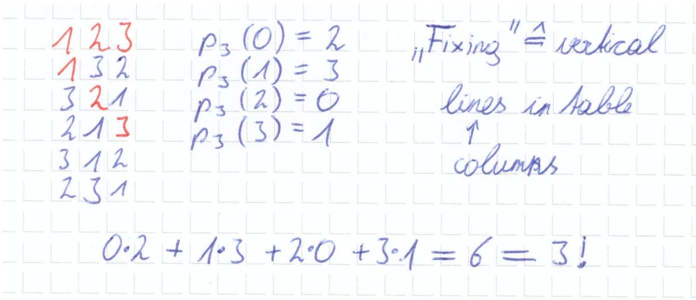
Exercise 5

Discrete Mathematics

November 12, 2020

Exercise 41

Note that the left hand side of the equality is the total number of fixed points in all permutations of $\{1, 2, \ldots, n\}$. To show that this number is equal to $n!$, note that there are $(n-1)!$ permutations of $\{1, 2, \ldots, n\}$ fixing 1,$(n-1)!$ permutations fixing 2, and so on, and $(n-1)!$ permutations fixing n. It follows that the total number of fixed points in all permutations is equal to $n \cdot (n-1)! = n!$.



Exercise 42

Consider the binomial theorem $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Substitute $x = -1$. As long as $n > 0$, we get 0 on the left side, so we obtain $0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$. Move the negative terms to the left side to get

$$\sum_{\substack{0 \leq k \leq n \\ k \text{ odd}}} \binom{n}{k} = \sum_{\substack{0 \leq k \leq n \\ k \text{ even}}} \binom{n}{k}.$$

The sum on the left side is the number of odd-sized subsets and the sum on the right is the number of even-sized subsets.

See Exam 3 in http://math.colorado.edu/ jonathan.wise/teaching/math2001-fall-2014/

Exercise 43

1. We can place the first rook anywhere on the board, i.e. $n \cdot n$ choices. The second rook cannot be placed in the row or column of the first one, i.e. $(n-1) \cdot (n-1)$ choices. This way, the last rook has exactly $1 \cdot 1$ choice. This gives us $\Pi_{i=1}^{n} i^2 = (n!)^2$. However, we still have to take into account, that the rooks are not labeled. Therefore the final result is
$$\frac{(n!)^2}{n!} = n!$$

2. Each rook has to be on a different row. There is only one such way to assign rooks to rows (rooks are indistinguishable). Then for the first rook, there is a choice of n columns, for the second of $n-1$ columns, for the third of $n-2$ columns and so on so in total there are $n!$ possibilities.

Exercise 44

https://math.stackexchange.com/q/3013488

We use double counting.

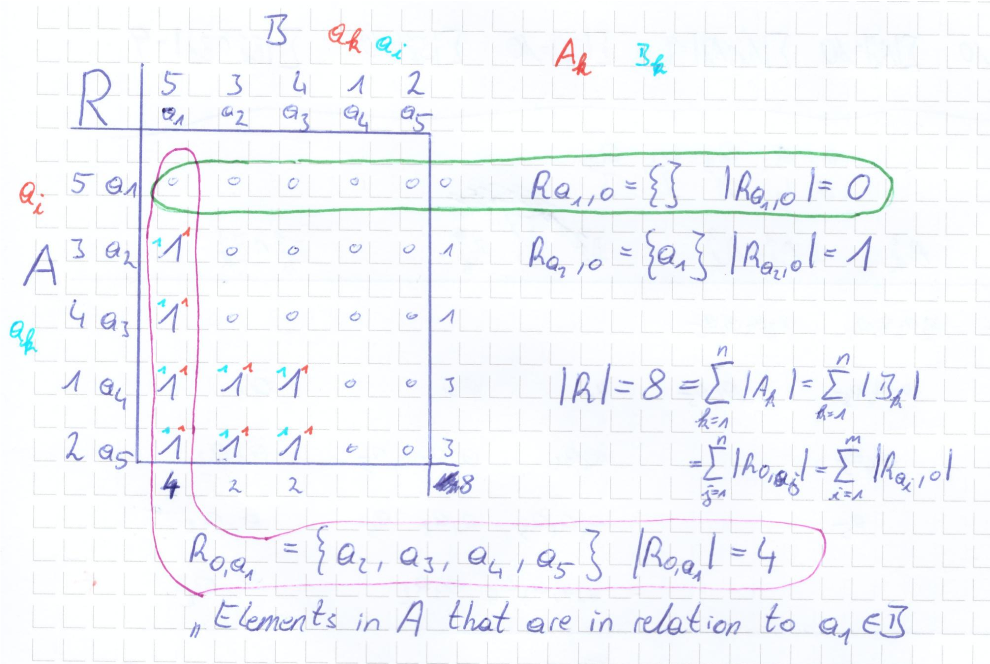
Let $A = B = \{a_1, a_2, \ldots, a_n\}, R \subseteq A \times B$.

$$R_{i,0} = \{b \in B : (a_i, b) \in R\}, \quad R_{0,j} = \{a \in A : (a, b_j) \in R\}$$

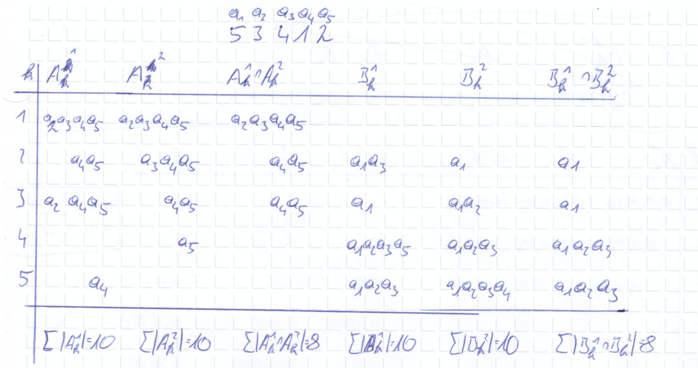
and define R as

$$(a_i, a_k) \in R :\Longleftrightarrow a_i < a_k, i > k \quad 1 \le k \le n$$

the result is proven in the image



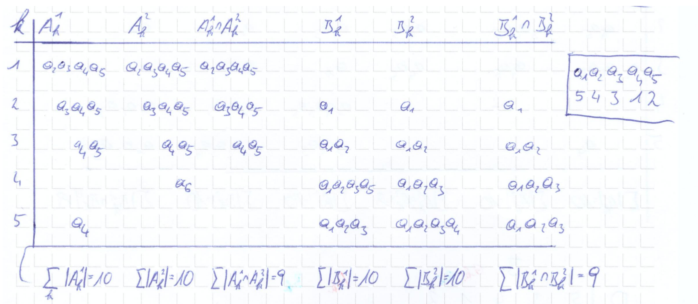
This example is relevant for the double counting explanation



	a_1	a_2	a_3	a_4	a_5
	5	3	4	1	2

k	A_k^1	A_k^2	$A_k^1 \cap A_k^2$	B_k^1	B_k^2	$B_k^1 \cap B_k^2$
1	$a_2a_3a_4a_5$	$a_2a_3a_4a_5$	$a_2a_3a_4a_5$			
2	a_4a_5	$a_3a_4a_5$	a_4a_5	a_1a_3	a_1	a_1
3	$a_2a_4a_5$	a_4a_5	a_4a_5	a_1	a_1a_2	a_1
4		a_5		$a_1a_2a_3a_5$	$a_1a_2a_3$	$a_1a_2a_3$
5	a_4			$a_1a_2a_3$	$a_1a_2a_3a_4$	$a_1a_2a_3$
	$\sum\lvert A_k^1\rvert=10$	$\sum\lvert A_k^2\rvert=10$	$\sum\lvert A_k^1\cap A_k^2\rvert=8$	$\sum\lvert B_k^1\rvert=10$	$\sum\lvert B_k^2\rvert=10$	$\sum\lvert B_k^1\cap B_k^2\rvert=8$

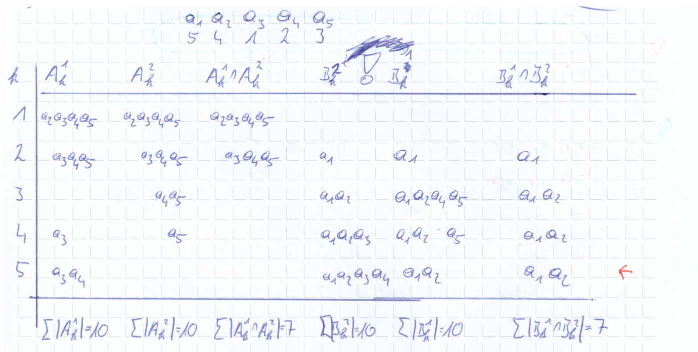
This is just another one



k	A_k^1	A_k^2	$A_k^1 \cap A_k^2$	B_k^1	B_k^2	$B_k^1 \cap B_k^2$
1	$a_2a_3a_4a_5$	$a_2a_3a_4a_5$	$a_2a_3a_4a_5$			
2	$a_3a_4a_5$	$a_3a_4a_5$	$a_3a_4a_5$	a_1	a_1	a_1
3	a_4a_5	a_4a_5	a_4a_5	a_1a_2	a_1a_2	a_1a_2
4		a_6		$a_1a_2a_3a_5$	$a_1a_2a_3$	$a_1a_2a_3$
5	a_4			$a_1a_2a_3$	$a_1a_2a_3a_4$	$a_1a_2a_3$
	$\sum_k\lvert A_k^1\rvert=10$	$\sum\lvert A_k^2\rvert=10$	$\sum\lvert A_k^1\cap A_k^2\rvert=9$	$\sum\lvert B_k^1\rvert=10$	$\sum\lvert B_k^2\rvert=10$	$\sum\lvert B_k^1\cap B_k^2\rvert=9$

a_1	a_2	a_3	a_4	a_5
5	4	3	1	2

Mind the accidental switch in columns B_k^1, B_k^2



	a_1	a_2	a_3	a_4	a_5
	5	4	1	2	3

k	A_k^1	A_k^2	$A_k^1 \cap A_k^2$	B_k^2	B_k^1	$B_k^1 \cap B_k^2$
1	$a_2a_3a_4a_5$	$a_2a_3a_4a_5$	$a_2a_3a_4a_5$			
2	$a_3a_4a_5$	$a_3a_4a_5$	$a_3a_4a_5$	a_1	a_1	a_1
3		a_4a_5		a_1a_2	$a_1a_2a_4a_5$	a_1a_2
4	a_3	a_5		$a_1a_2a_3$	$a_1a_2a_5$	a_1a_2
5	a_3a_4			$a_1a_2a_3a_4$	a_1a_2	a_1a_2
	$\sum\lvert A_k^1\rvert=10$	$\sum\lvert A_k^2\rvert=10$	$\sum\lvert A_k^1\cap A_k^2\rvert=7$	$\sum\lvert B_k^2\rvert=10$	$\sum\lvert B_k^1\rvert=10$	$\sum\lvert B_k^1\cap B_k^2\rvert=7$

Exercise 45

Define 10 sets

$$
\begin{aligned}
&\{x \in A \mid x \equiv 1 \pmod{20} \vee x \equiv 19 \pmod{20}\} \\
&\{x \in A \mid x \equiv 2 \pmod{20} \vee x \equiv 18 \pmod{20}\} \\
&\ \vdots \\
&\{x \in A \mid x \equiv 9 \pmod{20} \vee x \equiv 11 \pmod{20}\} \\
&\{x \in A \mid x \equiv 10 \pmod{20}\}
\end{aligned}
$$

As there are 11 integers $x \in A$ and 10 sets, by the pigeonhole principle there are two integers $a, b \in A$ (let wlog $b \leq a$) that are element of the same set.

Case 1 $a \equiv b \pmod{20}$. Then $20 \mid (a - b)$.

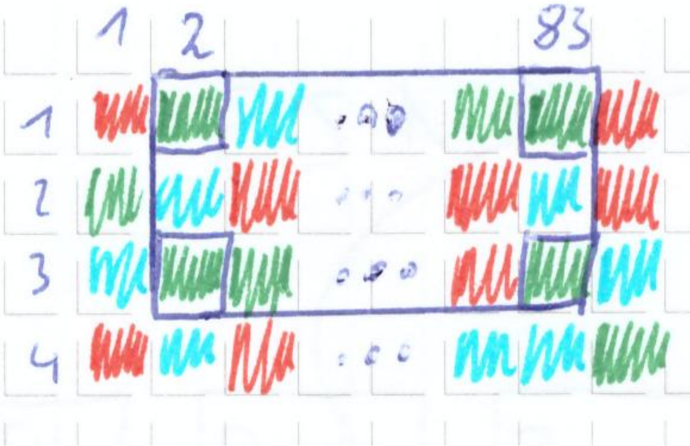
Case 2 $a \not\equiv b \pmod{20}$. Then $20 \mid (a + b)$.

Examples:

$$
\begin{aligned}
401 \equiv 1 \pmod{20}, 19 \equiv 19 \pmod{20} &\implies \{401, 19\} \text{ and } 20 \mid 420 = 401 + 19 \\
401 \equiv 1 \pmod{20}, 21 \equiv 1 \pmod{20} &\implies \{401, 21\} \text{ and } 20 \mid 380 = 401 - 21 \\
418 \equiv 18 \pmod{20}, 118 \equiv 18 \pmod{20} &\implies \{418, 118\} \text{ and } 20 \mid 300 = 418 - 118
\end{aligned}
$$

Exercise 46

For each column, there are 4 points and 3 possible colors per point, for a total of $3^4 = 81$ possible colorings. As there are 4 points per column and 3 possible colors, by the Pigeonhole Principle some color appears twice in a single column. With $81 + 1$ columns, by the Pigeonhole Principle, there are two columns with the same coloring. From each of the two columns, take some corresponding two points of a color that appears twice. These form a rectangle all of whose vertices are the same color.



Exercise 47

https://math.stackexchange.com/questions/1519803/proving-identities-using-combinatorial-interpretation-of-binomial-coefficients

1. Let X be an n-element set. Then $\binom{n}{k}$ is the number of k-element subsets of X. If $x \in X$ is a fixed element of X, then we can divide the k-element subsets of X into two classes: those which contain x and those that do not. The k-element subsets not containing x are precisely the k-element subsets of $X \setminus \{x\}$, and there are $\binom{n-1}{k}$ such sets. Then k-element subsets of X which do contain x are all of the form $\{x\} \cup Y$, where Y is a $(k-1)$-element subset of $X \setminus \{x\}$; there are $\binom{n-1}{k-1}$ such sets. Thus, in total, there are $\binom{n-1}{k} + \binom{n-1}{k-1}$ total k-element subsets of X, proving that
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

2. Let $X = X_1 \cup X_2$ with $|X_1| = |X_2| = n$. Then $\binom{2n}{n}$ is the number of n-element subsets of X. Let Y be one such subset. It chooses k elements of X_1, with $0 \le k \le n$. There are $\binom{n}{k}$ such subsets. Then Y has to choose $n-k$ elements from X_2. There are $\binom{n}{n-k}$ such subsets. Consequently, there are $\binom{n}{k} \cdot \binom{n}{n-k} = \binom{n}{k}^2$ possible choices for Y. Summing up all possible k gives the identity
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Exercise 48

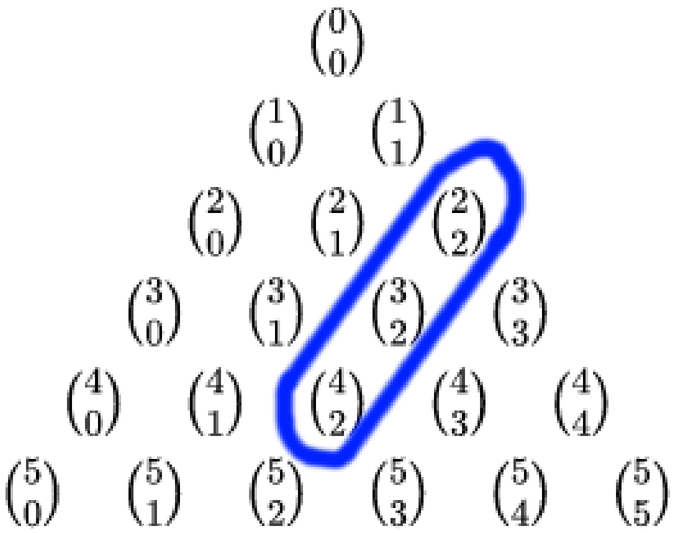
Combinatorial proof using double counting. Note that the left side counts bit strings of $n+1$ length with $r+1$ ones. We show that the right side counts the same objects. The final one must occur at position $r+1$ or $r+2$ or ... or $n+1$ Assume that it occurs at the m^{th} bit, where $r+1 \leq m \leq n+1$.

1. Thus, there must be r ones in the first $m-1$ positions
2. Thus, there are $\binom{m-1}{r}$ such strings of length $m-1$

As m can be any value from $r+1$ to $n+1$, the total number of possibilities is

$$\sum_{m=r+1}^{n+1} \binom{m-1}{r} = \sum_{m=r}^{n} \binom{m}{r}$$

which is illustrated for $r = 2, n = 4$



Exercise 49

https://math.stackexchange.com/a/1523108/844881

The second equation from the task description and a new notation [1] shows

$$\binom{x}{k} = \frac{x^{\underline{k}}}{k!} = \frac{x(x-1)(x-2)\ldots(x-(k-1))}{k(k-1)(k-2)\ldots 1} = \prod_{i=1}^{k} \frac{x+1-i}{i}$$

With $j = n-1-i$ in the second line (in one product the factors are increasing and in the other they are decreasing), we calculate

$$\begin{aligned}
(-x)^{\underline{k}} &= \prod_{i=0}^{k-1}(-x-i) \\
&= \prod_{j=0}^{k-1}(-x-k+1+j) \\
&= \prod_{j=0}^{k-1}(-(x+k-1-j)) \\
&= (-1)^k \prod_{j=0}^{k-1}(x+k-1-j) \\
&= (-1)^k (x+k-1)^{\underline{k}}
\end{aligned}$$

and thus

$$\binom{-x}{k} = \frac{(-x)^{\underline{k}}}{k!} = \frac{(-1)^k (x+k-1)^{\underline{k}}}{k!} = (-1)^k \binom{x+k-1}{k}$$

[1] https://en.wikipedia.org/wiki/Binomial_coefficient#Multiplicative_formula

Exercise 50

https://math.stackexchange.com/questions/580435/number-of-2n-letter-words-using-double-n-letter-alphabet-without-consecutiv

Definitions

We know from the lecture that the number of arrangements of a multiset $\{b_1^{k_1}, b_2^{k_2}, \dots, b_m^{k_m}\}$ of cardinality n is

$$\frac{n!}{k_1!k_2!\dots k_m!} \tag{1}$$

[1] Let S be a set of properties that the elements of some other set A may or may not have. For any subset $T \subseteq S$, denote by

- $f_=(T)$ the number of elements of A that have *exactly* the properties in T.
- $f_\geq(T)$ the number of elements of A that have *at least* the properties in T.

From that definition we get

$$f_\geq(T) = \sum_{Y \supseteq T} f_=(Y). \tag{2}$$

By Theorem 2.1.1 from [1] the existence and form of the following inverse function follows

$$f_=(T) = \sum_{Y \supseteq T} (-1)^{|Y \setminus T|} f_\geq(Y) \tag{3}$$

Of which the following (the number of elements having *none* of the properties in S) is a special case

$$f_=(\emptyset) = \sum_{Y \supseteq T} (-1)^{|Y|} f_\geq(Y)$$

where Y ranges over all subsets S.

Examples demonstrating the application of 2 and its inverse 3 :

1. Suppose $S = \{s_1, s_2\}$ and $T = \{s_1\}$ Then

$$\begin{aligned} f_=(\{s_1\}) &= (-1)^{|\{\}|} f_\geq(\{s_1\}) + (-1)^{|\{s_2\}|} f_\geq(\{s_1, s_2\}) \\ &= f_\geq(\{s_1\}) - f_\geq(\{s_1, s_2\}) \\ &= f_=(\{s_1\}) + f_=(\{s_1, s_2\}) - f_=(\{s_1, s_2\}) \\ &= f_=(\{s_1\}) \end{aligned}$$

   We think of $f_\geq(\{s_1\})$ as being a first approximation to $f_=(\{s_1\})$. We then substract $f_\geq(s_1, s_2)$ to get a better approximation. This reasoning brings us to the terminology "Inclusion-Exclusion".

2. Suppose $S = \{s_1, s_2, s_3\}$. Then

$$\begin{aligned} f_{\geq}(\{s_1, s_2\}) &= f_{=}(\{s_1, s_2\}) + f_{=}(\{s_1, s_2, s_3\}) \\ &= (-1)^{|\{\}|} f_{\geq}(\{s_1, s_2\}) + (-1)^{|\{s_3\}|} f_{\geq}(\{s_1, s_2, s_3\}) + (-1)^{|\{\}|} f_{\geq}(\{s_1, s_2, s_3\}) \\ &= f_{\geq}(\{s_1, s_2\}) - f_{=}(\{s_1, s_2, s_3\}) + f_{=}(\{s_1, s_2, s_3\}) \\ &= f_{\geq}(\{s_1, s_2\}) \end{aligned}$$

A special case of the Principle of Inclusion-Exclusion occurs when the function $f_=$ satisfies $f_=(T) = f_=(T')$ whenever $|T| = |T'|$. Then also $f_{\geq}(T)$ depends only on $|T|$. We set $a(n-i) = f_=(T)$ and $b(n-i) = f_{\geq}(T)$ whenever $|T| = i$. From equations 2 and 3 then follows the equivalence of the following formulas

$$b(m) = \sum_{i=0}^{m} \binom{m}{i} a(i), \quad 0 \leq m \leq n \tag{4}$$

$$a(m) = \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} b(i), \quad 0 \leq m \leq n \tag{5}$$

Examples:

1. Suppose $A = \{a_1, a_2\}, S = \{s_1, s_2\}$ where the properties are assigned $a_1 : s_1$ and $a_2 : s_2$. Then $f_=(\{s_1\} = f_=(\{s_2\}) = 1$ and $f_=(\{\}) = f_=(\{s_1, s_2\}) = 0$. For $n = 2, |T| = i = 1$ we get by application of equations 4 and 5

$$\begin{aligned} b(2-1) &= \binom{1}{0} a(0) + \binom{1}{1} a(1) \\ &= \binom{1}{0} \left[\binom{0}{0} \cdot (-1)^0 b(0) \right] + \binom{1}{1} \left[\binom{1}{0} \cdot (-1)^1 b(0) + \binom{1}{1} \cdot (-1)^0 b(1) \right] \\ &= 1 \cdot [1 \cdot 1 \cdot b(0)] + 1 \cdot [1 \cdot (-1) \cdot b(0) + 1 \cdot 1 \cdot b(1)] \\ &= b(0) - b(0) + b(1) = b(1) \end{aligned}$$

2. Suppose $A = \{a_1, a_2, a_3, a_4\}, S = \{s_1, s_2\}$ where the properties are assigned $a_1 : s_1$ and $a_2 : s_2$ and $a_3 : s_1, s_2$ and $a_4 : s_1, s_2$. Then $f_=(\{s_1\} = f_=(\{s_2\}) = 1$ and $f_=(\{\}) = 0$ and $f_=(\{s_1, s_2\}) = 2$.

Solution

This can be reduced to the problem of the number of permutations of the multiset $M_n = \{1, 1, 2, 2, \ldots, n, n\}$ (or in equivalent notation $M_n = \{1^2, 2^2, \ldots, n^2\}$) with no two consecutive terms equal [2]. We use the principle of inclusion and exclusion.

[2] 2

Let $h(n)$ be the number of permutations of M_n with no two consecutive terms equal. Thus $h(0) = 0, h(1) = 0$, and $h(2) = 2$ (corresponding to the permutations 1212 and 2121). For $1 \leq i \leq n$, let P_i be the property that the permutation w of M_n has two consecutive i's. Hence, we seek $f_=(\emptyset) = h(n)$.

It follows by symmetry that for fixed n, $f_\geq(T)$ depends only on $i = |T|$, so write $g(i) = f_\geq(T)$. Then $g(i)$ is equal to the number of permutations w of the multiset $\{1, 2, \ldots, i, (i+1)^2, \ldots, n^2\}$ (replace any $j \geq i$ appearing in w by two consecutive j's), so

$$g(i) = \frac{(i + 2(n-i))!}{2^{n-i}} = \frac{(2n-i)!}{2^{n-i}},$$

a special case of equation 1 with $k_1! = k_2! = \cdots = k_i! = 1$ and $k_{i+1} = k_{i+2} = \cdots = k_n! = 2$

Now we have $b(i) := g(n-i) = \frac{(n+i)!}{2^i}$. From equation 5 we then get

$$h(n) = \sum_{i=0}^{n} \binom{n}{i} \frac{(-1)^{n-i}(n+i)!}{2^i}$$

Example

Let A be the permutations of $M_n = M_2 = \{1, 1, 2, 2\}$. We look for the number of permutations $h(2)$ of M_2 with no consecutive terms equal.

The set of properties S is given by properties like: P_1 is the property that a permutation has two consecutive 1's.

$f_=(\emptyset)$ is the number of elements of A having none of the properties in S, that is the number of elements of A with no consecutive terms equal. Therefore, $f_=(\emptyset) = h(2)$.

We have n fixed to $n = 2$, so $f_\geq(T)$ (the number of elements of M_2 that have at least the properties in T), depends only on $|T| = i$. That means that, for example, the number of permutations of M_2 that have the property P_1 or P_2 or both depends only on $|\{P_1, P_2\}| = 2$ So we write $f_\geq(T) = g(i)$ and get, for example, for $i = 1$ a multiset $\{1, 2, 2\}$ the number of its permutations is

$$g(1) = \frac{(2 \cdot 2 - 1)!}{2^{2-1}} = \frac{3!}{2} = 3.$$

We verify this by enumerating the permutations of $\{1, 2, 2\}$: 122,212,221.

For the same $i = 1$ we get $b(1) = g(2-1) = \frac{(2+1)!}{2^1}$. Looking again at the general case, we finally get

$$\begin{aligned}
h(2) &= \sum_{i=0}^{2} \binom{2}{i} \frac{(-1)^{2-i}(2+i)!}{2^i} \\
&= \binom{2}{0} \frac{(-1)^2(2+0)!}{2^0} + \binom{2}{1} \frac{(-1)^1(2+1)!}{2^1} + \binom{2}{2} \frac{(-1)^0(2+2)!}{2^2} \\
&= 1 \cdot \frac{1 \cdot 2}{1} + 2 \cdot \frac{(-1) \cdot 3 \cdot 2}{2} + 1 \cdot \frac{1 \cdot 4 \cdot 3 \cdot 2}{4} \\
&= 2 - 6 + 6 && = 2
\end{aligned}$$

Which we verify again by enumerating the two permutations 1212 and 2121 that fulfill the requirement.

References

[1] Richard P. Stanley and Gian-Carlo Rota. *Enumerative Combinatorics*. Vol. 1. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997. DOI: 10.1017/CBO9780511805967.