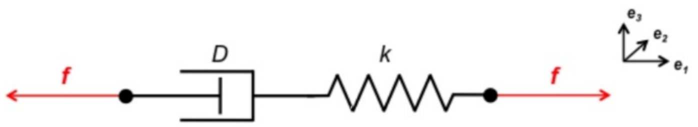
1 Mechanical elements - Maxwell body

Consider the following lumped parameter model of the Maxwell body:



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Assuming the extension to instantaneously rise from “no extension” to a magnitude x_0 (relaxation experiment):

- derive an expression for the force $f_1(t)$ of the body
- sketch the progression of $f_1(t)$ qualitatively

constant length

In a second example, assume the force to instantaneously rise to a magnitude f_0, where it stays constant (creep experiment):

- derive an expression for the extension $x_1(t)$ of the body
- sketch the progression of $x_1(t)$ qualitatively

constant force

Hint: The extension in the dashpot and spring is different, whereas the force acting on them is the same

To check your results: In your sketches of $f_1(t)$ and $x_1(t)$, how does the system behave for $t=0$? How does the system behave for $t\to\infty$? Is this behaviour what you would intuitively expect to happen?

1) $f(t,x) = \overbrace{k\,x_k}^{f_k} + \overbrace{D \cdot \frac{dx_d}{dt}}^{f_D}$

$x_k + x_d = x_0 \;|\; \dot{x}_k + \dot{x}_d = 0$ (coupled to each other) $\Rightarrow f_k = f_D = f_1$

$\frac{1}{k} \cdot \dot{f}_1 + \frac{1}{D} \cdot f_1 = 0$

General approach: $f_1 = Ce^{\lambda t}$

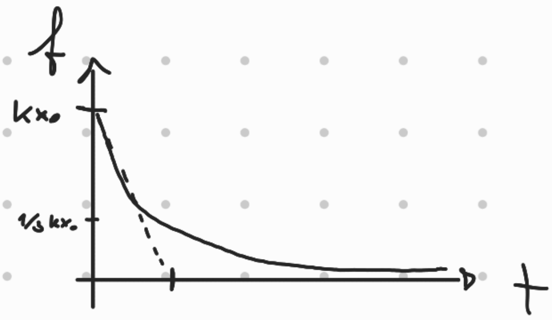
$\dot{f}_1 = -\frac{k}{D} f_1$

$\lambda \cancel{Ce^{\lambda t}} = -\frac{k}{D} \cancel{Ce^{\lambda t}} \Rightarrow \lambda = -\frac{k}{D}$

$\Rightarrow f_1(t) = Ce^{-\frac{k}{D}t}$

$f_1(0) = C = k \cdot x_0$

$\Big\} f_1 = k \cdot x_0 e^{-\frac{k}{D}t}$



2) $x_k = \frac{f_k}{k}$

$x_D = \frac{1}{D} \int f_D \, dt = \frac{1}{D} \cdot f_D \cdot t + c$

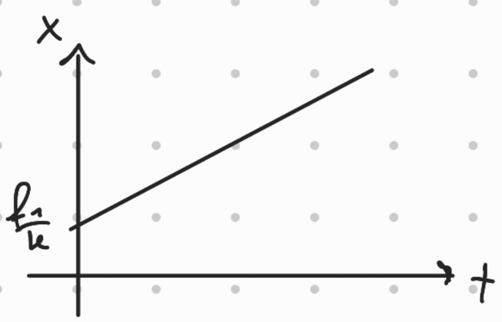
$x_D(0) = 0 = c$

$\Rightarrow x_D = \frac{f_D}{D} \cdot t$

$f_k = f_D = f_1$

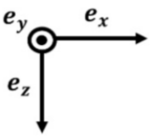
$x_1(t) = x_k + x_D = f_1 \left(\frac{1}{k} + \frac{1}{D} \cdot t \right)$

$x_1(0) = \frac{f_1}{k}$



2 Statics - Beams

Determine the required reaction forces and moments of the weightless beam. Sketch the internal forces n(x) (normal force) and v(x) (shear force), and the moment curve M(x).



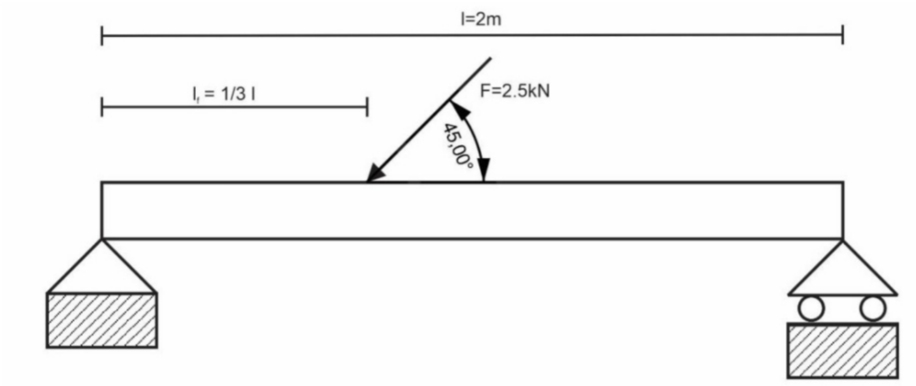
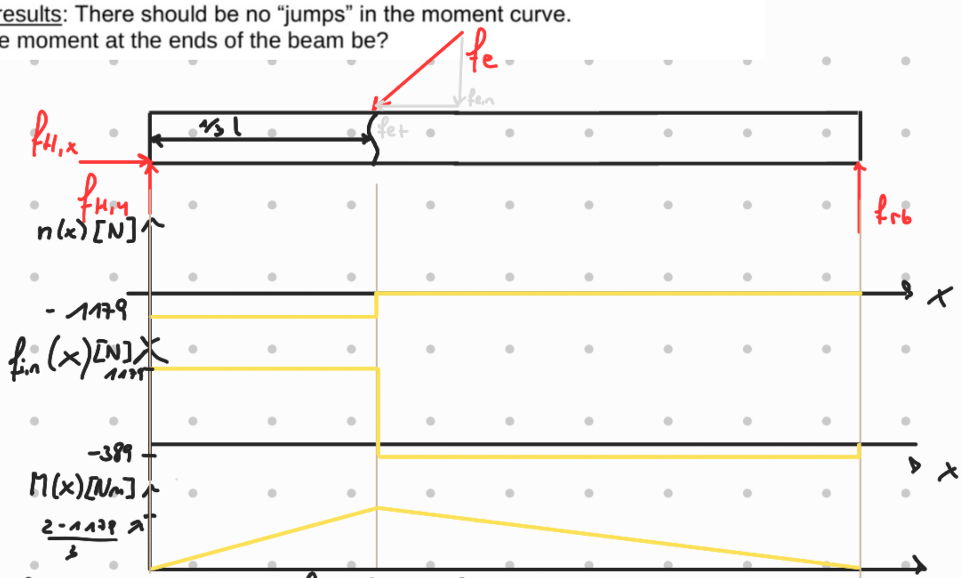


Figure 1: Beam

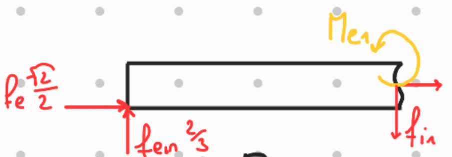
<u>To check your results</u>: There should be no "jumps" in the moment curve. What should the moment at the ends of the beam be?

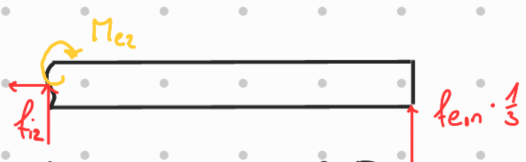


$$f_{en} = f_e \cdot \sin(45^\circ) = f_e \cdot \frac{\sqrt{2}}{2}$$
$$f_{et} = f_e \cdot \cos(45^\circ) = f_e \cdot \frac{\sqrt{2}}{2}$$
$$f_{e,n} = f_{e,t} = 1768\,N$$

$$x: \quad \Sigma f = 0: \quad 0 = f_{H,x} - f_{e,t} \Rightarrow f_{H,x} = f_e \cdot \frac{\sqrt{2}}{2}$$
$$y: \quad \Sigma f = 0: \quad 0 = f_{e,n} - f_{H,y} - f_{rb}$$
$$\Sigma M = 0: \quad f_{e,n} \cdot \frac{l}{3} = f_{rb} \cdot l \Rightarrow f_{rb} = f_{e,n} \cdot \frac{1}{3}$$
$$\Rightarrow f_{H,y} = f_{en} - f_{en}\frac{1}{3} = \frac{2}{3} f_{en}$$

right hand rule





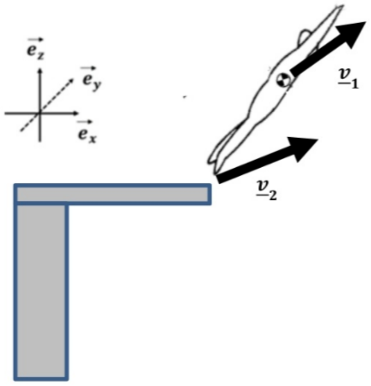
$$f_{i1,x} = -f_e \frac{\sqrt{2}}{2}$$
$$f_{i1,y} = \frac{2}{3} f_{en}$$
$$M_1 = \frac{2}{3} f_{en} \cdot x$$

$$f_{i2,x} = f_{i1,x} = -f_e \frac{\sqrt{2}}{2} \; ?$$
$$f_{i2,y} = -f_{en} \cdot \frac{1}{3}$$
$$M_2 = \frac{1}{3} f_{en} \cdot (l - x)$$

3 Dynamics

3.1 Jump

A diver jumps down from a platform conducting a forward flip. Right after take-off, his centre of mass (COM) moves with a velocity $\underline{v_1}$. The outer edge of his foot is located 32 cm to the left of the COM and 80 cm below it, as described by the vector $\underline{d}$.The outer edge of the divers' foot moves with a velocity $\underline{v_2}$.



$$\underline{v_1} = \begin{pmatrix} 2 \\ 0 \\ 3.8 \end{pmatrix} m/s\,, \quad \underline{v_2} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} m/s$$

$$\overline{X_{COM}X_{FOOT}} = \underline{d} = \begin{pmatrix} -0.32 \\ 0 \\ -0.8 \end{pmatrix} m$$

Figure 2: Diver jumping from a platform

While the diver is in the air, no forces are acting on his body, apart from gravity (e.g. air drag can be neglected), and his body does not deform.

a) Calculate the angular velocity vector of the divers' rotation around his centre of mass during the flip.
   Hint 1: You can use the right-hand rule, to calculate the angular velocity vectors direction.
   Hint 2: $\hat{e_z} = \hat{e_r} \times \hat{e_\Theta}$, (depending on how the example is solved, you might not need both hints).

b) At take-off the COM is 2 m above the ground. Which maximum height above ground will the COM reach during flight?

To check your results: b) 2.74 m

a) $v = \omega \times r$

$V_2 - V_1 = \omega \times d$

$$\begin{pmatrix} -3 \\ 0 \\ 1{,}2 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_y \\ 0 \end{pmatrix} \times \begin{pmatrix} -0{,}32 \\ 0 \\ -0{,}8 \end{pmatrix} = \begin{pmatrix} -0{,}8\omega_y \\ 0 \\ 0{,}32\omega_y \end{pmatrix} \Rightarrow \omega_y = \frac{3}{0{,}8} = \frac{1{,}2}{0{,}32} = 3{,}75 \frac{rad}{s}$$

Alternative way:

$\vec{V}_{rel} = \vec{V}_2 - \vec{V}_1 = \begin{pmatrix} -3 \\ 0 \\ 1{,}2 \end{pmatrix}$

$\|\vec{\omega}\| = \frac{\|V_{rel}\|}{\|d\|}$

b) $E_{ges} = E_{pot} + E_{kin}$

in $t=0$:

$0 = mg(h-h_0) + \frac{m v_z^2}{2}$

$0 = -9{,}81 \frac{m}{s^2}(h-h_0) + \frac{3{,}8^2}{2}$

$\Rightarrow h = \frac{3{,}8^2}{2} \cdot \frac{1}{9{,}81} + 2 = 2{,}74\,m$

alternative:

$v = v_0 + at \;/\,^2$

$\left.\begin{array}{l} v^2 = v_0^2 + 2v_0at + a^2t^2 \\ \text{and } \Delta x = v_0t + \frac{at^2}{2} \end{array}\right\}$

$\Rightarrow v_z^2 = v_0^2 - 2g(h_{max} - h_0)$

$h_{max} = 2{,}74\,m$

3.2 Moments of Inertia

From the general expression for the moment of inertia around COM

$$I = \int_V^{\square} \underline{r}_\perp^{\,2} \rho(\underline{r})\, dV$$

(with $\underline{r}_\perp$ being the normal distance between point $\underline{r}$ and the rotation axis)

a) Derive the formula for the moment of inertia of a homogeneous cylinder, rotating around its symmetry axis. **
Hint: use cylindrical coordinates!

b) Calculate the moment of inertia for the objects depicted below, around the axis of rotation denoted in the sketch with ×. Give your result in SI units!

- bicycle wheel: - use your result from a) - approximate as a hollow cylinder (centre of the cylinder at the wheel hub) with average density $\varrho = 1$ g/cm³
  outer diameter $d_1 = 74$ cm, inner diameter $d_2 = 70$ cm, cylinder height = thickness of the wheel = 2 cm
- sledgehammer: - use a formula collection -
  head: cylinder, height $h_1 = 14$ cm, diameter $d_1 = 10$ cm, $\varrho_{steel} = 8$ g/cm³
  handle: thin cylinder – height $h_2 = 100$ cm, diameter $d_2 = 4$ cm, $m = 1$ kg
  The axis of rotation is around the end of the wooden part as indicated in the sketch.

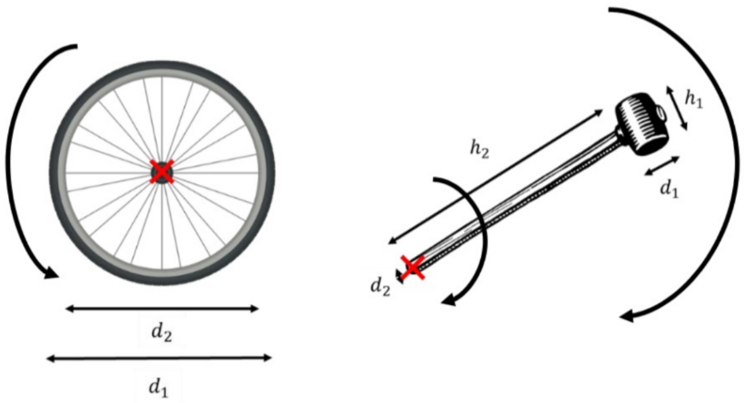
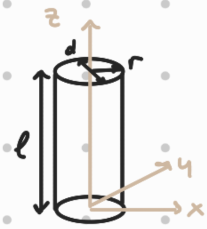


Figure 3. Bicycle wheel and sledgehammer

**You don´t need to derive the formula of the moment of inertia in the exam, but you need to understand where it comes from

To check your results: a) $I = 1/2\ m\ R^2$
b) wheel: $I = 0.117$ kgm², sledgehammer $I = 10.05$ kgm²

a) homogeneous: $\rho(r) \to \rho$



$$I_z = \rho \int_V r^2\, dV \qquad \text{(cylinder coordinates)}$$

$$= \rho \int_0^{2\pi}\int_0^{\ell}\int_0^{r} r'^3\, dr'\, dz\, d\varphi$$

$$= \rho \int_0^{2\pi}\int_0^{\ell} \frac{r'^4}{4}\, dz\, d\varphi = \rho \frac{r'^4}{4} \cdot \ell \cdot 2\pi = \frac{1}{2} \rho r'^4 \cdot \ell \cdot \pi$$

$$r = \frac{d}{2} \Rightarrow \frac{1}{2} \rho \cdot d^4 \cdot \frac{1}{16} \cdot \ell \cdot \pi$$

$$I_z = \frac{1}{32} \rho \cdot d^4 \ell \pi \qquad \left(m = V \cdot \rho = \frac{d^2}{4} \cdot \pi \cdot \ell \cdot \rho\right)$$

$$= \frac{md^2}{8} = \frac{mr^2}{2}$$

b) bicycle wheel:

$$m = \left(\frac{d_a^2}{4} - \frac{d_i^2}{4}\right)\pi \cdot \ell \cdot \rho = 0{,}904\,\text{kg}$$

$$I = \frac{m}{2}\left(\frac{r_a^2}{4} + \frac{r_i^2}{4}\right) = 0{,}1173\ \text{kgm}^2$$

sledgehammer:

$$m_{head} = 8{,}796\,\text{kg}$$

$$m_{handle} = 1\,\text{kg}$$

from formula collection

$$\left.\begin{aligned} I_{head} &= \frac{1}{4} m r^2 + \frac{1}{12} m \ell^2 = 9{,}72\ \text{kgm}^2 \\ I_{handle} &= \frac{1}{3} m \ell^2 = 0{,}\dot{3}\ \text{kgm}^2 \\ I_{head,shifted} &= I_{head} + M \underset{\uparrow\ h_2 + \frac{d_1}{2}}{d^2} = 9{,}72\ \text{kgm}^2 \end{aligned}\right\} I_{ges} = 10{,}05\ \text{kgm}^2$$