VU Discrete Mathematics

Exercises for 28th October 2025

13) Prove that a graph G with n vertices and $\chi(G) = k$ has at least $\binom{k}{2}$ edges.

14) Show the following inequality for Ramsey numbers: If $r \geq 3$ then

$$R(n_1, \ldots, n_{r-2}, n_{r-1}, n_r) \leq R(n_1, \ldots, n_{r-2}, R(n_{r-1}, n_r))$$

Hint: Let $n = R(n_1, \ldots, n_{r-2}, R(n_{r-1}, n_r))$ and consider an edge colouring of K_n with r colours, say $c_1, \ldots, c_r$. Identify the colours c_{r-1} and c_r and apply the Ramsey property for $r-1$ colours.

15) In how many ways can the letters a, a, b, b, c, d, e be listed such that the letters c and d are not in consecutive positions?

16) Let $p_n(k)$ be the number of permutations of $\{1, 2, \ldots, n\}$ having exactly k fixed points. Use the method of double counting to prove the identity $\sum_{k=0}^{n} k p_n(k) = n!$.

17) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, \ldots, 100\}$. All points of the plane having coordinates (x, y) which satisfy $(x, y) \in A \times B$ are coloured with one of the colours red, green or blue. Prove that there exists a monochromatic rectangle.

Remark: A rectangle is called monochromatic if all its four vertices have the same colour.

18) Let $n \in \mathbb{N}$. Prove the identities

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1 \qquad \text{and} \qquad \sum_{k=1}^{n} (n-k) 2^{k-1} = 2^n - n - 1$$

by using a combinatorial interpretation.