1. Leg – joint reaction forces and moments

Given figure 1 below calculate the joint reaction forces at the knee and the hip at the 50% stance phase of the gait cycle.

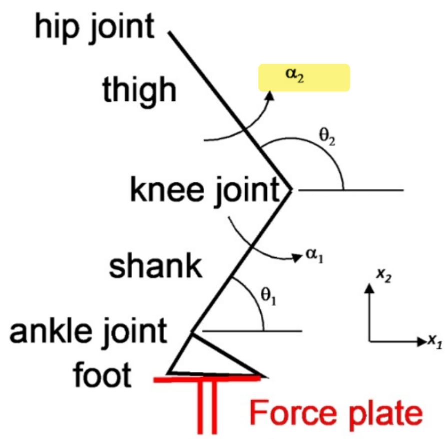


Figure 1

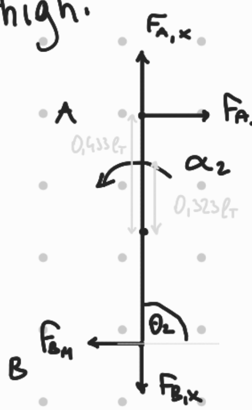
Use the following data for your calculations:

Anthropometric data

Limb segment	Segment length	Segment mass*	Centre of mass**	Radius of gyration r_g #
Thigh	32 cm	0.1 M	0.433	0.323
Shank	43 cm	0.0465 M	0.433	0.302

* Expressed as a fraction of the total body mass, M, where M = 70 kg
** Expressed as a fraction of the total segment length, measured from the proximal end of the segment
\# Expressed as a fraction of the segment length and for rotation around the center of mass

thigh:



$$I_T = m r_g^2 =$$

$$\Rightarrow M_{\alpha 2} = I_T \cdot \alpha_2 =$$

1)

$M = 70\,kg$ $\qquad g = 9{,}81\,\frac{m}{s^2}$

$m_T = 0{,}1\,M$ $\ell_T = 32\,cm$ $\ell_{cT} = 0{,}433\,\ell_T$ $r_T = 0{,}323\,\ell_T$ $\theta_2 = 88{,}7°$ ← thigh angle	$a_{T1} = 1{,}15\,\frac{m}{s^2}$ ← thigh acc. x1 $a_{T2} = -1{,}46\,\frac{m}{s^2}$ ← thigh acceleration x_1 $\alpha_2 = 0{,}01\,\frac{rad}{s^2}$ ← thigh acceleration α_1
$m_s = 0{,}0465\,M$ $\ell_s = 43\,cm$ $\ell_{cs} = 0{,}433\,\ell_s$ $r_s = 0{,}302\,\ell_s$ $\theta_1 = 78{,}7°$ ↑ shank angle	$a_{S1} = 1{,}21\,\frac{m}{s^2}$ ← shank acc. x_2 $a_{S2} = -0{,}95\,\frac{m}{s^2}$ ← shank acc. x_2 $\alpha_1 = -0{,}22\,\frac{rad}{s^2}$ ← shank acc. α_1 $R_{A1} = 21{,}6\,N$ reaction force x_1 $R_{A2} = -453{,}5\,N$ reaction force x_2 $M_A = -30{,}3\,Nm$ muscle moment

Shank:

X_1: $R_{K1} - R_{A1} = m_s \cdot a_{s1}$

$R_{K1} = m_s \cdot a_{s1} + R_{A1}$

$\approx 25{,}539\,N$

X_2: $R_{K2} - R_{A2} - m_s \cdot g = m_s \cdot a_{s2}$

$R_{K2} = m_s \cdot a_{s2} + R_{A2} + m_s \cdot g$

$\approx -424{,}661\,N$

$I_s = m_s r_s^2$

$M_1 = I_s \cdot \alpha_1 = m_s r_s^2 \cdot \alpha_1 \approx -0{,}012\,Nm$

$M_R = R_{A2}(\ell_s - \ell_{cs})\cos\theta_1 - R_{A1}(\ell_s - \ell_{cs})\sin\theta_1 \approx -26{,}830\,Nm$

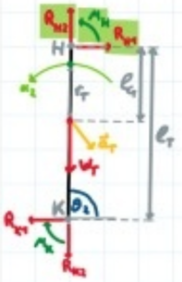
$M_{R_{KT}} = R_{K2}\,\ell_{cs}\cos\theta_1 - R_{K1}\,\ell_{cs}\sin\theta_1 \approx -20{,}156\,Nm$

$M_K - M_A + M_R + M_{R_{KT}} = M_1$

$M_K = M_1 + M_A - M_R - M_{R_{KT}}$

$\approx 16{,}673\,Nm$

Thigh:



X_1: $R_{H1} - R_{K1} = m_T\,a_{T1}$

$R_{H1} = m_T\,a_{T1} + R_{K1}$

$\approx 33{,}589\,N$

X_2: $R_{H2} - R_{K2} - m_T g = m_T\,a_{T2}$

$R_{H2} = m_T\,a_{T2} + R_{K2} + m_T g$

$\approx -366{,}211\,N$

$I_T = m_T r_T^2$

$M_2 = I_T \cdot \alpha_2 = m_T r_T^2 \alpha_2 \approx 7{,}478 \cdot 10^{-4}\,Nm$

$M_{R_{KS}} = R_{K2}(\ell_T - \ell_{cT})\cos\theta_2 - R_{K1}(\ell_T - \ell_{cT})\sin\theta_2 \approx -6{,}381\,Nm$

$M_{R_H} = R_{H2}\,\ell_{cT}\cos\theta_2 - R_{H1}\,\ell_{cT}\sin\theta_2 \approx -5{,}804\,Nm$

$M_H - M_K + M_{R_{KS}} + M_{R_H} = M_2$

$M_H = M_2 + M_K - M_{R_{KS}} - M_{R_H}$

$\approx 28{,}859\,Nm$

2. Strain Gauge Rosette

A strain gauge rosette measurement on a human femoral neck during a compression experiment (Figure 2) gives the following output:

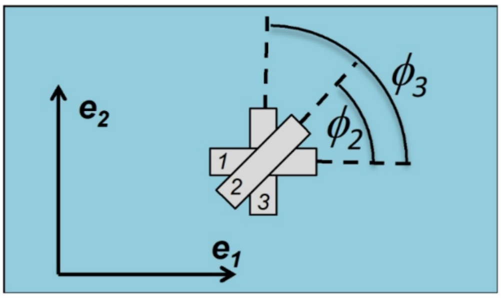


Figure 2

$\varepsilon_{n1} = 0.004$ $\qquad$ $\varepsilon_{n2} = 0.002$ $\qquad$ $\varepsilon_{n3} = -0.001$

Gauge 1 is aligned with the e_1 direction whereas gauges 2 and 3 are at angles of $\Theta_2 = 45°$ and $\Theta_3 = 90°$, respectively.

Using this information determine the 2D strain tensor at the measurement point.

$$\varphi_1 = 0° \rightarrow n_1 = \begin{pmatrix} \cos(\varphi_1) \\ \sin(\varphi_1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\varphi_2 = 45° \rightarrow n_2 = \begin{pmatrix} \cos(\varphi_2) \\ \sin(\varphi_2) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\varphi_3 = 90° \rightarrow n_3 = \begin{pmatrix} \cos(\varphi_3) \\ \sin(\varphi_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{\varepsilon_{ni} = n_i \cdot \underline{\underline{\varepsilon}}\, n_i}$$

$$\varepsilon_{n1} = n_1 \cdot \underline{\underline{\varepsilon}} \cdot n_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \end{pmatrix} = \varepsilon_{11}$$

$$\varepsilon_{n3} = n_3\, \underline{\underline{\varepsilon}}\, n_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{12} \\ \varepsilon_{22} \end{pmatrix} = \varepsilon_{22}$$

$$\varepsilon_{n2} = n_2\, \underline{\underline{\varepsilon}}\, n_2 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{11} + \varepsilon_{12} \\ \varepsilon_{12} + \varepsilon_{22} \end{pmatrix} = \frac{1}{2} (\varepsilon_{11} + 2\varepsilon_{12} + \varepsilon_{22})$$

$$= \frac{1}{2} (\varepsilon_{n1} + \varepsilon_{n3}) + \varepsilon_{12}$$

$$\Rightarrow \varepsilon_{12} = \varepsilon_{n2} - \frac{1}{2} (\varepsilon_{n1} + \varepsilon_{n3})$$

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{n1} & \varepsilon_{n2} - \frac{1}{2}(\varepsilon_{n1} + \varepsilon_{n3}) \\ \varepsilon_{n2} - \frac{1}{2}(\varepsilon_{n1} + \varepsilon_{n3}) & \varepsilon_{n3} \end{pmatrix} \approx \begin{pmatrix} 0{,}004 & 0{,}0005 \\ 0{,}0005 & -0{,}001 \end{pmatrix}$$

3. Strain and Stress Transformation

A rotation matrix $\mathbf{R}(\theta) = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ produces a rotation in a counter-clockwise fashion (as shown in figure 4).

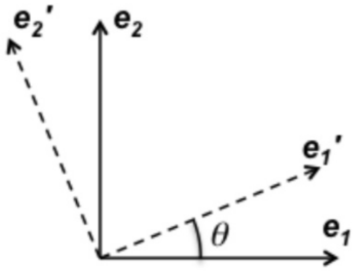


Figure 4

Using this rotation matrix within the transformation law for 2nd rank tensors:

$$\underline{\underline{\varepsilon}}' = \mathbf{R}^T \underline{\underline{\varepsilon}} \mathbf{R}$$

i) Calculate the strain tensor $\underline{\underline{\varepsilon}}'$ in the rotated reference frame.

ii) Further consider how a state of pure shear strain $\begin{pmatrix} 0 & \varepsilon_{12} \\ \varepsilon_{12} & 0 \end{pmatrix}$ transforms upon a rotation of Θ = 45°.

i) $\varepsilon = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \qquad R^T = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$

$$\Rightarrow \varepsilon' = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \varepsilon_{11}\cos(\theta) + \varepsilon_{12}\sin(\theta) & -\varepsilon_{11}\sin(\theta) + \varepsilon_{12}\cos(\theta) \\ \varepsilon_{21}\cos(\theta) + \varepsilon_{22}\sin(\theta) & -\varepsilon_{21}\sin(\theta) + \varepsilon_{22}\cos(\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta)(\varepsilon_{11}\cos(\theta) + \varepsilon_{12}\sin(\theta)) + \sin(\theta)(\varepsilon_{21}\cos(\theta) + \varepsilon_{22}\sin(\theta)) & \cos(\theta)(-\varepsilon_{11}\sin(\theta) + \varepsilon_{12}\cos(\theta)) + \sin(\theta)(-\varepsilon_{21}\sin(\theta) + \varepsilon_{22}\cos(\theta)) \\ -\sin(\theta)(\varepsilon_{11}\cos(\theta) + \varepsilon_{12}\sin(\theta)) + \cos(\theta)(\varepsilon_{21}\cos(\theta) + \varepsilon_{22}\sin(\theta)) & -\sin(\theta)(-\varepsilon_{11}\sin(\theta) + \varepsilon_{12}\cos(\theta)) + \cos(\theta)(-\varepsilon_{21}\sin(\theta) + \varepsilon_{22}\cos(\theta)) \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon_{11}\cos^2(\theta) + (\varepsilon_{12}+\varepsilon_{21})\sin(\theta)\cos(\theta) + \varepsilon_{22}\sin^2(\theta) & (\varepsilon_{22}-\varepsilon_{11})\sin(\theta)\cos(\theta) + \varepsilon_{12}\cos^2(\theta) - \varepsilon_{21}\sin^2(\theta) \\ \sin(\theta)\cos(\theta)(\varepsilon_{22}-\varepsilon_{11}) - \varepsilon_{12}\sin^2(\theta) + \varepsilon_{21}\cos^2(\theta) & \varepsilon_{11}\sin^2(\theta) - (\varepsilon_{21}+\varepsilon_{12})\sin(\theta)\cos(\theta) + \varepsilon_{22}\cos^2(\theta) \end{pmatrix}$$

ii) $\varepsilon_{21} = \varepsilon_{12}, \quad \varepsilon_{11} = \varepsilon_{22} = 0, \quad \sin(\theta) = \cos(\theta) = \frac{1}{\sqrt{2}}$

$$\Rightarrow \varepsilon' = \begin{pmatrix} 2\varepsilon_{12}\cdot\frac{1}{2} & 0 \\ 0 & -2\varepsilon_{12}\cdot\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \varepsilon_{12} & 0 \\ 0 & -\varepsilon_{12} \end{pmatrix}$$

4. Generalised Hooke's law – shear component

Consider a pure shear test, i.e. a stress state: $\underline{\underline{\sigma}} = \begin{pmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{e_1,e_2}$

Upon a rotation of $\theta = 45\mathrm{deg}$ we get a stress state similar to the state in exercise 4 ii. Using Hooke's law in the rotated coordinate system (including the Poisson effect) calculate the corresponding strains and rotate these back to the original configuration.

Verify the shear components of the generalised Hooke's law for isotropic, homogeneous materials:

$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}$

$$R_z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$\underline{\underline{\sigma}}' = R^T \underline{\underline{\sigma}} R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \sigma_{12} & \sigma_{12} & 0 \\ \sigma_{12} & -\sigma_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2\sigma_{12} & 0 & 0 \\ 0 & -2\sigma_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{12} & 0 & 0 \\ 0 & -\sigma_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} \sigma_{11}' & \sigma_{12}' & 0 \\ \sigma_{21}' & \sigma_{22}' & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_{11}' \\ \varepsilon_{22}' \\ 2\varepsilon_{12}' \end{pmatrix} = \frac{1}{E}\begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{pmatrix}\begin{pmatrix} \sigma_{11}' \\ \sigma_{22}' \\ \sigma_{12}' \end{pmatrix}$$

$$= \frac{1}{E}\begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{pmatrix}\begin{pmatrix} \sigma_{12} \\ -\sigma_{12} \\ 0 \end{pmatrix} = \frac{1}{E}\begin{pmatrix} \sigma_{12}(1+\nu) \\ -\sigma_{12}(1+\nu) \\ 0 \end{pmatrix} = \frac{1+\nu}{E}\begin{pmatrix} \sigma_{12} \\ -\sigma_{12} \\ 0 \end{pmatrix}$$

$$\underline{\underline{\varepsilon}}' = \frac{1+\nu}{E}\begin{pmatrix} \sigma_{12} & 0 & 0 \\ 0 & -\sigma_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{\varepsilon}} = R\,\underline{\underline{\varepsilon}}'\,R^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\frac{1+\nu}{E}\begin{pmatrix} \sigma_{12} & 0 & 0 \\ 0 & -\sigma_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$= \frac{1+\nu}{2E}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \sigma_{12} & \sigma_{12} & 0 \\ \sigma_{12} & -\sigma_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1+\nu}{2E}\begin{pmatrix} 0 & 2\sigma_{12} & 0 \\ 2\sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1+\nu}{E}\begin{pmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \varepsilon_{12} = \sigma_{12}\frac{1+\nu}{E}$$

5. Forearm fracture

A common cause for distal forearm fractures is a fall on the hyperextended hand (Fig. 1A). Here, the radius bone is "locked" at the wrist (Fig. 1B), and experiences loading as shown in the free body diagram (Fig. 1D), where 80% of the force F_L is assumed to act on the radius.

Model the radius as a hollow cylinder of cortical bone with a constant cross-sectional area and an inner radius r_i and outer radius r_o (Fig. 1C).

r_o = 8 mm - outer radius; r_i = 4 mm - inner radius; a= 22 cm - length of the radius ; θ = 15° - the angle between loading force and the axis of the bone (e_x)

i.) Assume that $|F_L|$ = 10 kN, and calculate the bending moment in the radius as a function of x. Where is the maximum bending moment?

ii.) Calculate the maximum tensile stress and the maximum compressive stress in the radius. Where are these located?

iii.) The Young's modulus of cortical bone is given as 17 GPa (assumed to be the same for tension and compression) the ultimate strain as 1.6%. Assuming linear elastic deformation until failure, will the radius fracture?

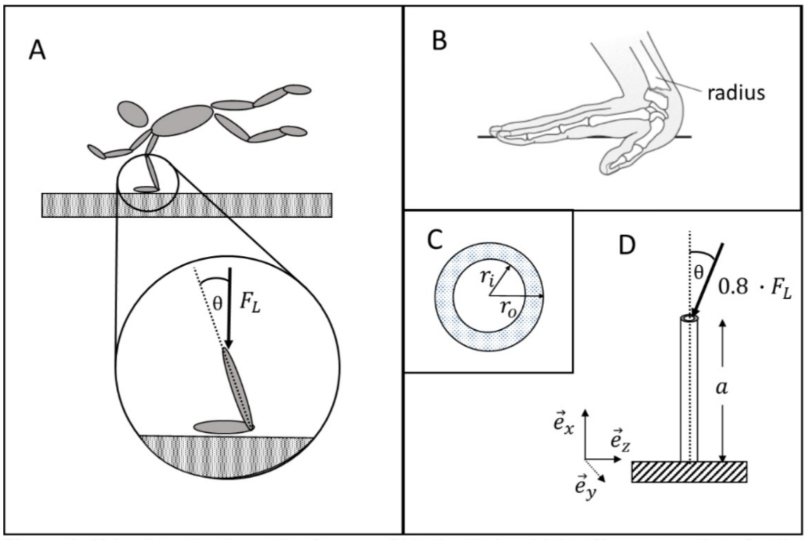
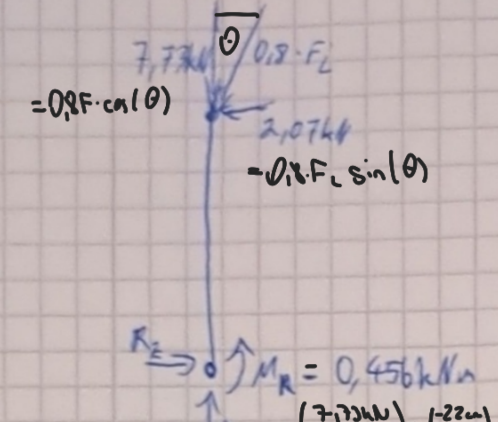


Figure 1: A) loading situation at the forearm, B) locked in hand joint, C) cross-section of radius bone D) free body diagram.

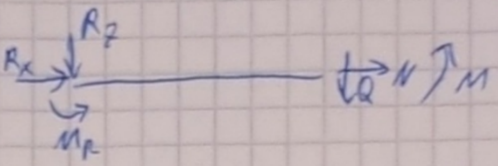
5.) 1)

$F_L = 10 kN$



$$\begin{pmatrix} 7{,}73kN \\ 2{,}07kN \\ 0 \end{pmatrix} \times \begin{pmatrix} -22cm \\ 0 \\ 0 \end{pmatrix}$$

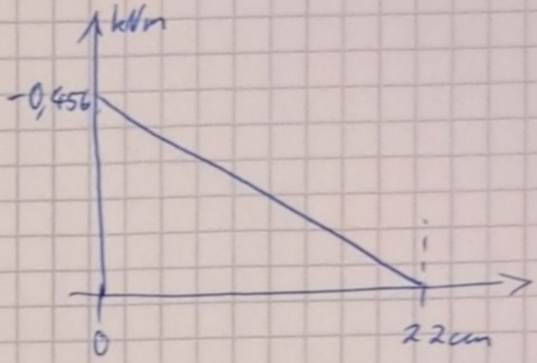
$= 2{,}07kN \cdot 22cm$

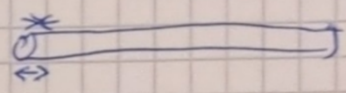


$M = -M_R + Q \cdot x$

$-0{,}456 + (-2{,}07kN)x$

$(0-22cm)$





2.) $\sigma = \frac{M c}{I}$

$I = \frac{\pi}{4}(R^4 - r^4) = \cancel{32{,}169 cm^4}$ (8mm 4mm)

$3{,}0159 e^{-9} m^4$

$\sigma = \frac{456 Nm \cdot 0{,}008}{I} = 1{,}2096 GPa$ (outer radius)

$\sigma_N = \frac{F}{A} = \frac{0{,}8 \cdot F_L \cdot \cos(\theta)}{\pi(r_o^2 - r_i^2)}$

⊕ $\sigma_{max} = 1{,}163 GPa$

3.) $\varepsilon = \frac{1{,}2 GPa}{17 GPa} = 7\% > 1{,}6\% \Rightarrow$ fracture, breaks!

$\sigma = E\varepsilon = 17GPa \cdot 0{,}016 = 272 MPa$ <

$\sigma_{min} = \sigma - \sigma_N = -1{,}266 GPa$