VU Discrete Mathematics

Exercises for 14th October 2025

1) Let $G = (V, E)$ be a graph we define an order on V by writing $V = \{v_1, \ldots, v_n\}$. The adjacency matrix of G with respect to this order is denoted by A. Furthermore, let $a_{i,j}^{(k)}$ be the entry of the i-th row and the j-th column of A^k, the k-th power of A.
Prove by induction that $a_{i,j}^{(k)}$ is the number of walks of length k from v_i to v_j!

Consider now the graph $G = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{1-2, 1-3, 1-4, 2-3, 2-4, 3-4\}$. Use the adjacency matrix of G to compute the number of triangles (*i.e.* cycles of length 3) of G!

2) Prove that a simple undirected graph G with n vertices and more than $(n-1)(n-2)/2$ is connected!

3) Prove that the edge set of an undirected simple graph can be partitioned into cycles if, and only if, every vertex has even degree.

Hint: To prove the existence of a cycle, consider a maximal path and use the even degree condition, i.e. the fact that all vertices have even degree.

4) Let $G = (V, E)$ and $G' = (V', E')$ be two undirected graphs. A graph isomorphism is a bijective mapping $\phi : V \to V'$ such that two vertices $x, y \in V$ are adjacent if and only if $\phi(x)$ and $\phi(y)$ are adjacent. The two graphs G and G' are called isomorphic, if there exists an isomorphism $\phi : V \to V'$.
Prove the following statements: If $G = (V, E)$ and $G' = (V', E')$ are isomorphic graphs and $\phi : V \to V'$ is an isomorphism, then $d_G(x) = d_{G'}(\phi(x))$ for all $x \in V$.
If, on the other hand, $\phi : V \to V'$ is a bijective mapping satisfying $d_G(x) = d_{G'}(\phi(x))$ for all $x \in V$, then G and G' are not necessarily isomorphic.

5) Let $G = (V, E)$ be an undirected graph with n vertices which does not have any cycle of length 3. Prove:

1. If $xy \in E$ then $d(x) + d(y) \leq n$.
2. The previous inequality $d(x) + d(y) \leq n$ implies that $\sum_{v \in V} d(v)^2 \leq n|E|$.
3. The graph has at most $n^2/4$ edges. Hint: Use the handshaking lemma, the Cauchy-Schwarz inequality $\left(\sum_{i=1}^{r} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{r} a_i^2\right)\left(\sum_{i=1}^{r} b_i^2\right)$, and what you have proved so far.

6) Use the matrix tree theorem to compute the number of spanning forests of the graph below!