207.) $f(t) = \begin{cases} -1 & (t \leq 1) \\ t & (t > 1) \end{cases}$

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} \int_0^x -1\,dt & (x \leq 1) \\ \int_0^1 -1\,dt + \int_1^x t\,dt & (x > 1) \end{cases} = \begin{cases} (-t + C_1)\Big|_0^x & (x \leq 1) \\ (-t + C_2)\Big|_0^1 + \left(\frac{t^2}{2} + C_3\right)\Big|_1^x & (x > 1) \end{cases} =$$

$$= \begin{cases} -x & (x \leq 1) \\ -1 + \frac{x^2}{2} - \frac{1}{2} & (x > 1) \end{cases} = \begin{cases} -x & (x \leq 1) \\ \frac{x^2 - 3}{2} & (x > 1) \end{cases}$$

Stetigkeit:

$$\left.\begin{array}{l} F(1) = -1 \\ \lim\limits_{x \to 1^-} F(x) = -1 \\ \lim\limits_{x \to 1^+} F(x) = \frac{1-3}{2} = -1 \end{array}\right\} \checkmark \Rightarrow \text{stetig}$$

Differenzierbarkeit:

$$\lim_{x \to 1^-} \frac{F(x) - F(1)}{x - 1} = \lim_{x \to 1^-} \frac{-x + 1}{x - 1} = -1$$

$$\lim_{x \to 1^+} \frac{F(x) - F(1)}{x - 1} = \lim_{x \to 1^+} \frac{x^2 - 3 + 2}{2(x-1)} = \lim_{x \to 1^+} \frac{(x+1)(x-1)}{2(x-1)} = \lim_{x \to 1^+} \frac{x+1}{2} = 1$$

$\Rightarrow$ NICHT differenzierbar an Stelle $x_0 = 1$.

210.) $f(x) = x^2$, $F(x) = \int_1^2 x^2 \, dx$

$$O_z(f) = \sum_{i=1}^{n} \max(f(x_{i-1}), f(x_i)) \cdot (x_i - x_{i-1})$$

Wir wissen:

- $f(x)$ monoton wachsend in $[1,2] \Rightarrow \max(f(x_{i-1}), f(x_i)) = f(x_i)$ in $x_i \in [1,2]$
- $\Delta x := x_i - x_{i-1} = \frac{2-1}{n} = \frac{1}{n}$
- $f(x_i) = x_i^2$
- $x_i = 1 + \frac{i}{n}$

$$\left.\begin{array}{l} f(x_i) = x_i^2 \\ x_i = 1 + \frac{i}{n} \end{array}\right\} f(x_i) = \left(1 + \frac{i}{n}\right)^2 = 1 + \frac{2i}{n} + \frac{i^2}{n^2}$$

$$\Rightarrow O_z(f) = \sum_{i=1}^{n} f(x_i) \cdot \frac{1}{n} = \frac{1}{n} \cdot \sum_{i=1}^{n} \left(1 + \frac{2i}{n} + \frac{i^2}{n^2}\right) =$$

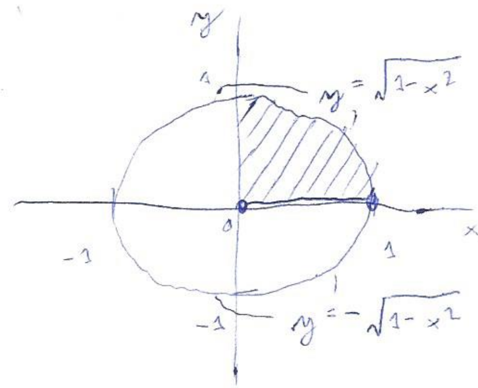
$$= \frac{1}{n}\left(n + \frac{\cancel{2}}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}} + \frac{1}{n^2} \cdot \frac{1}{6}\cancel{n}(n+1)(2n+1)\right) = 1 + \frac{n+1}{n} + \frac{2n^2+3n+1}{6n^2}$$

$$F(x) = \int_1^2 x^2 \, dx = \lim_{n\to\infty} O_z = \lim_{n\to\infty} \left(1 + \frac{n+1}{n} + \frac{2n^2+3n+1}{6n^2}\right) = 1 + 1 + \frac{1}{3} = \frac{7}{3}.$$

215.) $$\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n}\sqrt{n^2-k^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{n}\, n\sqrt{1-\left(\frac{k}{n}\right)^2} = \lim_{n\to\infty}\underbrace{\frac{1}{n}}_{\Delta x}\sum_{k=1}^{n}\sqrt{1-\underbrace{\left(\frac{k}{n}\right)}_{x_k}^2} =$$

$$= \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$

I



II $$\int_0^1 \sqrt{1-x^2}\,dx \left| \begin{array}{l} x = \sin u \\ dx = \cos u\, du \end{array} \right| = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 u}\cdot \cos u\, du =$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 u\, du = \int_0^{\frac{\pi}{2}} \cos u \cdot \cos u\, du =$$

$$= \sin u \cos u \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\sin^2 u\, du =$$

$$= \sin u \cos u \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (1-\cos^2 u)\, du =$$

$$= \sin u \cos u \Big|_0^{\frac{\pi}{2}} + u\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos^2 u\, du$$

$$2\int_0^{\frac{\pi}{2}} \cos^2 u\, du = \sin u \cos u \Big|_0^{\frac{\pi}{2}} + u\Big|_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 u\, du = \frac{1}{2}\left(\sin u \cos u \Big|_0^{\frac{\pi}{2}} + u\Big|_0^{\frac{\pi}{2}}\right) =$$

$$= \frac{1}{2}\left(1\cdot 0 - 0\cdot 1 + \frac{\pi}{2} - 0\right) = \frac{\pi}{4}$$

218.) zu zeigen: $\displaystyle\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C$

Beweis:

$$\int \frac{u'(x)}{u(x)}\,dx = \int \frac{1}{u(x)} \cdot u'(x)\,dx \;\Bigg|\; \begin{array}{l} v = u(x) \\ dv = u'(x)\,dx \end{array} =$$

$$= \int \frac{1}{v}\,dv = \ln|v| + C = \ln|u(x)| + C \quad \blacksquare$$

Beispiel:

$$\int \frac{dx}{x \ln x} = \int \frac{1}{\ln x} \cdot \frac{1}{x}\,dx \;\Bigg|\; \begin{array}{l} v = \ln x \\ dv = \frac{1}{x}\,dx \end{array} = \int \frac{1}{v}\,dv =$$

$$= \ln|v| + C = \ln|\ln x| + C.$$

224.)

$$\int \frac{x}{x^3+1}\,dx = \int -\frac{\frac{1}{3}}{x+1}dx + \int \frac{\frac{1}{3}x+\frac{1}{3}}{x^2-x+1}\,dx = -\frac{1}{3}\int\frac{1}{x+1}dx + \frac{1}{3}\int\frac{x+1}{x^2-x+1}dx$$

· Partialbruchzerlegung:

$$\frac{x}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$Ax^2 - Ax + A + Bx^2 + Bx + Cx + C = x$$

$$(A+B)x^2 + (-A+B+C)x + (A+C) = x$$

$$\begin{cases} A+B=0 \\ -A+B+C=1 \\ A+C=0 \end{cases}$$

$B = C = -A$

$-3A = 1$

$A = -\frac{1}{3}$

$B = C = \frac{1}{3}$

für: $\frac{1}{x+1}$

$u_1 = x+1$

$du_1 = dx$

für $\frac{x+1}{x^2-x+1}$

$u_2 = x^2-x+1$

$du_2 = (2x-1)\,dx$

Bruch umformen!

$$\frac{x+1}{x^2-x+1} = \frac{1}{2}\left(\frac{2x-1}{x^2-x+1} + \underbrace{\frac{3}{x^2-x+1}}_{\text{umformen!}}\right)$$

$$\frac{3}{x^2-x+1} = 3\cdot\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} = \cancel{3}\cdot\frac{1}{\frac{3}{4}\left(\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1\right)} = 4\cdot\frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}$$

$u_3 = \frac{2x-1}{\sqrt{3}}$

$du_3 = \frac{2}{\sqrt{3}}\,dx$

$$\int\frac{x}{x^3+1}dx = -\frac{1}{3}\int\frac{1}{u_1}du_1 + \frac{1}{6}\int\frac{1}{u_2}du_2 + \int\frac{\cancel{4}^{2}}{3\cdot\cancel{2}}\cdot\frac{1}{u_3^2+1}\cdot\frac{\sqrt{3}}{\cancel{2}}du_3 =$$

$$= -\frac{1}{3}\ln|u_1| + \frac{1}{6}\ln|u_2| + \frac{\sqrt{3}}{3}\arctan u_3 + C =$$

$$= -\frac{1}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| + \frac{\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C =$$

$$= \frac{1}{6}\left(\ln|\underbrace{x^2-x+1}_{>0}| - \underbrace{2\ln|x+1|}_{(>0\ ?!)}\right) + \frac{\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C =$$

$$= \frac{1}{6}\ln\left(\frac{x^2-x+1}{\underbrace{(x+1)^2}_{>0}}\right) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

244.)

$$\int \frac{dx}{(1+x)\sqrt{x}} = \int \frac{2\cancel{u}\,du}{(1+u^2)\cancel{u}} = 2\int \frac{1}{1+u^2}\,du = 2\arctan u + C =$$

$$= 2\arctan\sqrt{x} + C.$$

$u = \sqrt{x} \Rightarrow u^2 = x$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u} \Leftrightarrow dx = 2u\,du$

251.) $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx =$

$\underline{I} = \int_0^{\frac{\pi}{2}} \frac{\cos 2x + 1}{2} \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos 2x \, dx + \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 \, dx = \frac{1}{2} \left(\frac{\sin 2x}{2} \Big|_0^{\frac{\pi}{2}} + x \Big|_0^{\frac{\pi}{2}} \right) =$

$\left(\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1 \iff \cos^2 x = \frac{\cos 2x + 1}{2} \right)$

$= \frac{1}{2} \left(\sin x \cos x \Big|_0^{\frac{\pi}{2}} + x \Big|_0^{\frac{\pi}{2}} \right) = \frac{1}{2} \left(1 \cdot 0 - 0 \cdot 1 + \frac{\pi}{2} - 0 \right) = \frac{\pi}{4}$

$\underline{II} = \int_0^{\frac{\pi}{2}} \cos x \cdot \cos x \, dx = \sin x \cos x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \cdot (-\sin x) \, dx =$

$= \sin x \cos x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \sin x \cos x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \, dx =$

$= \sin x \cos x \Big|_0^{\frac{\pi}{2}} + x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$2 \cdot \int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \sin x \cos x \Big|_0^{\frac{\pi}{2}} + x \Big|_0^{\frac{\pi}{2}}$

$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{1}{2} \left(\sin x \cos x \Big|_0^{\frac{\pi}{2}} - x \Big|_0^{\frac{\pi}{2}} \right) = \frac{1}{2} \left(1 \cdot 0 - 0 \cdot 1 + \frac{\pi}{2} - 0 \right) = \frac{\pi}{4}.$