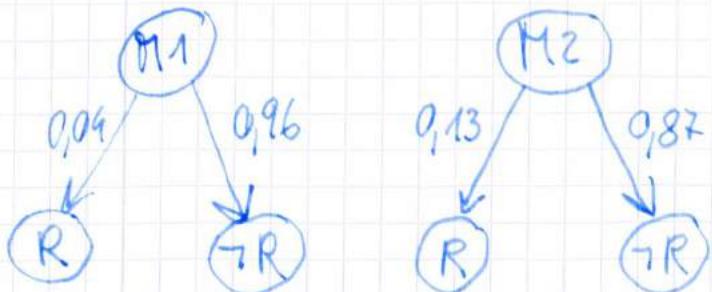
6. $S_1 \neq S_2$

(3-5)

7. $S_1 = S_2$, because $S_1 \subseteq S_2$ must hold and $S_1 \neq S_2$ (1-6)

3.5] A factory produces an item on two machines simultaneously. The produced items from machine 1 contain 4% rejects; those from machine 2 contain 13% rejects. All the items from the factory are mixed in a ratio of 3 : 1 and from this mixture one item is drawn randomly. What is the probability that a reject is drawn? What is the (conditional) probability that the item was produced by the first machine, if it is a reject?

a)



$P(R)$ denotes the probability that a drawn item is a rejected one.

$$P(R) = 0,75 \cdot 0,04 + 0,25 \cdot 0,13 = 0,0625 = 6,25\%$$

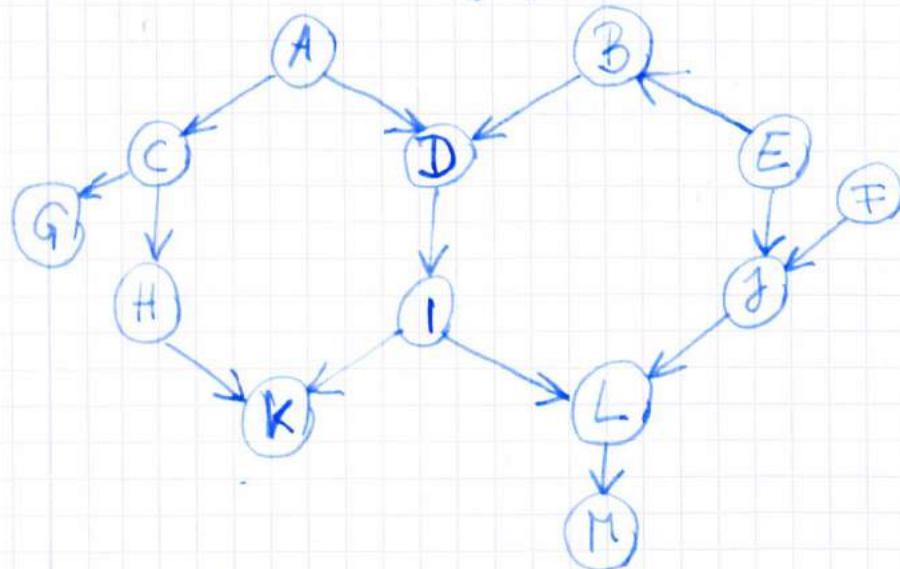
b) $P(M_1 | R)$ denotes the probability that an item was produced by machine 1 and rejected

$$P(M_1 | R) = \frac{P(M_1 \cap R)}{P(R)} = \frac{P(R | M_1) \cdot P(M_1)}{P(R)}$$

$$= \frac{0,04 \cdot 0,25}{0,0625} = 0,48$$

$$\Rightarrow P(M_1 | R) = 48\%$$

3.6] Consider the following graph of a Bayesian network:



Answer the following questions and give an explanation of your answer:

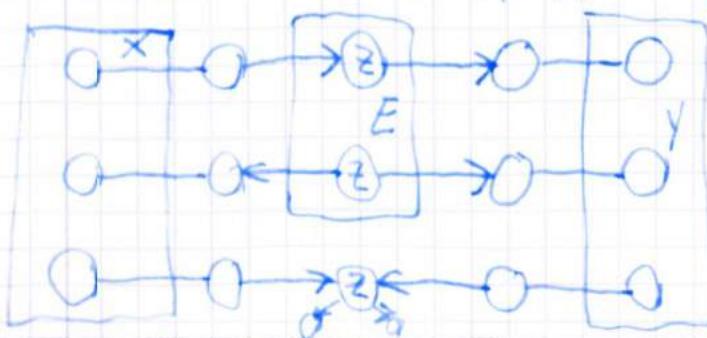
- (i) Is B conditionally independent of G?
- (ii) Is B conditionally independent of G given evidence of H?
- (iii) Which evidences are needed such that G is conditionally independent of M?
- (iv) Which evidences are needed such that A is conditionally independent of L?

Knowledge: V is conditionally independent of \vec{V}_i by given evidence \vec{E} iff $P(V|\vec{V}_i, \vec{E}) = P(V|\vec{E})$.

$P(V|\vec{W}, \text{Parents}(V)) = P(V|\text{Parents}(V))$ for any set \vec{W} of nodes which are neither parents nor descendants of V .

D-separation: Set \vec{E} of nodes path P . P is blocked relative to \vec{E} if there is some node $Z \in P$ such that one of the following cond. holds:

- 1) $Z \in \vec{E}$, one arc incoming & other outgoing
- 2) $Z \in \vec{E}$ and both arcs outgoing
- 3) Neither Z nor a descendant of Z is in \vec{E} , both arcs on P incoming



(i) B is conditionally independent of G, because all paths between B & G are blocked (D-separation (3)). D separates path $P_1 = \{B, D, A, C, G\}$ and K separates path $P_2 = \{B, D, I, H, C, G\}$. L and K separate path $P_3 = \{B, E, J, L, I, K, H, C, G\}$ and L blocks path $P_4 = \{B, E, J, L, I, D, A, C, G\}$.

(ii) B is not conditionally independent of G given evidence M, because path $P = \{B, E, J, L, I, D, A, C, G\}$ is not blocked. The evidence M make L not separating the path P anymore.

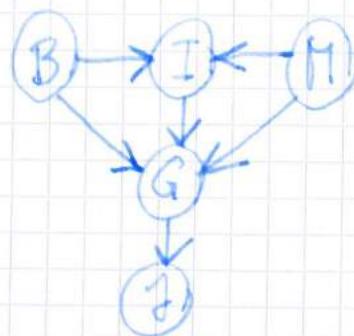
(iii) To make G conditionally independent from M, the evidence $E = \{L\}$ is needed. L then separates the paths $P_1 = \{G, C, A, D, I, L, M\}$, $P_2 = \{G, C, A, D, B, E, J, L, M\}$, $P_3 = \{G, C, H, K, I, D, B, E, J, L, M\}$ and $P_4 = \{G, C, H, K, I, L, M\}$. Furthermore, D separates P_2 and K separates P_3, P_4 .

(iv) To make A conditionally independent from L, the evidence $E = \{I\}$ is needed. I then separates $P_1 = \{A, D, I, L\}$, K separates $P_2 = \{A, C, H, K, I, L\}$ and E separates $P_3 = \{A, D, B, E, J, L\}$.

3.7 Given the following Bayesian network with Boolean variables
 B = Broke Election Law, I = Indicted, M = Politically Motivated Prosecutor,
 G = Found Guilty and J = Jailed,
where the probabilities for B and M are $P(B) = 0,9$ and $P(M) = 0,1$, respectively. The conditional probabilities for $P(G|B,I,M)$, $P(J|G)$ and $P(I|B,M)$ are follows:

B	M	$P(I B,M)$
0	0	0,1
0	1	0,5
1	0	0,5
1	1	0,9

G	$P(J G)$
1	0,9
0	0,0



B	I	M	$P(G B,I,M)$
0	0	0	0,0
0	0	1	0,0
0	1	0	0,1
0	1	1	0,2
1	0	0	0,0
1	0	1	0,0
1	1	0	0,8
1	1	1	0,9

- (i) Which, if any, of the following relations hold in view of the "network structure alone" (ignoring the conditional probability tables for now)?
- $P(B,I,M) = P(B)P(I)P(M)$.
 - $P(J|G) = P(J|G,I)$.
 - $P(M|G,B,I) = P(M|G,B,I,J)$.

(ii) Calculate $P(b,i,\neg m,g,j)$.

(iii) Calculate the probability that someone goes to jail given that he or she broke the law, has been indicted and faces a political motivated prosecutor.

(i)(a) The relation $P(B,I,M) = P(B)P(I)P(M)$ holds because the nodes B , I and M don't depend on any others, they have no incoming arcs.

(i)(b) ~~$P(J|G,I) = P(J|G,I)$ holds because~~

Knowledge:

1) Each node is conditionally independent of its non-descendants, given its parents. That is $P(V | \bar{W}, \text{Parents}(V)) = P(V | \text{Parents}(V))$, for any set \bar{W} of nodes which are neither parents nor descendants of V .

2) The network provides full information about the JPD $P(V_1, \dots, V_n)$ in the following way: $P(V_1, \dots, V_n) = \prod_{i=1}^n P(V_i | \text{Parents}(V_i))$.

(i) a) $P(B, I, M) = P(B)P(I)P(M)$ doesn't hold, because due to (2) $P(B, I, M) = P(B)P(M)P(I|M, B)$ and $P(I|M, B) = \frac{P(I, M, B)}{P(M, B)} \neq P(I)$.

b) $P(J|G) = P(J|G, I)$ holds, because node J is neither parent nor descendant of J , so respective (a) we have $\bar{W} = \{I\}$ and $P(J|\bar{W}, G) = P(J|G)$ holds.

c) $P(M|G, B, I) = P(M|G, B, I, J)$ doesn't hold, because J is a descendant of M and is conditional dependent. Only node B is neither parent nor descendant of M .

$$\begin{aligned} \text{(ii)} \quad P(b, i, \neg m, g, j) &= P(b) \cdot P(i|b, m) \cdot P(\neg m) \cdot P(g|b, i, \neg m) \cdot P(j|g) = \\ &= 0,9 \cdot 0,5 \cdot 0,9 \cdot 0,8 \cdot 0,9 = \underline{\underline{0,2916}} \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad P(j, b, i, m) &= P(j, b, i, m, g) + P(j, b, i, m, \neg g) = \\ &= P(j|g) \cdot P(b) \cdot P(i|b, m) \cdot P(m) \cdot P(g|b, i, m) + \\ &\quad P(j|\neg g) \cdot P(b) \cdot P(i|b, m) \cdot P(m) \cdot P(\neg g|b, i, m) = \\ &= 0,9 \cdot 0,9 \cdot 0,9 \cdot 0,1 \cdot 0,9 + 0 = \underline{\underline{0,06561}} \end{aligned}$$

$$P(j|b, i, m) = \frac{P(j, b, i, m)}{P(b, i, m)} = \frac{0,06561}{0,081} = 0,81 = \underline{\underline{81\%}}$$

$$P(b, i, m) = P(b) \cdot P(i|b, m) \cdot P(m) = 0,1 \cdot 0,9 \cdot 0,9 = \underline{\underline{0,081}}$$