1	2	3	4	5	Σ	Grade
25	15	20	10	25	95	1

Examination for "Logic and Reasoning in Computer Science"

June 26th, 2024 **1st Exam for SS 2024**

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This exam sheet consists of five problems, yielding a total of 100 points. Good luck!

Problem 1. (25 points) Consider the formula :

$$(\neg q \wedge r) \wedge \neg((p \leftrightarrow r) \vee (p \rightarrow r))$$

(a) Which atoms are pure in the above formula?

(b) Compute a clausal normal form C of the above formula by applying the CNF transformation algorithm with naming and optimization based on polarities of subformulas;

(c) Decide the satisfiability of the computed CNF formula C by applying the DPLL method to C. If C is satisfiable, give an interpretation which satisfies it.

Problem 2. (20 points) Formalize the following arguments and verify whether they are correct:

1. I must be punished only if I am guilty; I'm guilty. Thus I must be punished
2. If I'm guilty, I must be punished; I'm not guilty. Thus I must not be punished.

Note that verifying whether an argument is correct means to prove a statement of the form: from the hypothesis P_1 and .. and P_n, the conclusion Q follows (that is, you need to either formally prove $P_1, \ldots, P_n \models Q$, or exhibit a counterexample for the statement).

Problem 3. (20 points) Let A be a propositional, well-formed formula using $n \geq 1$ propositional variables such that

- A is not a propositional atom, and
- A is built from propositional atoms using only $\neg$ and $\leftrightarrow$.

How many branches does a splitting tree of A have? Provide a sufficiently detailed explanation of your answer.

Problem 4. (10 points) Provide either a tableau proof or a counterexample for the statement

$$\neg\exists x A(x) \models \exists x \neg(A(x) \vee A(f(x)))$$

If you provide a counterexample, you have to show that it is in fact a counterexample.

Problem 5. (25 points) Consider the formula:

$$\underbrace{a = b - 4}_{P_1} \wedge \underbrace{f(b+1) = c}_{P_2} \wedge (\underbrace{f(a+5) \neq c}_{P_3} \vee \underbrace{read(write(A, a+2, 3), b-3) = 1}_{P_4})$$

where b, c are constants, f is a unary function symbols, A is an array constant, $read, write$ are interpreted in the array theory, and $+, -, 1, 2, 3, \ldots$ are interpreted in the standard way over the integers.

Use the Nelson-Oppen decision procedure for reasoning in the combination of the theories of arrays, uninterpreted functions, and linear integer arithmetic. Use the decision procedures for the theory of arrays and the theory of uninterpreted functions and use simple mathematical reasoning for deriving new equalities among the constants in the theory of linear integer arithmetic. If the formula is satisfiable, give an interpretation that satisfies the formula.

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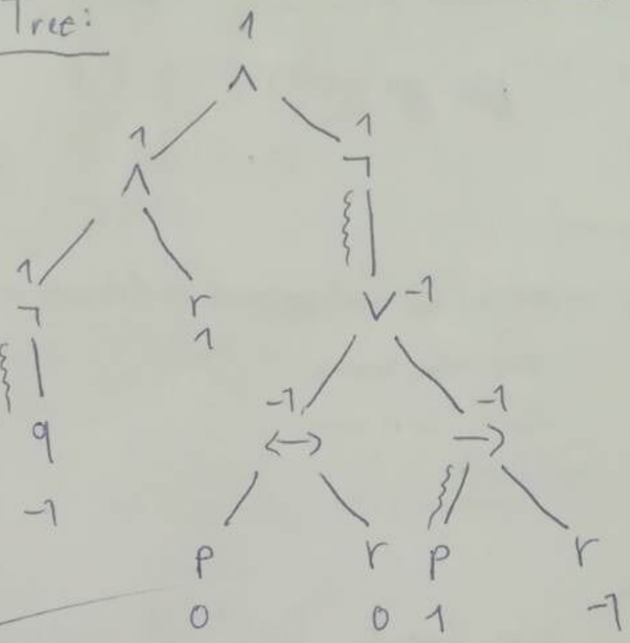
Problem 1: $(\neg q \wedge r) \wedge \neg((p \leftrightarrow r) \vee (p \rightarrow r))$

a) q is pure negative ✓

Syntax Tree:



b)

	~~subformula~~
~~n_1~~	~~$(\neg q \wedge r) \wedge \neg((p \leftrightarrow r) \vee (p \rightarrow r))$~~
~~n_2~~	
~~n_3~~	
~~n_4~~	
~~n_5~~	
~~n_6~~	

b)

Name	subformula	definition	CNF
n_1	$(\neg q \wedge r) \wedge \neg((p \leftrightarrow r) \vee (p \rightarrow r))$	$n_1 \rightarrow (n_2 \wedge n_4)$	$\neg n_1 \vee n_2$ $\quad n_1$ $\neg n_1 \vee n_4$
n_2	$\neg q \wedge r$	$n_2 \rightarrow (n_3 \wedge r)$	$\neg n_2 \vee n_3$ $\neg n_2 \vee r$
n_3	$\neg q$	$n_3 \rightarrow \neg q$	$\neg n_3 \vee \neg q$
n_4	$\neg((p \leftrightarrow r) \vee (p \rightarrow r))$	$n_4 \rightarrow \neg n_5$	$\neg n_4 \vee \neg n_5$
n_5	$(p \leftrightarrow r) \vee (p \rightarrow r)$	$(n_6 \vee n_7) \rightarrow n_5$	$\neg n_6 \vee n_5$ $\neg n_7 \vee n_5$
n_6	$p \leftrightarrow r$	$(p \leftrightarrow r) \rightarrow n_6$	$p \vee r \vee n_6$ $\neg p \vee \neg r \vee n_6$
n_7	$p \rightarrow r$	$(p \rightarrow r) \rightarrow n_7$	$p \vee n_7$ $\neg r \vee n_7$

$\Rightarrow C = \{ \neg n_1 \vee n_2, \neg n_1 \vee n_4, \neg n_2 \vee n_3, \neg n_2 \vee r, \neg n_3 \vee \neg q, \neg n_4 \vee \neg n_5, \neg n_6 \vee n_5, \neg n_7 \vee n_5,$
$p \vee r \vee n_6, \neg p \vee \neg r \vee n_6, p \vee n_7, \neg r \vee n_7, n_1 \}$

✓

Problem 1:

[redacted]

c) DPLL:

n_1
$\neg n_1 \vee n_2$
$\neg n_1 \vee n_4$
$\neg n_2 \vee n_3$
$\neg n_2 \vee r$
$\neg n_3 \vee \neg q$
$\neg n_4 \vee \neg n_5$
$\neg n_6 \vee n_5$
$\neg n_7 \vee n_5$
$p \vee r \vee n_6$
$\neg p \vee \neg r \vee n_6$
$p \vee n_7$
$\neg r \vee n_7$

$I = \{n_1 \mapsto 1\}$

Unit Propagation $\Rightarrow$

n_2
n_4
$\neg n_2 \vee n_3$
$\neg n_2 \vee r$
$\neg n_3 \vee \neg q$
$\neg n_4 \vee \neg n_5$
$\neg n_6 \vee n_5$
$\neg n_7 \vee n_5$
$p \vee r \vee n_6$
$\neg p \vee \neg r \vee n_6$
$p \vee n_7$
$\neg r \vee n_7$

$I = \{n_1 \mapsto 1, n_2 \mapsto 1, n_4 \mapsto 1\}$

Unit Propagation $\Rightarrow$

n_3
r
$\neg n_3 \vee \neg q$
$\neg n_5$
$\neg n_6 \vee n_5$
$\neg n_7 \vee n_5$
$p \vee r \vee n_6$
$\neg p \vee \neg r \vee n_6$
$p \vee n_7$
$\neg r \vee n_7$

Unit Propagation:

$I = \{n_1 \mapsto 1, n_2 \mapsto 1, n_4 \mapsto 1, n_3 \mapsto 1, r \mapsto 1, n_5 \mapsto 0\}$ $\Rightarrow$

$\neg q$
$\neg n_6$
$\neg n_7$
$\neg p \vee n_6$
$p \vee n_7$
$n_7 \vee \square$

$\Rightarrow$ Unit Propagation:

$I = \{n_1 \mapsto 1, n_2 \mapsto 1, n_3 \mapsto 1, n_4 \mapsto 1, n_5 \mapsto 0, r \mapsto 1, q \mapsto 0, n_6 \mapsto 0, n_7 \mapsto 0\}$

$\neg P$
P
$\square$

contains $\square$ return unsat $\Rightarrow$ formula is unsat.

Problem 2:

1. I must be punished only if I am ~~…~~ guilty. I'm guilty. Thus I must be punished.

punished ... p

guilty ... g

~~…~~ $g \leftrightarrow p, \quad g \models p$

× only if!

Tableau proof:

Assume the entailment is not correct and derive contradiction:

1. $t: g \leftrightarrow p$ — by assump.
2. $t: g$ — " —
3. $f: p$ — " —

~~…~~

4. $f: g$ from 1. contradiction with 2.

4. $t: p$ from 1. contradiction with 3. (here I only used one of the 2 conclusions of the Tableau rules)

good proof, bad statement!

closed tableau hence valid entailment.

2. If I'm guilty, I must be punished. I'm not guilty. Thus I must not be punished.

same variables:

$g \to p, \ \neg g \models \neg p$ ✓

Counterexample:

$I = \{g \mapsto 0, \ p \mapsto 1\}$ ✓

All hypothesis are sat:

$g \to p \Rightarrow \bot \to \top \Rightarrow \top$

$\neg g \Rightarrow \neg \bot \Rightarrow \top$

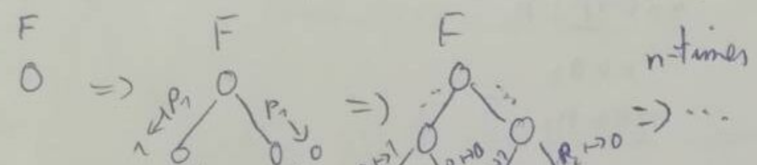
Conclusion is not satisfied: ✓

$\neg p \Rightarrow \neg \top \Rightarrow \bot$

Problem 3:

The splitting Tree has 2^n leaf nodes and assymtoticaly the same number of brances.

Argument: With each split where a propositional atom gets replaced by either $\top$ or $\bot$ the number of current leafnodes gets multiplied by 2. (assuming that we evaluate the formula in parallell and we splitt at the same depth.) Here is a visualisation:



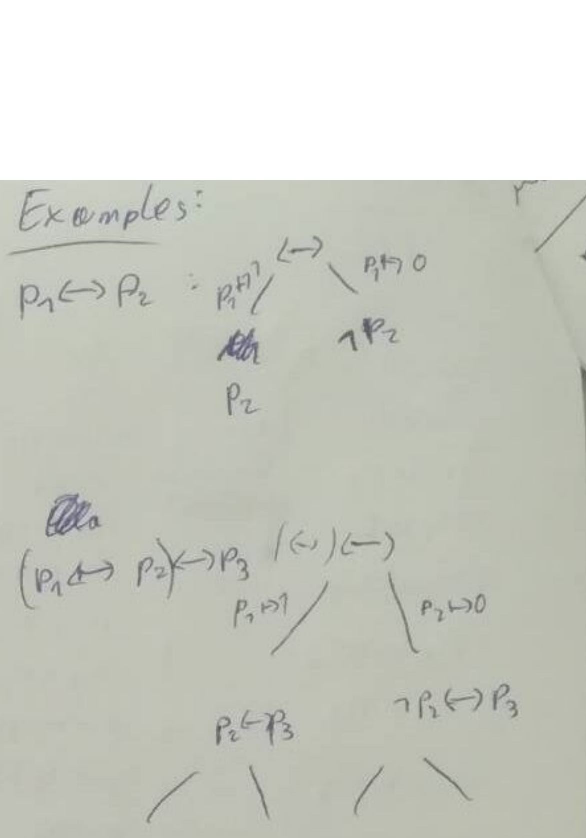
Note that the simplify method never leads to big simplifications (where we get $\top$ or $\bot$ as a result). We only either use $A \leftrightarrow \top \Rightarrow A$ or $\top \leftrightarrow A \Rightarrow A$ and $A \leftrightarrow \bot \Rightarrow \neg A$ or $\bot \leftrightarrow A \Rightarrow \neg A$.

The $\neg$ also never leads to big simplifications.

So we have to split untill all atoms are replaced and the result is only $\top$ or $\bot$.

Since each split (accross all nodes with the same depth) results in 2 times as many branches we get 2^n branches if we split for each atom, (Which is what we have to do, because there are no „big"-simplifications).

Examples:



Note: „big" simplifications are so important, because they lead to „cutting of the Tree", where we know the result (= whether it is sat or unsat) without using all of the atoms.

Without big simplification the only thing that happens is, that other subformulas might get negated, but never removed!

Problem 4: $\neg \exists x A(x) \models \exists x \neg (A(x) \vee A(f(x)))$

Tableaux proof:

Assume entailment is not correct and derive contradiction:

1. $t: \neg \exists x A(x)$ by assump.
2. $f: \exists x \neg (A(x) \vee A(f(x)))$ — " —
3. $f: \exists x A(x)$ from 1. $\neg$
4. $f: A(a)$ from 3. $\exists$

~~5. $\neg (A(a) \vee A(f(a)))$~~

5. $f: A(f(a))$ from 3. $\exists$
6. $f: \neg (A(a) \vee A(f(a)))$ from 2. $\exists$
7. $t: A(a) \vee A(f(a))$ from 6. $\neg$

8. $t: A(a)$ from 7. $\vee$
contradiction with 4.

8. $t: A(f(a))$ from 7. $\vee$
contradiction with 5.

closed Tableau, hence valide entailment

Problem 5

[redacted]

$P_1 \mapsto 1$, $P_2 \mapsto 1$, $P_3 \mapsto 1$, $P_4 \mapsto 1$ $\nearrow$ $a = b-4 \wedge f(b+1) = c \wedge f(a+5) = c \wedge read(write(A, a+2, 3), b-3) = 1$

same variables x_i $i \in \{1-6\}$

lin Alg	functions	Arrays	shared equalities
$a = b-4$ $x_1 = b+1$ $x_2 = a+5$ $x_3 = a+2$ $x_4 = 3$ $x_5 = b-3$ $x_6 = 1$	$f(x_1) = c$ $f(x_2) = c$	$read(write(A, x_3, x_4), x_5) = x_6$	1. $x_1 = x_2$
1. $x_1 = b+1 = a+5 = x_2 = x_5 + 4 =$ $= x_3 + 3$	2. $x_1 = x_2 \Rightarrow f(x_1) = f(x_2) = c$	3. assume $x_3 = x_5$ $\Rightarrow x_4 = x_6$	3. $x_3 = x_5$, $x_4 = x_6$
$x_4 = x_6 \Rightarrow 3 = 1$ contradiction			
		go on: $x_3 \neq x_5$ $read(write(A, x_3, x_4), x_5) =$ $read(A, x_5) = x_6$	

no further equalities and no contradiction $\Rightarrow$ Model: (constructed useing the known equalities)

$a' = 0 \Rightarrow b' = 4$, $x_2' = 5$, $x_3' = 2$, $x_4' = 3$, $x_6' = 1 \Rightarrow x_5' = 1$, $c' = 10$, $f'(D^n) \to \{10\}$

. $A'(u) = 1$ for all u, $x_1' = 5$

$\forall x$ $f': x \mapsto 10$

Model for original formula: $a' = 0$, $b' = 4$, $c' = 10$, $f'(u) = 10$ for all u, $A'(u) = 1$ for all u

Array interpreted as function $A' : \mathbb{Z} \to \mathbb{Z}$; $f' : \mathbb{Z} \to \mathbb{Z}$

Checking satisfied: $a = b-4 \wedge f(b+1) = c \wedge (f(a+5) \neq c \vee read(write(A, a+2, 3), b-3) = 1)$

replace constants: $0 = 4-4 \wedge f(5) = 10 \wedge (f(5) \neq 10 \vee read(write(A', 2, 3), 1) = 1$

$\Rightarrow 0 = 0 \wedge 10 = 10 \wedge (10 \neq 10 \vee 1 = 1) \Rightarrow$ sat.

Problem 5 [redacted]

$$a=b-4 \wedge f(b+1)=c \wedge (f(a+5)\neq c \vee read(write(A,a+2,3),b-3)=1)$$

Simplify for DPLL: $p_1: a=b-4$ $p_2: f(b+1)=c$ $p_3: f(a+5)=c$ $p_4: read(write(A,a+2,3)b-3)=1$

DPLL:

P_1		$P_1 \mapsto 1$
P_2	$\Rightarrow$	$P_2 \mapsto 1$
$\neg P_3 \vee P_4$		$P_3 \mapsto 0$
		$P_4 \mapsto 0$

$\Rightarrow a=b-4 \wedge f(b+1)=c \wedge f(a+5)\neq c \wedge read(write(A,a+2,3),b-3)\neq 1$

Introduce variables: $x_1=b+1$, $x_2=a+5$, $x_3=a+2$, $x_4=3$, $x_5=b-3$, $x_6=1$

Lin Alg.	functions	Arrays	shared equalities
$a=b-4$	$f(x_1)=c$	$read(write(A,x_3,x_4),x_5)\neq x_6$	
$x_1=b+1$	$f(x_2)\neq c$		
$x_2=a+5$			
$x_3=a+2$			
$x_4=3$			
$x_5=b-3$			
$x_6=1$			
1. $x_1=b+1=a+5=x_2$	2. $x_1=x_2 \Rightarrow f(x_1)=f(x_2)\Rightarrow c=c \wedge c\neq c$		1. $\Rightarrow x_1=x_2$

contradiction $\Rightarrow$ add clause to Sat Solver: $(\neg p_1 \vee \neg p_2 \vee p_3 \vee p_4)$ ✓

DPLL:

P_1		$P_1 \mapsto 1$
P_2	$\Rightarrow$	$P_2 \mapsto 1$
$\neg P_3 \vee P_4$		$P_3 \mapsto 0$
$\neg P_1 \vee \neg P_2 \vee P_3 \vee P_4$		$P_4 \mapsto 1$

$\Rightarrow a=b-4 \wedge f(b+1)=c \wedge f(a+5)\neq c \wedge read(write(A,a+2,3),b-3)=1$

same variables:

Lin Alg.	functions	Arrays	shared equalities
$a=b-4$	$f(x_1)=c$	$read(write(A,x_3,x_4)x_5)=x_6$	
$x_1=b+1$	$f(x_2)\neq c$		
$x_2=a+5$			
$x_3=a+2$			
$x_4=3$			
$x_5=b-3$			
$x_6=1$			
1. $x_1=b+1=a+5=x_2$	2. $x_1=x_2 \Rightarrow f(x_1)=f(x_2) \Rightarrow c=c \wedge c\neq c$ ↯		1. $\Rightarrow x_1=x_2$

contradiction add clause to Sat Solver $(\neg P_1 \vee \neg P_2 \vee P_3 \vee \neg P_4)$ ✓

DPLL:

P_1		$P_1 \mapsto 1$
P_2	$\Rightarrow$	$P_2 \mapsto 1$
$\neg P_3 \vee P_4$		$P_3 \mapsto 1$
$\neg P_1 \vee \neg P_2 \vee P_3 \vee P_4$		$P_4 \mapsto 1$
$\neg P_1 \vee \neg P_2 \vee P_3 \vee \neg P_4$		