Financial Management and Reporting

(330.215)

Summer Term 2019

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Lectures 3 and 4: Options

Content

1. Basic types of options
2. Payoff structures at maturity
3. Trading strategies
4. Valuation I: Upper and lower bounds
5. Put-Call Parity
6. Valuation II: Binominal Model
7. Valuation III: Black-Scholes-Merton Model

1. Basic types of options

Options are **conditional forward transactions**. They give the holder the **right** (but not the obligation) to **buy or sell** a specific amount of an underlying instrument.

The holder does **not have to exercise this right**. Whether the holder will exercise his right will mainly be **conditional on the price** of the underlying asset.

Typical underlyings are: stocks, bonds, stock indices, futures contracts, foreign exchange, commodities, interest rates, etc.

Basic types of options: Calls, Puts

- **Call option**

  Gives the **HOLDER** the **right** to **BUY** a particular **amount** of the **underlying,** at a **pre-specified price** at **any time up to** expiration or **on the expiration date**.

  Pre-specified price: exercise price, strike price

- **Put option**

  Gives the **HOLDER** the **right** to **SELL** a particular **amount** of the **underlying**, at a **pre-specified price** at **any time up to** expiration or **on the expiration date**.

- **American and European options**

  Options can be either American or European style, a distinction that has nothing to do with the geographical location.

European options:	Can be exercised **only on the expiration date**.
American options:	Can be exercised at **any time** up to the **expiration date**.

- There are **4 types of participants** in options markets

Buyer of a **call**:	**long** position, **holder** of an option (a call)
Buyer of a **put**:	**long** position, **holder** of an option (a put)
Seller of a **call**:	**short** position, **writer** of an option (a call)
Seller of a **put**:	**short** position, **writer** of an option (a put)

The **holder** (**long** position) has the **right** (but not the obligation) to exercise his option.

In contrast: The **writer** (short position) has the **obligation** to *buy the underlying for cash* (put option) or *sell the underlying for cash* (call option), if the holder exercises his/her right. She has to wait whether the holder (long position) exercises his/her right.

2. Payoff structures at maturity

2.1 Payoff of a call option at maturity

- The **value of a call option** at maturity (C_T) depends on 2 variables:

  The **value of the underlying** at maturity (S_T), and the **exercise price** (X).

- The holder of the call will **exercise** the right to buy the underlying for X only if

  $\mathbf{S_T > X}$: $C_T = [S_T - X]$ and **NOT** if $\mathbf{S_T \leq X}$**:** $C_T = 0$

Therefore: **Payoff of a long position** in a **call** option:

$$C_T = \max[S_T - X, \quad 0]$$

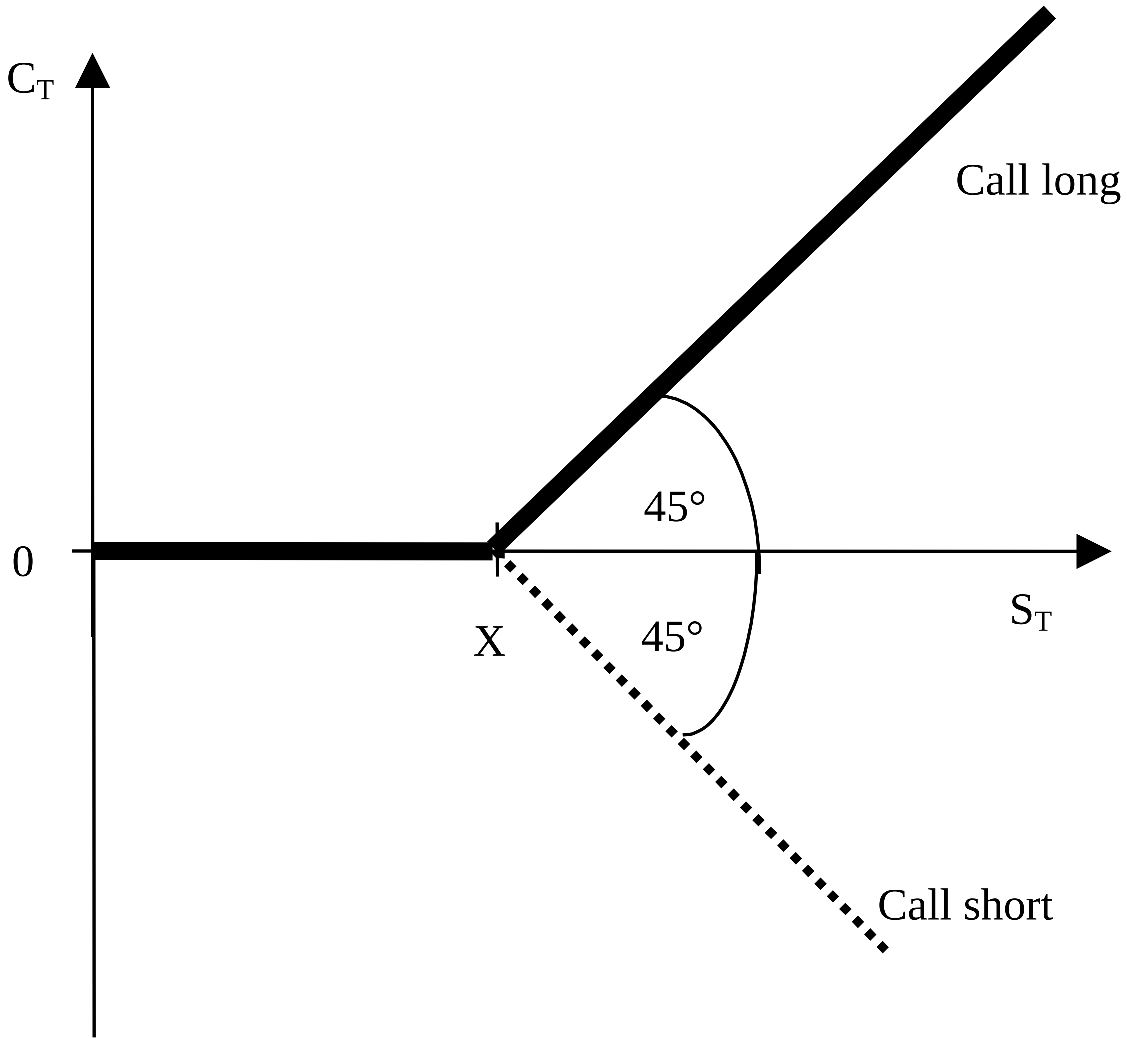
The **writer** (short position) of an option receives a **cash up front payment** (i.e. the option price/option premium) but has potential **liabilities later on**.

The writer's profit and loss is **reverse to that of the purchaser** (long position).

Payoff to the **short position** in a call option:

$$-\max[S_T - X, \quad 0] = \min[X - S_T, \quad 0]$$

Payoff of a call long and a call short position at maturity



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Example

Suppose an investor buys on January 9, 20xx, 100 February 20xx calls (= 1 contract) on **E.ON** with a strike price of €14 and pays €1.60 per call option. This contract expires on the 3rd Friday in February 20xx and trades on EUREX.

(a) How does the profit and loss profil at the expiration of this options contract look like?

(b) Suppose E.ON trades at the expiration of the February 20xx contract at (i) €18, or (ii) €12. What is the percentage profit for the investor?

2.2 Payoff of a put option at maturity

As the value of a call, the value of **a put option** at maturity (P_T) also depends on the **value of the underlying** at maturity (S_T), and the **exercise price** (X).

- The holder of the put will **exercise** his right to **sell** the underlying for X only if

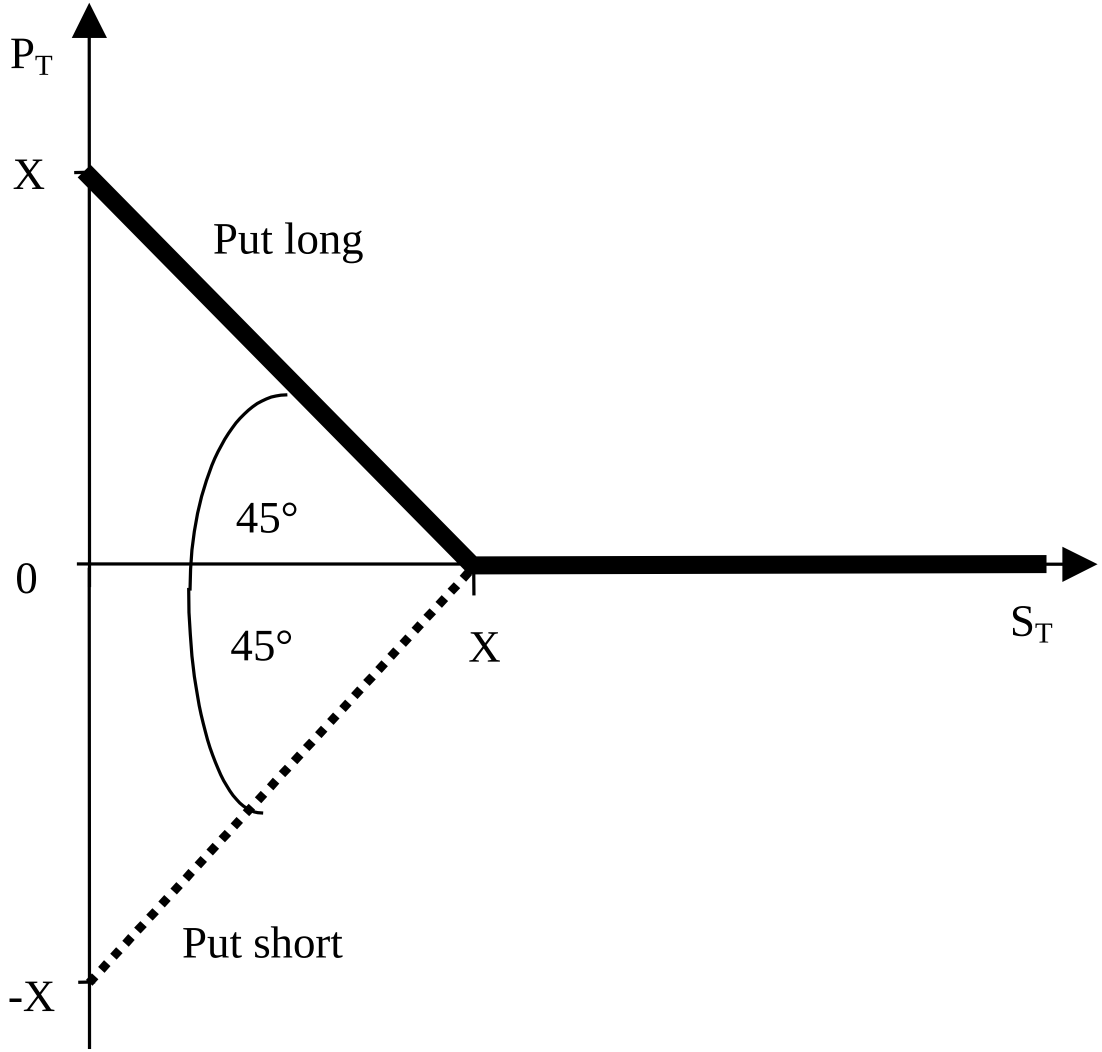
$$\mathbf{S_T < X}: \quad P_T = [X - S_T] \qquad \text{and } \textbf{NOT} \text{ if} \qquad \mathbf{S_T \geq X}: \; P_T = 0$$

Therefore: **Payoff of a long position** in a **put** option:

$$P_T = \max[X - S_T, \quad 0]$$

Payoff of the **short position** in a put option: $-\max[X - S_T, \quad 0] = \min[S_T - X, \quad 0]$

Payoff of a put long and a put short position at maturity



Example

January 9, 20xx: Suppose an investor buys 100 June 20xx puts (= 1 contract) on **Deutsche Post** with a strike price of €16 and pays €1.23 per put option. This contract expires on the 3rd Friday in June 20xx and trades on EUREX.

(a) How does the profit and loss profil at the expiration of this options contract look like?

(b) Suppose Deutsche Post trades at the expiration of the June 20xx contract at (i) €20, or (ii) €10. What is the percentage profit for the investor?

3. Trading Strategies

3.1 Straddle

A straddle consists of a **call and a put option** with the **same strike price** and **expiration date**.

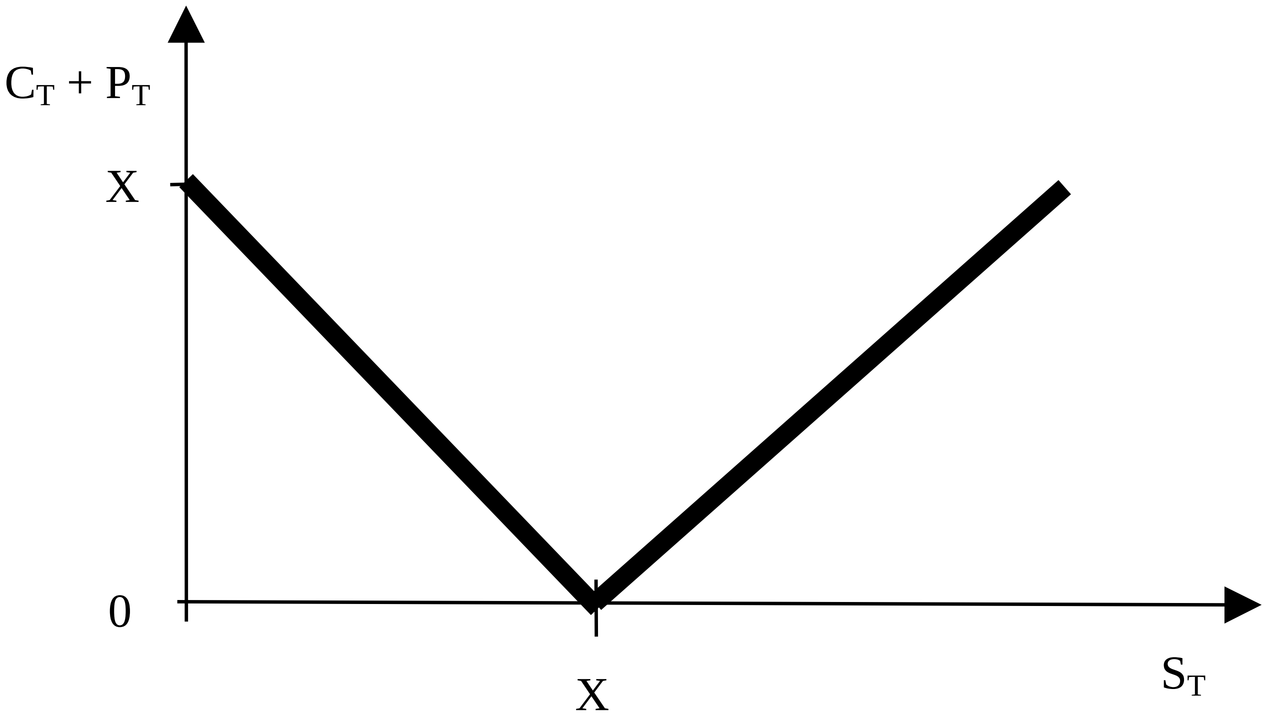
Straddle long: long call + long put

Straddle short: short call + short put

The price of a straddle equals the sum of the call and the put option. **At maturity,** the **value of a long straddle** is therefore:

$$C_T + P_T = \max[S_T - X, \quad 0] + \max[X - S_T, \quad 0]$$

Value of a long straddle at expiration



A **long straddle** is appropriate when an investor is **expecting a large move** in the value of the underlying but does not know in which direction the move will be (i.e. **increase in volatility**).

If the stock price is close to the strike price at expiration of the options, the straddle leads to a loss.

However, if there is a sufficiently large move in either direction, a significant profit will result.

A **short straddle** is appropriate if a **decreasing volatility** is expected.

It is a highly risky strategy. Large movements in either direction can lead to huge (unlimited) losses.

3.2 Strangle

A strangle consists of a **call and a put** option with the **same expiration date** but **different strike prices** X_C (Call) and X_P (Put), with

$$X_C > X_{P.}$$

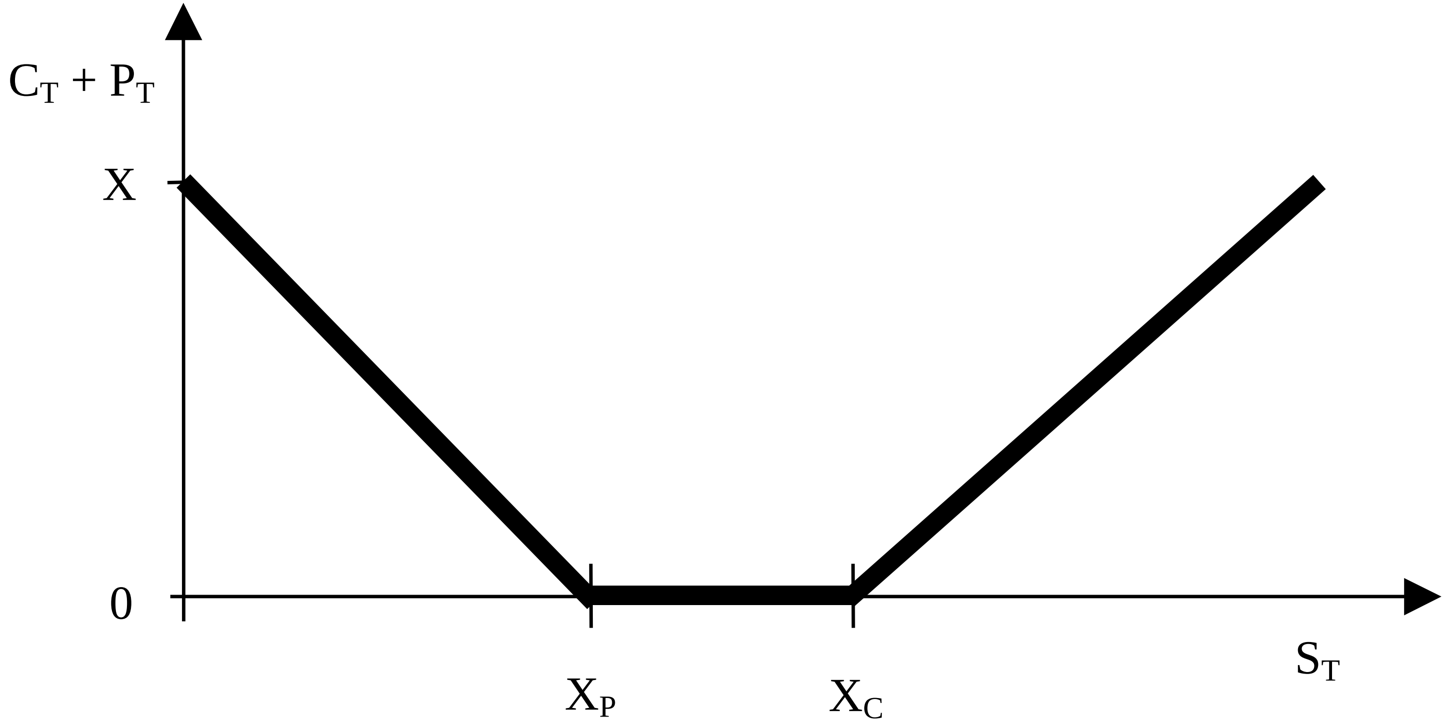
The investor in a **long strangle** is beating that there will be a **large price move** but is uncertain whether it will be an increase or a decrease (**strong increase in volatility**).

To be **profitable**, the stock price has to **move more in a long strangle** than in a long straddle. However, the **downside risk is less** with a long strangle.

The price of a strangle equals the sum of the call and the put option. At maturity the value of a long strangle is:

$$C_T + P_T = \max\left[S_T - X_C, \quad 0\right] + \max\left[X_P - S_T, \quad 0\right]$$

Value of a long strangle at expiration



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3.3 Protective put

A protective put is a strategy where a **long put** position is used to **hedge an existing position** in the underlying (e.g. a portfolio of stocks).

When the price of the underlying falls, profits in the put option will compensate for losses in the spot market position.

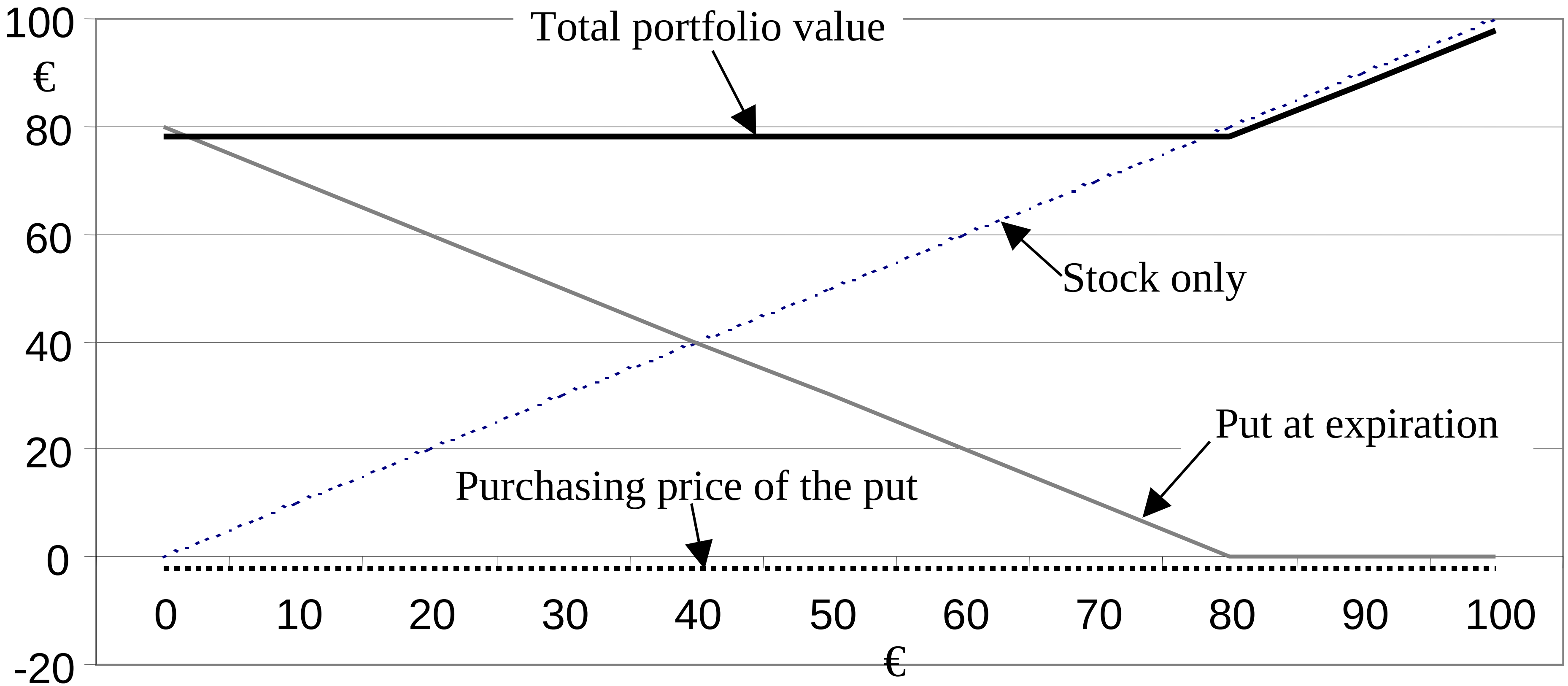
However, the put option requires an initial investment, reducing the upside profit when the price of the underlying increases.

Example

Suppose an investor owns a portfolio of OMV stocks trading at €80. She buys puts with an exercise price of €80 to hedge against decreasing stock prices. The price of the put is €2.

Profit and loss at maturity (long OMV, long €80 put) in €:

OMV share price	Purchasing price (put)	Value of the put at maturity	Value of total portfolio
72	-2	8	78
74	-2	6	78
76	-2	4	78
78	-2	2	78
80	-2	0	78
82	-2	0	80
84	-2	0	82



100
€
80
60
40
20
0
-20
Total portfolio value
Stock only
Put at expiration
Purchasing price of the put
0
10
20
30
40
50
60
70
80
90
100
€

3.4 Covered call writing

Covered call writing is the **sale of calls covered by an existing position** in the underlying asset (e.g. a stock).

It can be an interesting strategy for an investor who **anticipates stable or falling prices** but wants to hold his position in the underlying asset.

The investor can increase the return on the portfolio by the option price received.

However, covered call writing **limits the potential profit** on rising prices.

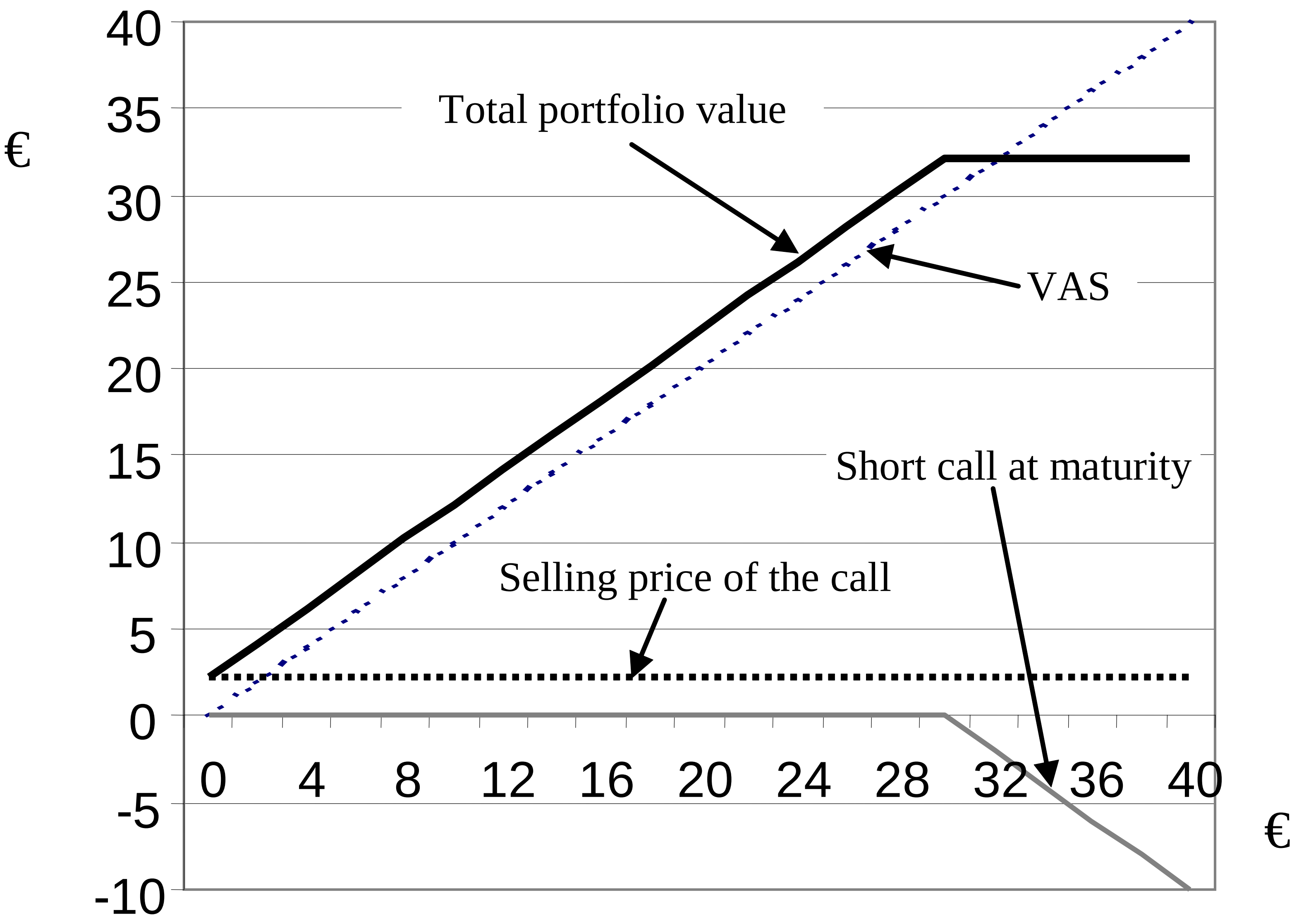
If, e.g. the price of the underlying is above the strike price of the sold call option, the long position in the call will exercise the option and buy the underlying for the strike price.

Example

An investor holds a VAS position with a current price of €31.50 per share. She expects stable prices and sells calls with a strike price of €30 at a price of €2.20.

Profit and loss at maturity (long VAS, short €30 call) in €:

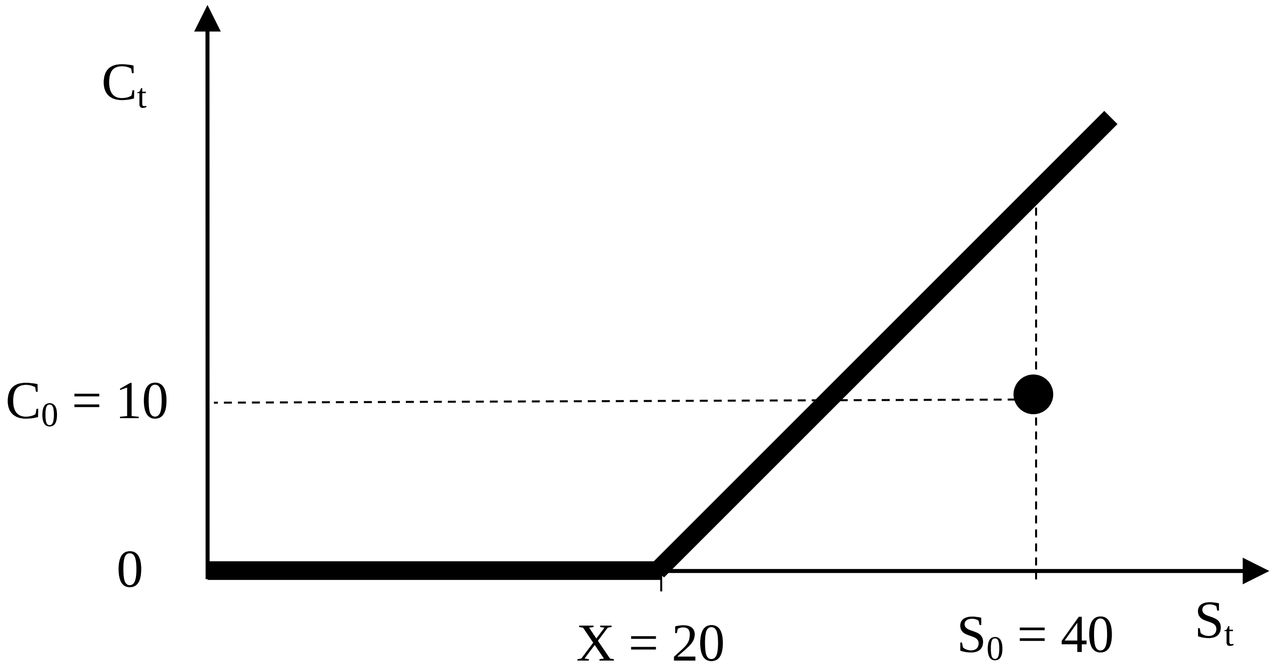
VAS share price	Selling price of the call	Value of the call short position at maturity	Value of the total portfolio
24	+2.20	0	26.20
26	+2.20	0	28.20
28	+2.20	0	30.20
30	+2.20	0	32.20
32	+2.20	-2	32.20
34	+2.20	-4	32.20
36	+2.20	-6	32.20


€
40
35
30
25
20
15
10
5
0
-5
-10
Total portfolio value
VAS
Short call at maturity
Selling price of the call
0
4
8
12
16
20
24
28
32
36
40
€

4. Valuation I: Upper and lower bounds

Example

An American call option has a current price of C_0 = €10. The exercise price is €20 and the underlying has a current price of S_0 = €40.



The figure shows that the current option price is **too low** compared to the current price of the underlying. Via buying the option and exercising it, an investor can buy the underlying at a total price below the current market price.

Arbitrage portfolio:

	t = 0
Call long	-10
Exercise (buying the underlying for X = 20)	-20
Selling the underlying	+40
Arbitrage profit	**+10**

Consequence: The price of the call option will increase, and
the price of the underlying will drop
untill arbitrage opportunities do not exist anymore.

Lower valuation boundary (for the moment):

$$C_0 \quad \geq \quad \max\left[S_0 - X, \quad 0\right]$$

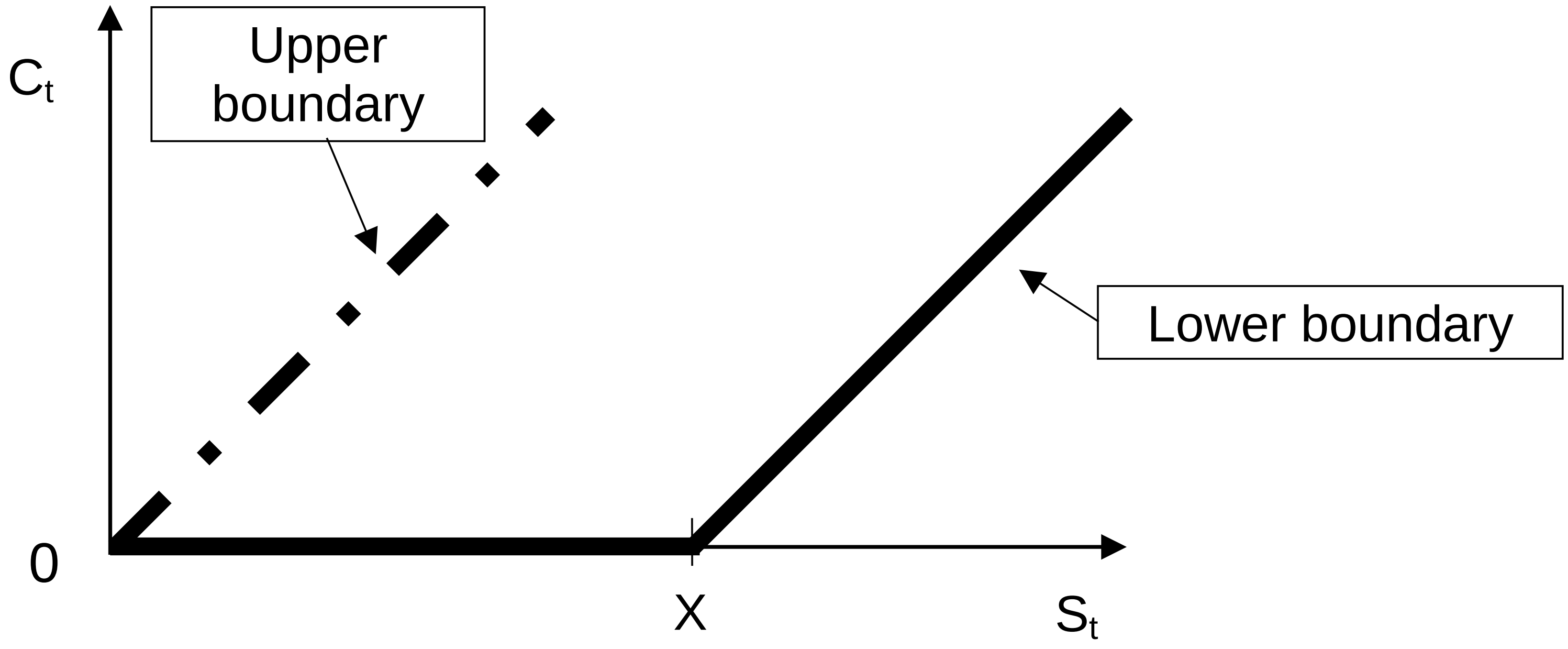
Upper valuation boundary:

A call never can be worth more than its underlying.

Example: Price = € 45

	t = 0
Call short	+45
Stock long	-40
Arbitrage profit	+5

Thus, the price of a call option only can lie in between the upper and the lower valuation boundary:



European options have the **disadvantage** that they cannot be exercised in advance (before maturity).

But still if the **boundaries are violated** also for **European options,** an **arbitrage portfolio** can be generated.

Example:

European Call option: Maturity T = 1 year, X = €100, S_0 = €200, C_0 = €100

(a): r = 0% p.a.

Arbitrage portfolio:

	(t = 0)	Value at maturity (t = T)	
Position	Portfolio	$S_T \leq 100$	$S_T > 100$
Call long	-100	0	S_T - 100
Stock short	+200	$-S_T$	$-S_T$
Investment (zero bond long)	-100 = X	100	100
Portfolio value	0	100 - $S_T \geq 0$	0

$$\Rightarrow \quad C_0 \geq \max[S_0 - X, \quad 0]$$

(b): r = 7% p.a.

r: Continuously compounded risk-free rate of interest for an investment maturing in time T

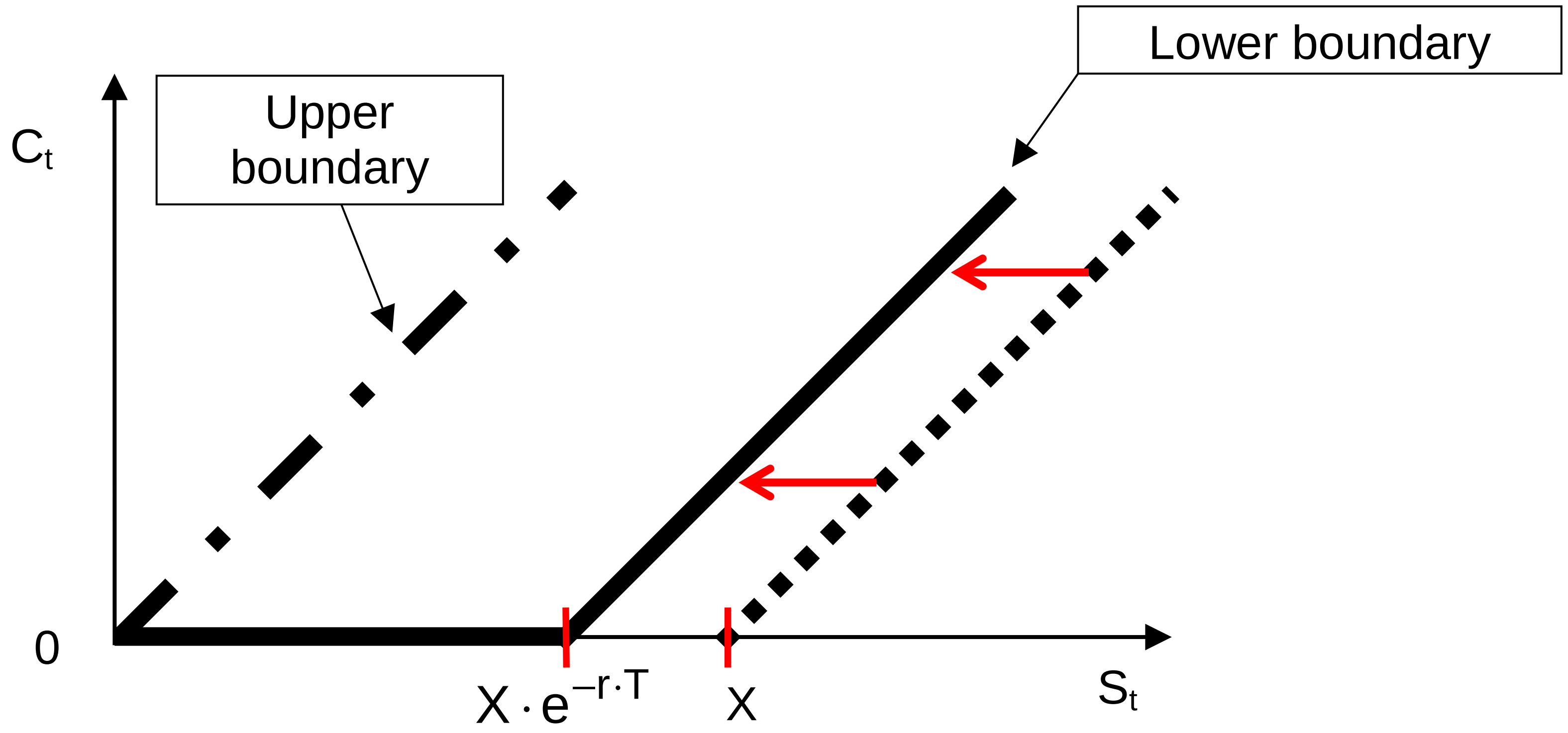
Arbitrage portfolio:

	(t=0)	Value at maturity (t=T=1 year)	
Position	Portfolio	$S_T \leq 100$	$S_T > 100$
Call long	-100	0	S_T - 100
Stock short	+200	$-S_T$	$-S_T$
Investment (zero bond long)	-93.24	100	100
Portfolio value	+6.76	$100 - S_T \geq 0$	0

And: $-93.24 = X \cdot e^{-r \cdot T} = 100 \cdot e^{-0.07 \cdot 1.0}$

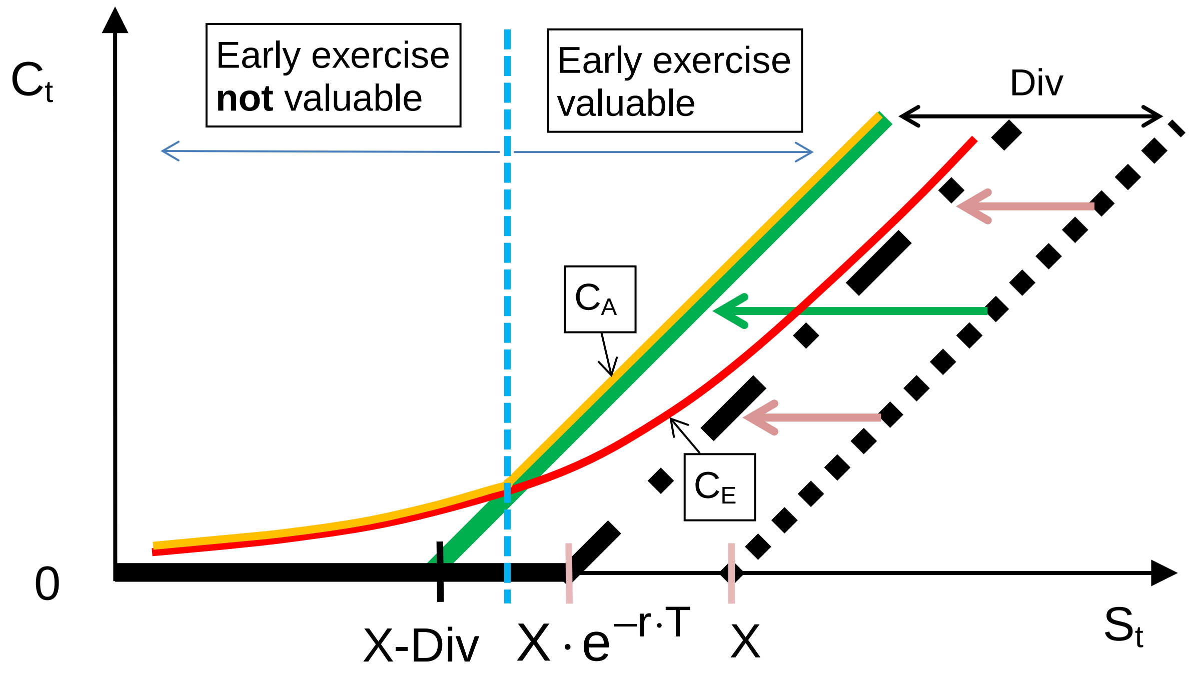
This leads to the lower boundary for European (C_E) and American (C_A) call options:

$$C_A \geq C_E \geq \max\left[S_0 - X \cdot e^{-r \cdot T},\ 0\right]$$



American Call **Early exercise** is valuable **only** if the dividend (Div) received (before maturity) when owning the stock is larger than

$$X - X \cdot e^{-r \cdot T}$$



European Put (P_E): Lower boundary

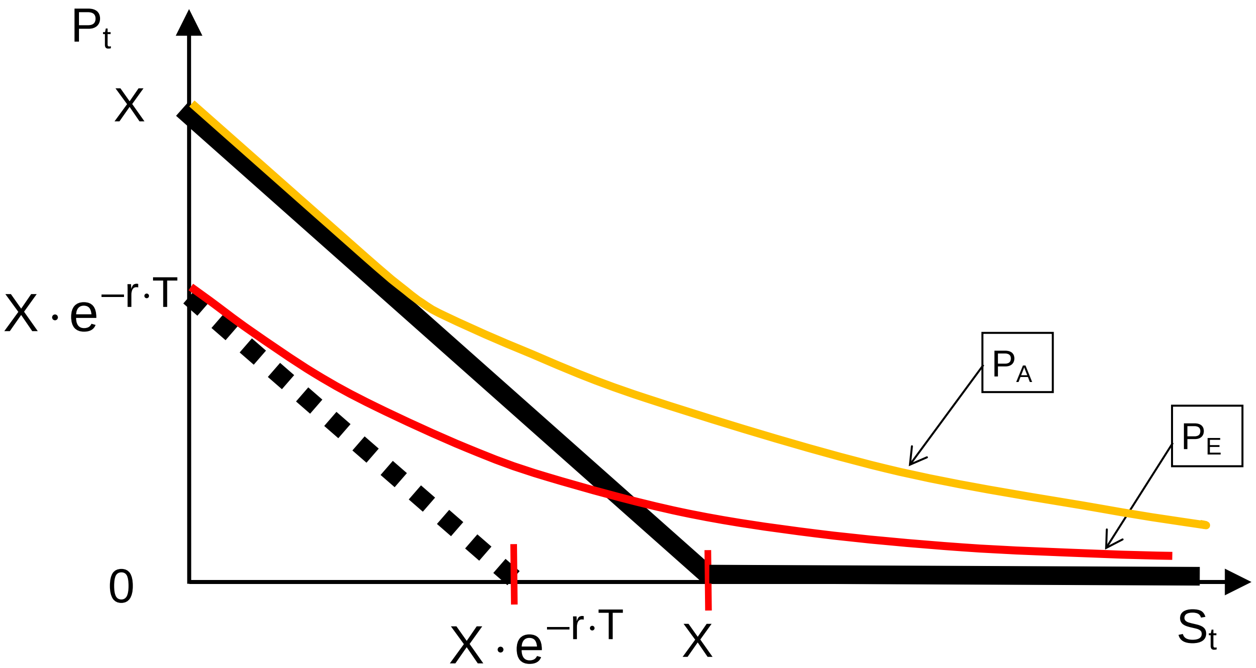
	(t = 0)	Value at maturity (t = T)	
Portfolio components	Portfolio	$S_T \leq X$	$S_T > X$
Put (P_E) long	$-P_E$	$X - S_T$	0
Loan (zero bond short)	$X \cdot e^{-r \cdot T}$	$-X$	$-X$
Stock long	$-S_0$	S_T	S_T
Portfolio value	$-P_E - S_0 + X \cdot e^{-r \cdot T}$	0	$S_T - X > 0$

➔ $-P_E - S_0 + X \cdot e^{-r \cdot T} \leq 0$

$P_E \geq X \cdot e^{-r \cdot T} - S_0$

Thus, at t = 0: $P_E \geq \max\left[0, \quad X \cdot e^{-r \cdot T} - S_0\right]$

European Put (P_E) and American Put (P_A): Lower boundary



Thus, at t = 0: $P_A \geq \max[0, \quad X - S_0]$

5. Put-Call Parity

The put-call parity describes an important **relationship** between the **value** of a European **put** and the value of a European **call** on the same underlying.

Let us consider the following portfolio:

(1) **Long** position: European **put** option with exercise price X and maturity t=T
(2) **Long** position: **Underlying** (e.g. a stock)
(3) **Short** position: **Zero bond** with a face value of X and a current (present) value of $X \cdot e^{-r \cdot T}$

	t = 0	t = T	
		$S_T < X$	$S_T \geq X$
Put **long**	$-P_0$	$X - S_T$	0
Underlying **long**	$-S_0$	S_T	S_T
Zero bond **short**	$+X \cdot e^{-r \cdot T}$	$-X$	$-X$
	$-P_0 - S_0 + X \cdot e^{-r \cdot T}$	0	$S_T - X$

➔ **Synthetic call** = $-P_0 - S_0 + X \cdot e^{-r \cdot T}$

As $(C_T = \max[0, S_T - X])$ and $(P_T = \max[X - S_T, 0])$ it follows that

$$C_T = P_T + S_T - X$$

and for t = 0:

$$C_0 = P_0 + S_0 - X \cdot e^{-r \cdot T}.$$

This relationship is known as the **put-call parity**. It shows that the **value** of a European **call** with a certain exercise price and exercise date is related to the **value** of a European **put** with the same exercise price and exercise date (and of course the same underlying).

For a **European put,** the corresponding portfolio consists, therefore, of the following components:

- Call **long**
- Underlying **short**
- Zero bond (face value = X) **long**

If the put-call parity does not hold, there are **arbitrage opportunities**.

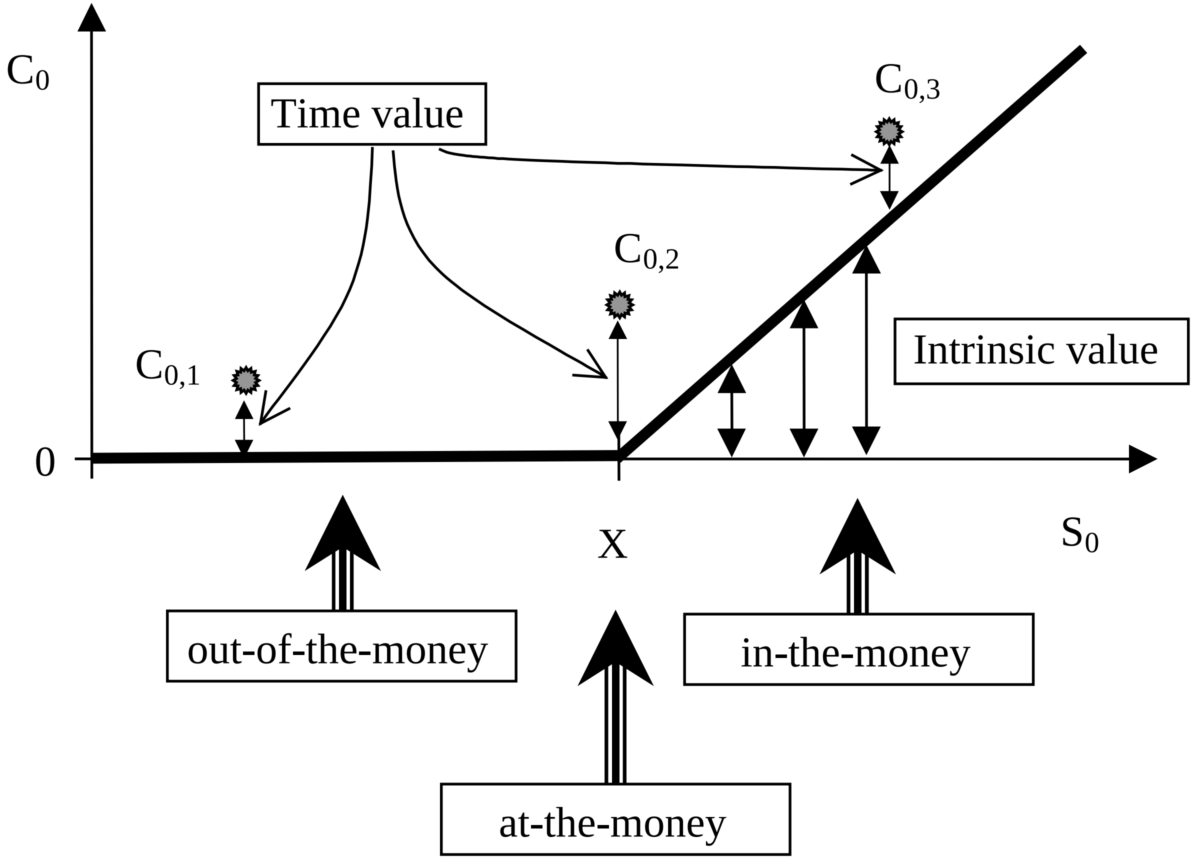
Example

Suppose that the current price of a European call is €12 and the price of the corresponding European put is €1.

Both options have a time to maturity of 6 months, an exercise price of €72 and an underlying trading currently at €80. The 6-months discount factor (P(T)) is 0.9535 (continuously compounded: $e^{-r \cdot T} = 0.9535$)

Are these prices arbitrage free? If not, how does the corresponding arbitrage portfolio look like?

❍ Intrinsic value and time value



Intrinsic value: Is the maximum of zero and the value the option would have if it is exercised immediately.

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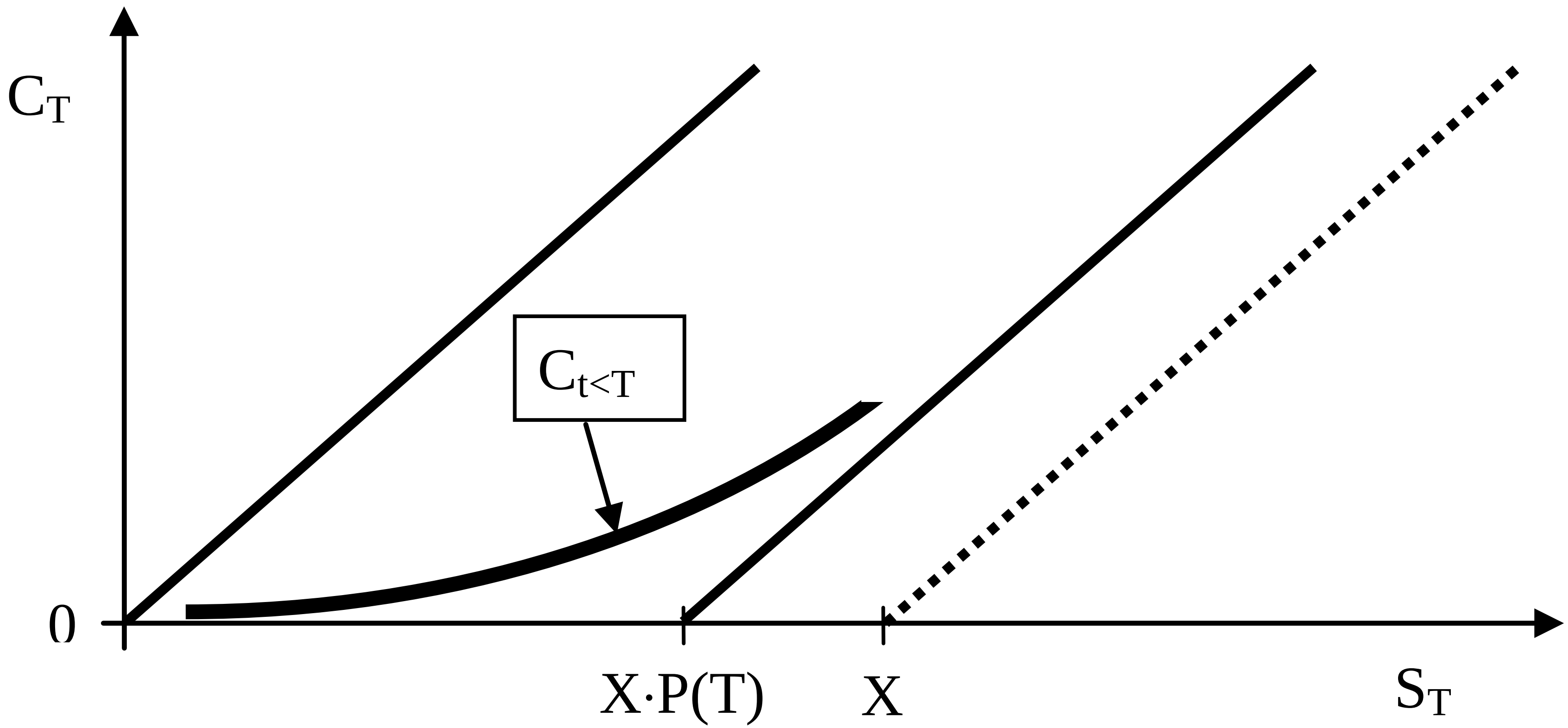
Factors affecting stock option prices

	Call	Put
current stock price, S_0	+	-
strike price, X	-	+
time to expiration, T	+	+
volatility of the stock price, σ	+	+
risk-free interest rate, r	+	-
dividends expected during the life of the option	-	+

❍ Option value before maturity



Example: Options on Siemens (11.2.2011, EUREX), stock price (close) = €94.68

Calls

Strike	Last	Settlement Price	Trading Volume	Open Interests
110		0.02	0	0
105		0.02	0	2,065
100		0.09	0	6,419
98	0.15	0.24	25	5,484
96	0.69	0.65	369	5,251
94	1.58	1.57	231	4,458
92		3.03	0	2,722
90		4.80	0	267
88		6.72	0	102
86		8.72	0	4
84		10.71	0	0
82		12.71	0	75
80		14.70	0	0

Example: Options on Siemens (11.2.2011, EUREX), stock price (close) = €94.68

Puts

Strike	Last	Settlement Price	Trading Volume	Open Interests
110		15.32	0	209
105		10.32	0	406
100		5.40	0	526
98		3.56	0	891
96		1.95	0	551
94	0.81	0.88	821	2,764
92	0.41	0.34	21	2,479
90		0.11	0	4,030
88	0.05	0.04	111	3,808
86	0.03	0.02	200	4,765
84		0.02	0	1,424
82		0.01	0	3,377
80	0.01	0.01	1	2,025

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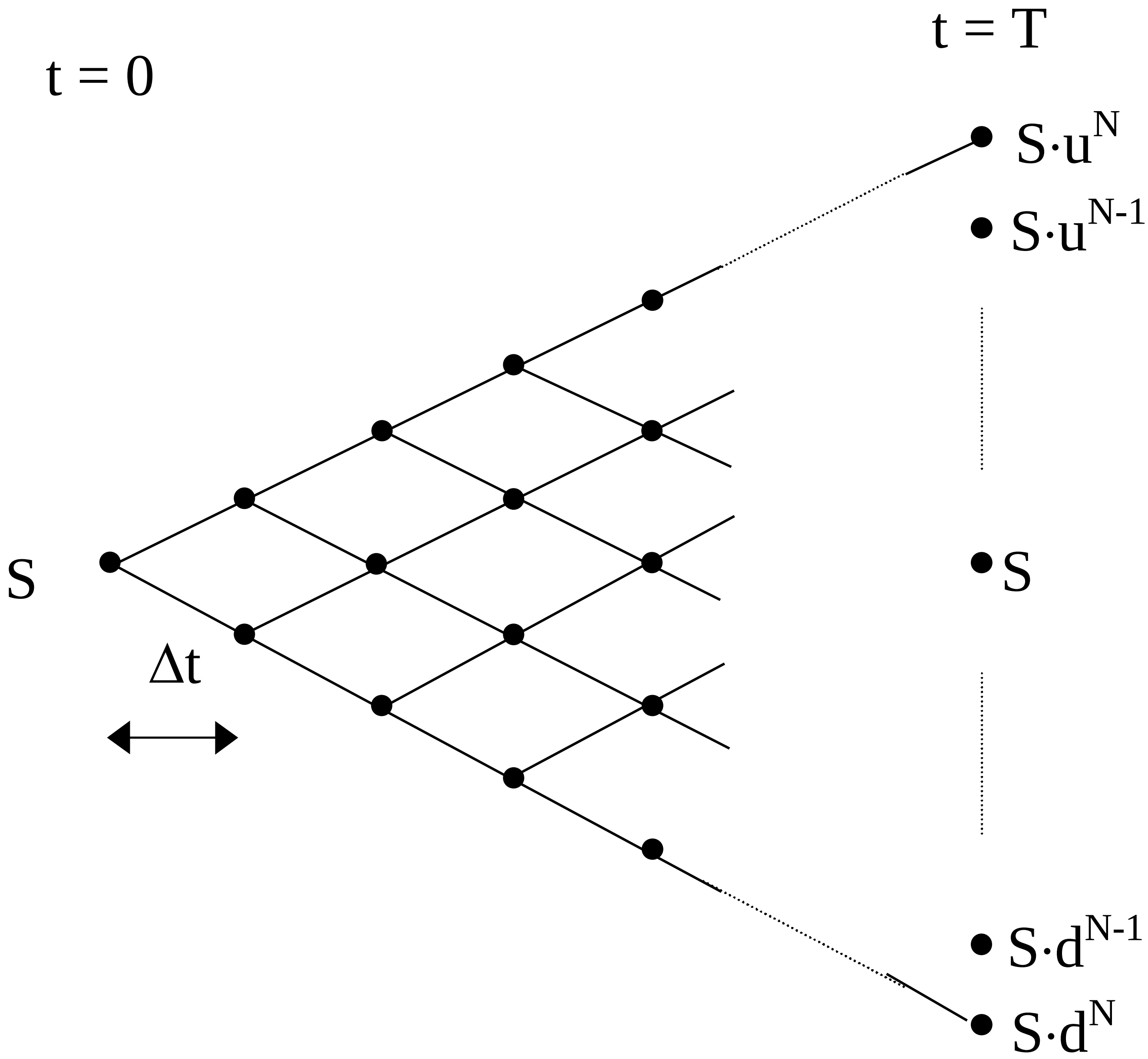
6. Valuation II: Binominal Model

- The (remaining) maturity is divided into (small) sub-periods
- In each sub-period: The price of the underlying can either increase by the factor *u* or decrease by the factor *d*:

$$u > 1, d < 1$$

- $(u - 1)$ = relative value change of the underlying in case of a price increase
- $(1 - d)$ = relative value change of the underlying in case of a price decrease
- For N sub-periods: 2^N possible trails for the price development of the under-lyings

  e.g.: 30 sub-periods ➔ more than one billion possible price trails


t = T
t = 0
$S \cdot u^N$
$S \cdot u^{N-1}$
S
S
Δt
$S \cdot d^{N-1}$
$S \cdot d^N$

6.1 One-step model

Example: Replication model

Call Option: Maturity = 3 months

Strike price = €100

S_0 = €100

$S_0 \cdot u$ = €130

$S_0 \cdot d$ = €80

Zero bond (ZB): ZB_0 = €99, ZB_1 = €100, maturity = 3 months

Expected probabilities: q = (1-q) = 50%

What is the arbitrage free value of the option?

t = 0 t = 3 Mon = T

Underlying:

$S_u = 100 \cdot 1.3 = 130$

$S_0 = 100$

$S_d = 100 \cdot 0.8 = 80$

Zero bond:

$ZB_u = 100$

$ZB_0 = 99$

$ZB_d = 100$

Call:

$C_u = \max[S_u - X, 0] = 30$

$C_0 = ?$

$C_d = \max[S_d - X, 0] = 0$

❍ Replication of C via S and ZB

System of equations**:**

$$
\begin{array}{rcccl}
S & + & ZB & = & C \\
130 \cdot h & + & 100 \cdot m & = & 30 \\
80 \cdot h & + & 100 \cdot m & = & 0 \\
\hline
50 \cdot h & + & 0 & = & 30
\end{array}
$$

h: Amount of the underlying in the replication portfolio (for one call)

m: Amount of the zero bond in the replication portfolio (for one call)

$$h = \frac{C_u - C_d}{S_u - S_d} = \frac{30}{50} = 0.6 \text{ units}$$

$$m = \frac{C_u - S_u \cdot h}{ZB_u} = \frac{C_d - S_d \cdot h}{ZB_d} = \frac{30 - 130 \cdot 0.6}{100} = \frac{-48}{100} = -0.48 \text{ units}$$

Replication portfolio:

0.60 · Underlying	LONG:	0.6·(-100) =	-60.00
0.48 · Zero bond	SHORT:	0.48·99 =	47.52
1.00 · Call	LONG:		**-12.48**

Thus, the arbitrage free value of the calls is €12.48.

Check:

	t = 0	t = T	
		S_d = 80	S_u = 130
h · Underlying LONG	-60	48	78
m · Zero bond SHORT	+47.52	-48	-48
= 1 · Call LONG	-12.48	**0**	**30**

or

	t = 0	t = T	
		S_d = 80	S_u = 130
h · Underlying LONG	-60	48	78
1 · Call SHORT	+12.48	0	-30
= m · Zero bond LONG	-47.52	**48**	**48**

Portfolio:
- h units of the underlying (LONG)
- one call option SHORT

Secure final wealth!

↳ Only can earn the risk free rate

Value of the portfolio:

$$h \cdot S_0 - C_0 = \left(h \cdot S_0 \cdot u - C_u\right) \cdot P(T) = \left(h \cdot S_0 \cdot d - C_d\right) \cdot P(T)$$

From this follows the value of the option:

$$C_0 = h \cdot S_0 - (h \cdot S_0 \cdot u - C_u) \cdot P(T)$$

or

$$C_0 = h \cdot S_0 - (h \cdot S_0 \cdot d - C_d) \cdot P(T)$$

P(T) = Discount factor for maturity *T* of the option ($P(T) = e^{-r \cdot T}$)

$$C_0 = 0.6 \cdot 100 - (0.6 \cdot 100 \cdot 1.3 - 30) \cdot \frac{99}{100} = 12.48$$

$$C_0 = 0.6 \cdot 100 - (0.6 \cdot 100 \cdot 0.8 - 0) \cdot \frac{99}{100} = 12.48$$

Thus, independent from the future share price development a secure end value is obtained. Through transformation, we receive the **hedge ratio** *h*:

$$h = \frac{C_u - C_d}{S_0 \cdot u - S_0 \cdot d}$$

Thus, if the portfolio contains *h* units of the underlying it is riskless and yields as return the riskless interest rate *r*.

To determine the current option value (C) based on **Cox/Ross/Rubinstein (1979),** a view more transformations are necessary:

$$\left(h \cdot S_0 \cdot u - C_u\right) \cdot e^{-r \cdot T} = h \cdot S_0 - C$$

$$\left(-C_u + h \cdot S_0 \cdot u\right) = \frac{1}{e^{-r \cdot T}} \cdot \left(-C + h \cdot S_0\right)$$

As

$$h \cdot S_0 = \frac{C_u - C_d}{u - d}$$

We get

$$\left(-C_u + \frac{C_u - C_d}{u - d} \cdot u\right) = \frac{1}{e^{-r \cdot T}} \cdot \left(-C + \frac{C_u - C_d}{u - d}\right)$$

$$C \cdot \frac{1}{e^{-r \cdot T}} = C_u - u \cdot \frac{C_u - C_d}{u - d} + \frac{1}{e^{-r \cdot T}} \cdot \frac{C_u - C_d}{u - d}$$

$$C \cdot \frac{1}{e^{-r \cdot T}} = \frac{\frac{1}{e^{-r \cdot T}} \cdot C_u}{u - d} - \frac{u \cdot C_u}{u - d} + C_u + \frac{u \cdot C_d}{u - d} - \frac{\frac{1}{e^{-r \cdot T}} \cdot C_d}{u - d}$$

$$C \cdot \frac{1}{e^{-r \cdot T}} = \frac{\frac{1}{e^{-r \cdot T}} - d}{u - d} \cdot C_u + \frac{u - \frac{1}{e^{-r \cdot T}}}{u - d} \cdot C_d$$

When the two ratios on the right side are replaced by ***p*** bzw. ***(1-p)***, thus:

$$p = \frac{\frac{1}{e^{-r \cdot T}} - d}{u - d} = \frac{e^{r \cdot T} - d}{u - d} \quad \text{and} \quad (1 - p) = \frac{u - \frac{1}{e^{-r \cdot T}}}{u - d} = \frac{u - e^{r \cdot T}}{u - d},$$

we receive the **arbitrage-free value of the European call option** according to the risk-neutral valuation based on Cox, Ross, and Rubinstein (1979):

$$C = e^{-r \cdot T} \cdot \left[p \cdot C_u + (1 - p) \cdot C_d \right]$$

Likewise, one receives for a **European put option**:

$$P = e^{-r \cdot T} \cdot \left[p \cdot P_u + (1 - p) \cdot P_d \right]$$

p: Probability for a price increase

(1-p): Probability for a price decrease

➔ In a risk-neutral world (pseudo-probabilities)

➔ Both valuation methods (replication portfolio and risk-neutral valuation) provide the same result.

(a) C_0 = €15:

The **call is too expensive.**

Strategy: **Buy *h* shares, sell one call option,** and finance the missing money (at t=0) via a loan.

	t = 0	t = T	
	Portfolio	$S_d = 80$	$S_u = 130$
h shares long	-60	48	78
1 Call short	+15	0	-30
Zero bond short (loan)	+45	-45.455	-45.455
	0	**+2.545**	**+2.545**

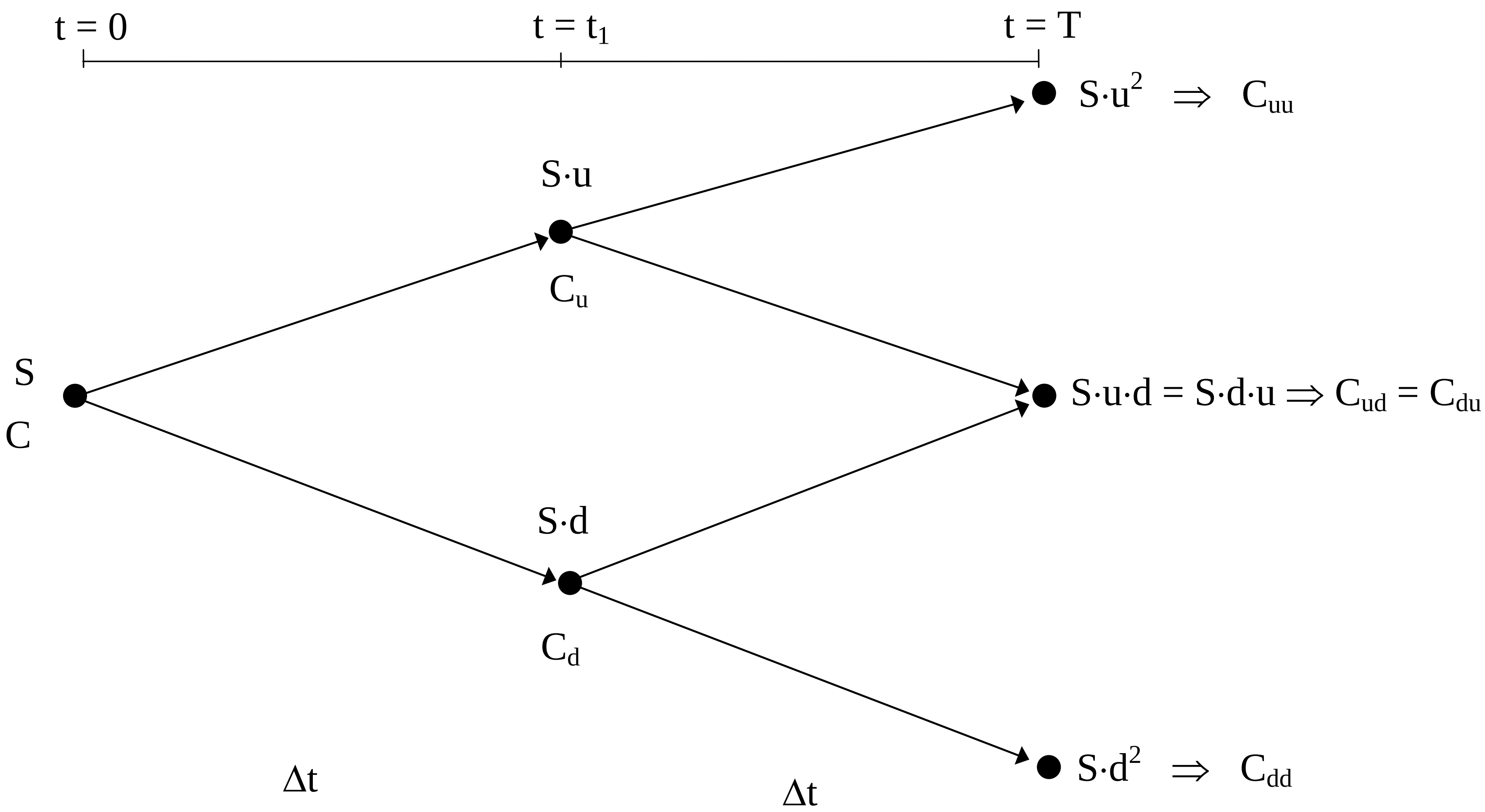
(b) C_0 = €10:

The **call is too cheap.**

Strategy: **Sell *h* shares, buy one call option,** and invest the remaining money (at t=0).

	t = 0	t = T	
	Portfolio	S_d = 80	S_u = 130
h shares short	60	-48	-78
1 Call long	-10	0	30
Zero bond long (investment)	-50	+50.505	+50.505
	0	**+2.505**	**+2.505**

6.2 Two-step model



- Relative change in the price of the underlying: modeled via *u* and *d* (identical in each sub-period)

- Sub-periods: in each case a length of Δt

  ➔ Risk-neutral probability *p* is the same in each node

Repeated application of the Cox/Ross/Rubinstein (1979) valuation model.

Starting point: Pay-off structure of *C* at maturity

Value of the option at $t = t_1$:

$$C_u = e^{-r\cdot\Delta t}\cdot\left[p\cdot C_{uu} + (1-p)\cdot C_{ud}\right]$$

$$C_d = e^{-r\cdot\Delta t}\cdot\left[p\cdot C_{ud} + (1-p)\cdot C_{dd}\right]$$

Value at t = 0:

$$C = e^{-r\cdot\Delta t}\cdot\left[p\cdot C_u + (1-p)\cdot C_d\right]$$

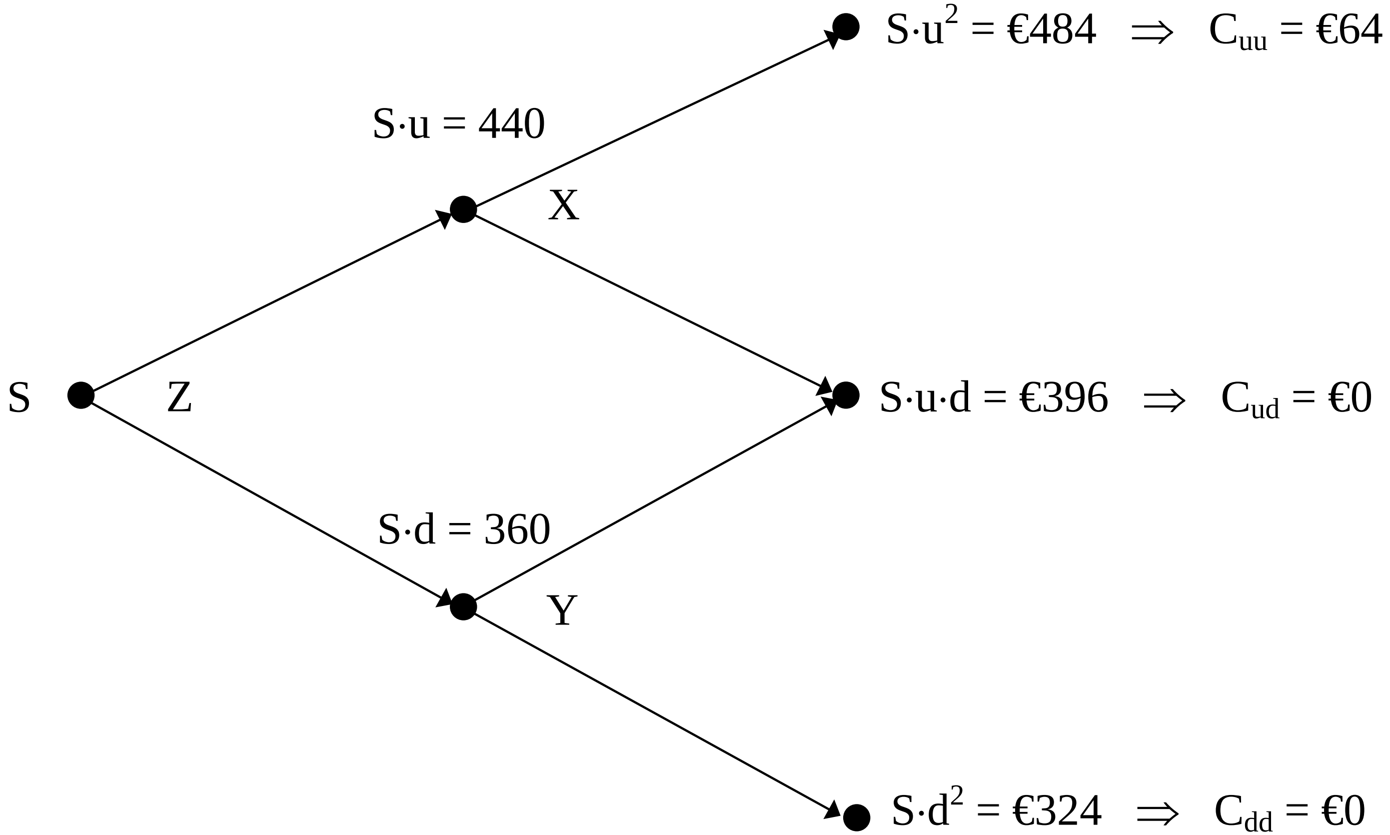
$$C = e^{-r\cdot\Delta t\cdot 2} \cdot \left\{ p^2 \cdot C_{uu} + 2 \cdot p \cdot (1-p) \cdot C_{ud} + (1-p)^2 \cdot C_{dd} \right\}$$

p^2	Probability that we reach at $t = T$ the upper node.
$2 \cdot p \cdot (1-p)$	Probability that we reach at $t = T$ the middle node.
$(1-p)^2$	Probability that we reach at $t = T$ the lower node.

The price of the call option corresponds to the expected value of the pay-off structure at maturity in a risk-neutral world, discounted with the risk-less rate (= risk-neutral valuation).

Example

S = €400, X = €420, (u - 1) = 10%, (1 – d) = -10%, $e^{-r \cdot \Delta t}$ = 0.995.



Calculation of the option prices for nodes *X* and *Y*. The risk-neutral probability *p* is:

$$p = \frac{e^{r\cdot\Delta t} - d}{u - d} = \frac{\frac{1}{e^{-r\cdot\Delta t}} - d}{u - d} = \frac{\frac{1}{0.995} - 0.9}{1.1 - 0.9} = 0.5251256$$

Node *X*:

$$\begin{aligned} C_u &= e^{-r\cdot\Delta t}\cdot\left[p\cdot C_{uu} + (1-p)\cdot C_{ud}\right] \\ &= 0.995\cdot\left[0.525126\cdot 64 + (1-0.525126)\cdot 0\right] \\ &= €33.44. \end{aligned}$$

Node *Y*:

$$\begin{aligned} C_d &= e^{-r\cdot\Delta t}\cdot\left[p\cdot C_{ud} + (1-p)\cdot C_{dd}\right] \\ &= 0.995\cdot\left[0.525126\cdot 0 + (1-0.525126)\cdot 0\right] \\ &= €0. \end{aligned}$$

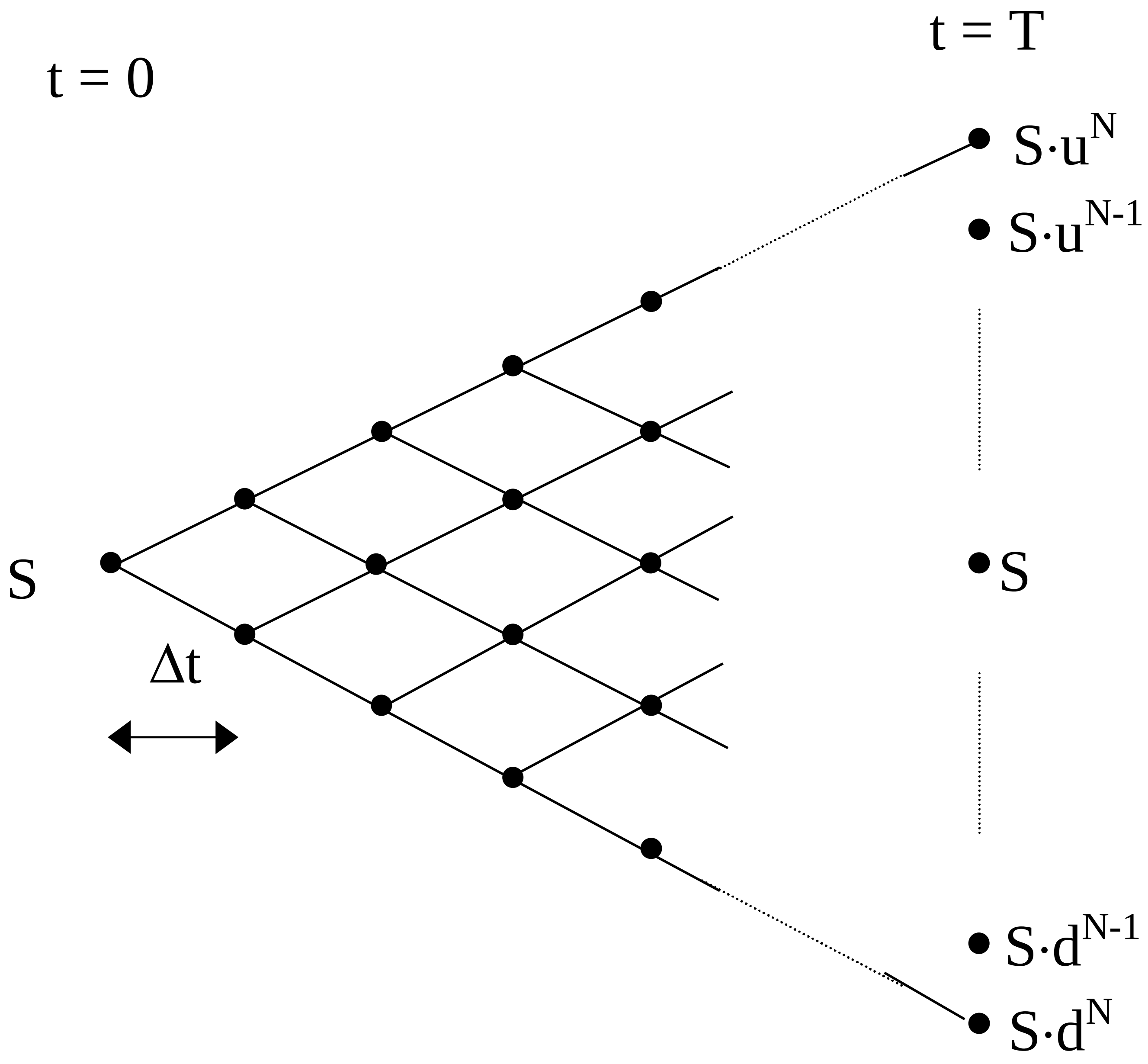
Current value of C:

$$
\begin{aligned}
C &= e^{-r\cdot\Delta t}\cdot\left[p\cdot C_u + (1-p)\cdot C_d\right]\\
&= 0.995\cdot\left[0.525126\cdot 33.44 + (1-0.525126)\cdot 0\right]\\
&= €17.47\,.
\end{aligned}
$$

If only the current option value is of interest:

$$
\begin{aligned}
C &= e^{-r\cdot\Delta t\cdot 2}\cdot\left\{p^2\cdot C_{uu} + 2\cdot p\cdot(1-p)\cdot C_{ud} + (1-p)^2\cdot C_{dd}\right\}\\
&= (0.995)^2\cdot\left\{(0.5251256)^2\cdot 64 + 0 + 0\right\}\\
&= €17.47
\end{aligned}
$$

6.3 N-step model



u and *d* depend on the volatility σ of the underlying. Based on the transformation of σ into *u* and *d* proposed by Cox, Ross, and Rubinstein (1979) we can calculate *u* and *d* as:

$$u = e^{\sigma \cdot \sqrt{\Delta t}} \qquad d = \frac{1}{u} = e^{-\sigma \cdot \sqrt{\Delta t}}$$

Value of a call option at **maturity**:

$$C_{N,j} = \max\lfloor 0, S_{N,j} - X \rfloor, \qquad 0 \le i \le N, 0 \le j \le i$$

N: Total number of sub-periods
i: Number of sub-periods **since *t = 0***
j: Number of sub-periods **with a price increase**

Valuation for **node (*i,j*)**: $C_{i,j} = e^{-r \cdot \Delta t} \cdot \left[p \cdot C_{i+1,j+1} + (1-p) \cdot C_{i+1,j} \right]$

6.4 American Put Option

At maturity: Value of an American option = Value of an European option

In nodes before maturity: The value of the American put option is the maximum of

- the value of an identical European option in the same node
- the proceeds if the option is exercised early

Thus, in each node it is necessary to check, whether an early exercise is favorable.

In node (*i,j*) an American put option has a value of:

$$P_{i,j} = \max\left\{\left(X - S \cdot u^{j} \cdot d^{(i-j)}\right),\, e^{-r \cdot \Delta t} \cdot \left[p \cdot P_{i+1,j+1} + (1-p) \cdot P_{i+1,j}\right]\right\}$$

Example

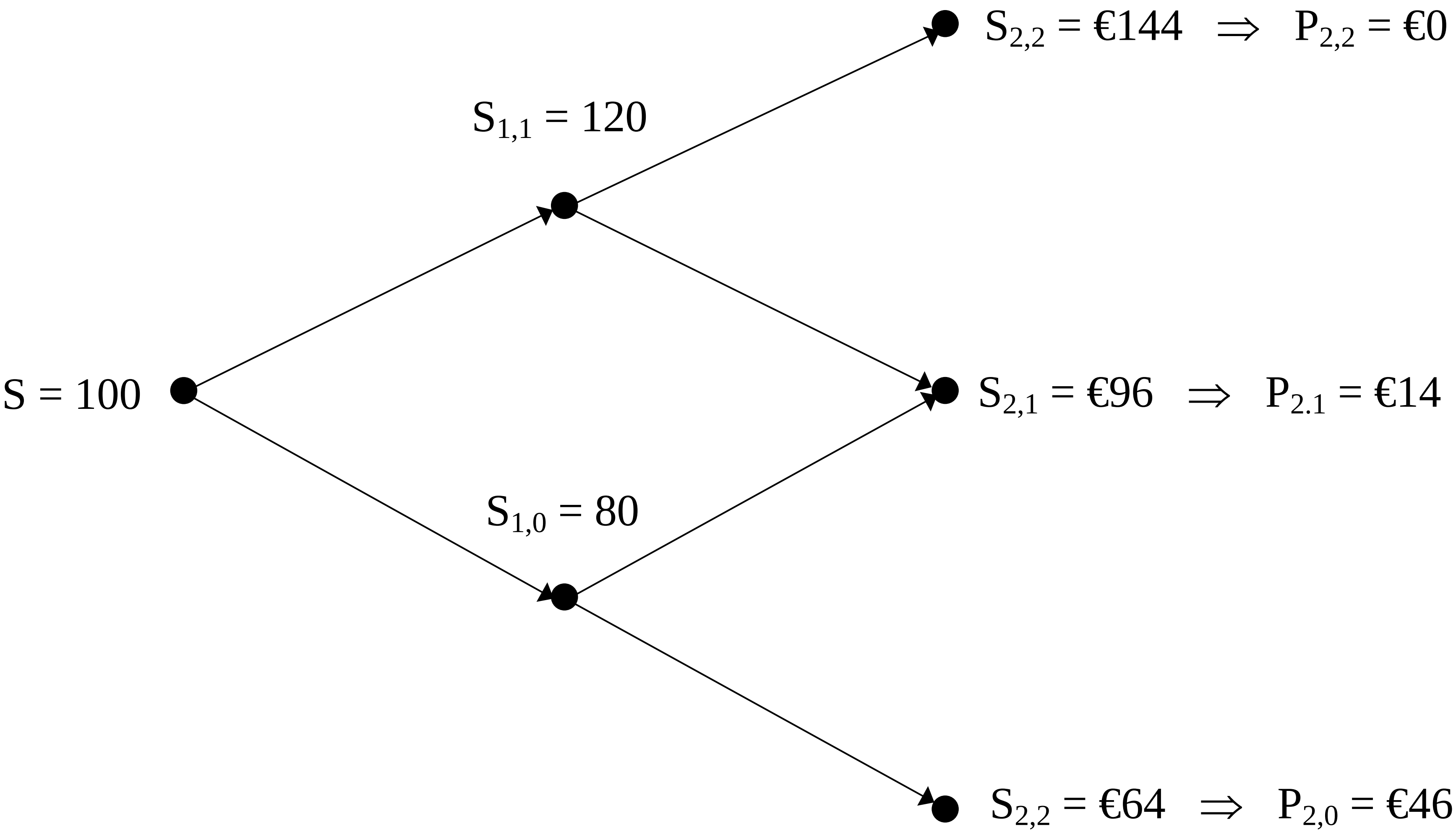
Valuation of an American put option: T = 2 years, X = €110, S = €100

2 sub-periods, share price ± 20% in each sub-period,
$P(\Delta t = 1\text{Jahr}) = 0.95 = e^{-r\cdot\Delta t}$

Risk-neutral probability *p*:

$$p = \frac{e^{r\cdot\Delta t} - d}{u-d} = \frac{\frac{1}{e^{-r\cdot\Delta t}} - d}{u-d} = \frac{\frac{1}{0.95} - 0.8}{1.2-0.8} = 0.63158$$

Share price development:



Value of the option at **node (1,1):**

$$P_{1,1} = \max\left\{\left(X - S \cdot u^{1} \cdot d^{0}\right),\ e^{-r\cdot\Delta t} \cdot \left[p \cdot P_{2,2} + (1-p) \cdot P_{2,1}\right]\right\}$$
$$= \max\left\{(110 - 120),\ 0.95 \cdot \left[0.63158 \cdot 0 + (1 - 0.63158) \cdot 14\right]\right\}$$
$$= \max\{-10,\ 4.90\} = €4.90$$

Thus, <u>early exercise</u> is <u>not useful</u> in node (1,1). Option value in **node (1,0)**:

$$P_{1,0} = \max\left\{\left(X - S \cdot u^{0} \cdot d^{1}\right),\ e^{-r\cdot\Delta t} \cdot \left[p \cdot P_{2,1} + (1-p) \cdot P_{2,0}\right]\right\}$$
$$= \max\left\{(110 - 80),\ 0.95 \cdot \left[0.63158 \cdot 14 + (1 - 0.63158) \cdot 46\right]\right\}$$
$$= \max\{30,\ 24.50\} = €30$$

<u>Early exercise</u> is <u>useful</u> in node (1,0). Instead of *24.50* (= value of a European put option) the value of the American put option amounts to €30 in this node.

The **current value of this option** (node (0,0)) can be calculated as

$$\begin{aligned} P_{0,0} &= \max\left\{\left(X - S \cdot u^0 \cdot d^0\right), e^{-r \cdot \Delta t} \cdot \left[p \cdot P_{1,1} + (1-p) \cdot P_{1,0}\right]\right\} \\ &= \max\left\{(110 - 100), 0.95 \cdot \left[0.63158 \cdot 4.90 + (1 - 0.63158) \cdot 30\right]\right\} \\ &= \max\{10, 13.44\} = €13.44 \end{aligned}$$

7. Valuation III: Black-Scholes-Merton Model

Fischer **Black** and Myron **Scholes** (1973, Nobel Prize in Economics 1997)

➔ Valuation equation for **European** **call** and **put** options ('standard model')

Cornerstones during the derivation:

(a) **Price development of the underlying** = **random**

(and can be characterized via stochastic differential equations)

(b) **Pay-off structure** of a call (put) option at maturity.

(c) Itô’s Lemma („algorithm“ to solve stochastic differential equations).

As a result, one gets for the value of a **European call option**:

$$C_0 = S_0 \cdot N\{d_1\} - X \cdot e^{-r \cdot T} \cdot N\{d_2\} \qquad \text{(BSM 1)}$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2} \cdot \sigma^2\right) \cdot T}{\sigma \cdot \sqrt{T}}, \quad d_2 = d_1 - \sigma \cdot \sqrt{T}$$

N{•}: Cumulative density of the standard normal distribution

Eq. (BSM 1) is also applicable for American calls on stocks if Div < (X - X·$e^{-r \cdot T}$)

For **European put options**:

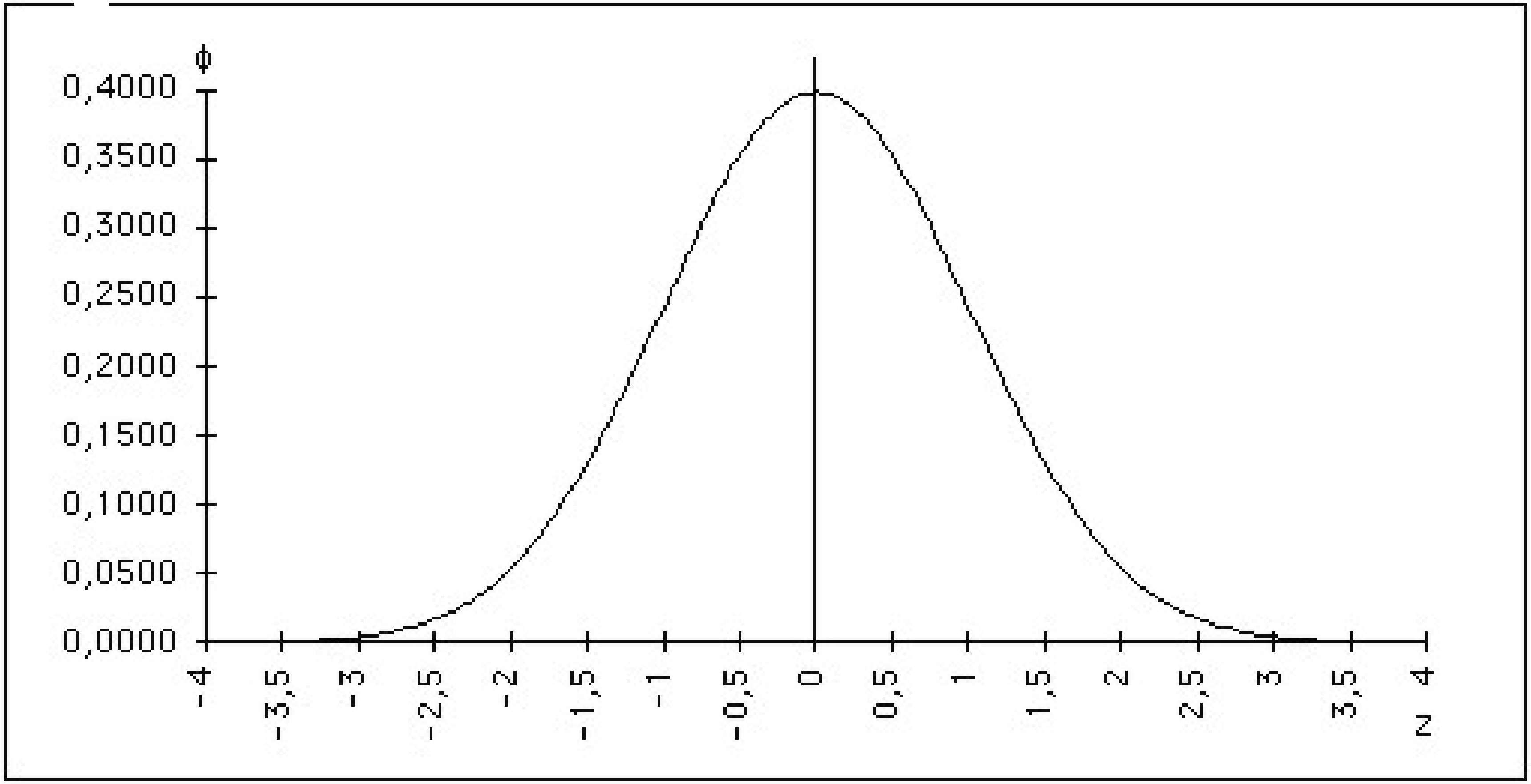
$$P_0 = X \cdot e^{-r \cdot T} \cdot N\{-d_2\} - S_0 \cdot N\{-d_1\} \qquad \text{(BSM 2)}$$

Important assumptions:

- Only for European options, especially not for American put options.
- No payouts during maturity.
- Constant volatility during maturity.
- Constant interest rate *r* during maturity.
- The price changes of the underlying are random.

❍ Cumulative density function of the standard normal distribution

$\Phi(z)_{0,1}$ ($\mu = 0$, $\sigma = 1$); e.g., N(1.0) = area from $z = -\infty$ until 1.0



❍ Cumulative density function of the standard normal distribution

$\Phi(z)_{0,1}$ (μ = 0, σ = 1); e.g., N(1.0) = 0.84134

z \ *	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.0*	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1*	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2*	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3*	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4*	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5*	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6*	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7*	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8*	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9*	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0*	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1*	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2*	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3*	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774

❍ Cumulative density function of the standard normal distribution

$\Phi(z)_{0,1}$ ($\mu = 0$, $\sigma = 1$) = probability that a variable with a standard normal distribution will be less than z

z \ *	0	1	2	3	4	5	6	7	8	9
1.4*	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5*	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6*	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7*	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8*	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9*	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0*	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1*	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2*	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3*	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4*	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5*	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6*	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7*	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736

❍ Cumulative density function of the standard normal distribution

$\Phi(z)_{0,1}$ ($\mu = 0$, $\sigma = 1$)

z \ *	0	1	2	3	4	5	6	7	8	9
2.8*	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9*	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0*	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1*	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2*	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3*	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4*	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5*	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6*	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7*	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8*	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995

Note: Negative values are not shown, as the distribution is symmetric. Thus, $\Phi(-z) = 1 - \Phi(z)$. The asterisk * is a wildcard for subsequent decimal places shown in the columns.