Aufgabe 1 (Gruppe A)

$x(t) = \frac{1}{2}\left(\cos(2t), \sin(2t), 2t\right)$, $t \in [0, \pi]$

a) $\dot{x}(t) = \left(-\sin(2t), \cos(2t), 1\right)$ (1p)

$\dot{x}(t) \neq 0$ in $[0, \pi]$ $\Rightarrow$ Param. ist regulär (1p)

b) $\|\dot{x}\| = \sqrt{-\sin^2(2t) + \cos^2(2t) + 1} = \sqrt{2}$;

$s(t) = \pm \int_0^t \|\dot{x}(t)\| \, dt = \pm \int_0^t \sqrt{2} \, dt = \pm \sqrt{2}\, t + c$

$\rightarrow$ wähle $\underline{s(t) = \sqrt{2}\, t}$ (1p) mit + und $c = 0$;

Länge der Kurve in $[0, \pi]$

$\ell = s(\pi) - s(0) = \sqrt{2}\, \pi$ (1p) ;

c) Bogenlängenparam.:

$s(t) = \sqrt{2}\, t \rightarrow t(s) = \frac{s}{\sqrt{2}}$ (1p)

$x(s) = \frac{1}{2}\left(\cos\left(\frac{2s}{\sqrt{2}}\right), \sin\left(\frac{2s}{\sqrt{2}}\right), \frac{2s}{\sqrt{2}}\right)$

oder (1p)

$= \frac{1}{2}\left(\cos(\sqrt{2}\, s), \sin(\sqrt{2}\, s), \sqrt{2}\, s\right)$

Aufgabe 2 (Gruppe A)

$$c(t) = \left(t, \frac{1}{2}t^2\right)$$

a) $\dot{c}(t) = (1, t)$ 1p , $\ddot{c}(t) = (0, 1)$ 1p

$$t(t) = \frac{(\dot{x}, \dot{y})}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} 1 \\ t \end{pmatrix} = \frac{(1, t)}{\sqrt{1+t^2}} \quad \text{1p}$$

$$n(t) = \frac{(-\dot{y}, \dot{x})}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{(-t, 1)}{\sqrt{1+t^2}} \quad \text{1p}$$

$$\kappa(t) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{1}{(t+1)^{3/2}} \quad \text{1p}$$

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b) Evolute: $e(t) = c(t) + \frac{1}{\kappa(t)} n(t)$

$$e(t) = \begin{pmatrix} t \\ \frac{1}{2}t^2 \end{pmatrix} + (t^2+1)^{3/2} \begin{pmatrix} -\frac{t}{(t^2+1)^{1/2}} \\ \frac{1}{(t^2+1)^{1/2}} \end{pmatrix} \quad \text{1p}$$

$$= \left(-t^3, \frac{3}{2}t^2 + 1\right) \quad \text{1p} \; ;$$

Aufgabe 3 (Gruppe B)

$$x(u,v) = \begin{pmatrix} \cos(u)\cosh(v) \\ \sin(u)\cosh(v) \\ v \end{pmatrix}$$

a) Erste Fundamentalform

$$x_u = \begin{pmatrix} -\sin(u)\cosh(v) \\ \cos(u)\cosh(v) \\ 0 \end{pmatrix} \quad (1p) \qquad x_v = \begin{pmatrix} \cos(u)\sinh(v) \\ \sin(u)\sinh(v) \\ 1 \end{pmatrix} \quad (1p)$$

$$E = x_u^T x_u = \sin^2(u)\cosh^2(v) + \cos^2(u)\cosh^2(v)$$
$$= \underbrace{(\sin^2 u + \cos^2 u)}_{1}\cosh^2 v = \underline{\underline{\cosh^2(v)}};$$

$$F = x_u^T x_v = -\sin(u)\cos(u)\cosh(v)\sinh(v) +$$
$$\cos(u)\sin(u)\cosh(v)\sinh(v) = \underline{\underline{0}};$$

$$G = x_v^T x_v = \cos^2(u)\sinh^2(v) + \sin^2(u)\sinh^2(v) + 1$$
$$= (\sin^2(u) + \cos^2(u))\sinh^2(v) + 1$$
$$= \sinh^2(v) + 1 = \underline{\cosh^2(v)};$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\underline{I} = \begin{bmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{bmatrix} \quad (2p)$$

b)

$$A = \iint \sqrt{EG - F^2}\, du\, dv$$

$$\rightarrow \int_{-1}^{1}\int_{0}^{2\pi} \sqrt{\cosh^4(v)}\, du\, dv \quad (1p) = \int_{-1}^{1}\int_{0}^{2\pi} \cosh^2(v)\, du\, dv$$

$$= 2\pi \int_{-1}^{1} \cosh^2(v)\, dv \quad (1p)$$

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Stammfunk. von $\int \cosh^2 x \, dx$ → partielle Integr.

$$\int \cosh^2 x \, dx = \int \cosh x \cosh x \, dx$$

$\boxed{\cosh^2 x - \sinh^2 x = 1}$

→ $f' = \cosh x \longrightarrow f = \sinh x$

$g = \cosh x \longrightarrow g' = \sinh x$

$\boxed{\begin{array}{l} \frac{d}{dx} \sinh x = \cosh x \\ \frac{d}{dx} \cosh x = \sinh x \end{array}}$

$$= \sinh x \cosh x - \int \sinh^2 x \, dx$$

$\boxed{\int f' g = f \cdot g - \int f \cdot g'}$

$$= \sinh x \cosh x - \int (\cosh^2 x - 1) \, dx$$

$$= \sinh x \cosh x - \int \cosh^2 x \, dx + \int 1 \, dx$$

$$= \sinh x \cosh x + x + c_1 - \int \cosh^2 x \, dx \quad \Big| + \int \ldots \quad \Big| \cdot \frac{1}{2}$$

$$= \boxed{\frac{1}{2}(\sinh x \cosh x + x) + c}$$

Stammfunk. einsetzen:

$$2\pi \int_{-1}^{1} \cosh^2(v) \, dv = 2\pi \left[\frac{1}{2} (\sinh x \cosh x + x) \right]_{-1}^{+1}$$

$$= 2\pi \left(\frac{1}{2} (\sinh(1) \cosh(1) - \sinh(-1) \cosh(-1)) + 1 \right)$$

$$= 2\pi \left(\frac{1}{2} (2 \cosh(1) \sinh(1) + 1) \right)$$

oder

1p

oder

$$= 2\pi \left(\frac{1}{2} \sinh(2) + 1 \right) = \boxed{\pi(\sinh(2) + 2)} \approx 17{,}6773$$

oder $\boxed{\frac{\pi}{2}(4 - e^{-2} + e^{2})}$ oder mit Substitution und $\boxed{\cosh x = \frac{1}{2}(e^x + e^{-x})}$