Chamber of torture

Step by step guide for task 1 of the nice exam

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Requiescas in pace

Let $1 \leq d_1 \leq d_2 \leq \ldots, \leq d_n \leq n-1$ be at least two integers such that $\sum_{i=1}^{n} d_i = 2n-2$. Prove that there is a tree with vertices $x_1, x_2, \ldots, x_n$ such that $d(x_i) = d_i$ for all i.

1. Read the question.
2. Realize that there is no question mark and that it is no question but a task.
3. Think that you have an idea of how to solve it.
4. Realize that this is not true and your idea is not really an idea.
5. Question your choice of university and field of study and especially your choice of doing this exam.
6. Question existence itself.
7. Remember that there was some lemma in the lecture called handshake lemma.

   You see

   $$\sum_{i=1}^{n} d_i = 2n - 2 = 2|E|.$$

   You also remember the condition for a tree $|V| = |E| + 1$, which is in this case equivalent to $|E| = n - 1$.
8. You see that there is no contradiction.
9. Think that you solved the task.
10. Celebrate, shortly.
11. Realize, that you did not yet prove, what you had to prove. You just show, that there is no contradiction in this condition. Not more. Postulating to have proven the task would be like saying that solving math exercises does not contradict the existence of reptiloids, thus they would have to exist. No, just no. In case you are actually a reptiloid: Do not eat other students!
12. Realize that again you do not have any idea of how to solve the task.
13. Despair.
14. Cry.
15. Think again.
16. And again.
17. Have the correct idea.
18. You might see that there is some n in the task and it is even a natural number. That is cool, there are far less natural numbers than real numbers. So you only have to prove it for countable infinite ones. Isn't that great? No? Okay, be sad.

19. Natural numbers, right? You might want to think about an induction. So for $n = 2$ it is easy, since $d_1 = d_2 = 1$ has to hold. But how to construct a tree with n vertices for the given degree sequence? The easiest way would be to just add some leave at some point, if that would be possible, that would be nice. But how would the tree has to look, which one wants to modify? Since it is some induction, you want to construct something out of $n - 1$ vertices, with some suitable condition for the rest.

20. So you see, the last one, the d_n is for sure at least equal to 2, if not, $n - 1$ could be equal to $2n - 2$, which would be strange. And not everything, which is strange, is also nice, only mathematicians are strange and nice (and evil).

21. So you might just subtract 1 from d_n and call it $\tilde{d}_n$.

22. You still have a n integers, so one has to die. You look at them. And you see on the left, there is something, it is the smallest integer of all original integers. You would want this integer to be equal to 1. But is this true?

23. You think about it.

24. You are still thinking about it.

25. Stop thinking.

26. Yes, it is. It is the smallest one, and if it would not be equal to one, then all others would also be at least equal to 2, but $2n > 2n - 2$.

27. The decision is made, d_1 has to be sacrificed for the higher good.

28. Kill d_1 and admire your work. Only $n - 1$ integers are left. And now there was some condition on the integers. Verify it.

29. See that it still holds. The sum is equal to $2n - 2 - 2 = 2(n - 1) - 2$.

30. Celebrate your brain shortly.

31. You celebrated too long, you forgot what you wanted to do.

32. Panic.

33. De-panic. Breath in, breath out.

34. Continue breathing.

35. Remember that you wanted to do an induction and already did the base case for $n = 1$, so you might assume that what you want to prove holds for all $k < n$ and some $n \in \mathbb{N}$. Now you know, that you have some integers $(n - 1)$ $d_2, ..., \tilde{d}_n$, they are not necessarily ordered, because of $\tilde{d}_n$. But that is not a big deal. Just order it somehow. Then use the induction hypothesis to construct your tree with this degree sequence. Now search for the special vertex associated with $\tilde{d}_n$ and add a leaf to it. Note this is like a resurrection of the d_1 you killed.

36. See what you have done. See your fantastic work of evilness.
37. Confuse yourself a bit and think you did something wrong.
38. Panic.
39. Calm and realize that you did everything correctly.
40. Write everything down in a readable and formal way.
41. See that this was only the first task.
42. Despair again.