Problem Set 5

Problem 5.1 Consider the following joint pdf of two random variables x and y:

$$f_{\mathsf{x},\mathsf{y}}(x,y) = \begin{cases} x+y, & (x,y) \in [0,1] \times [0,1] \\ 0, & \text{otherwise} \end{cases}.$$

a) Sketch $f_{\mathsf{x},\mathsf{y}}(x,y)$.

b) Calculate and sketch the marginal pdfs $f_{\mathsf{x}}(x)$ and $f_{\mathsf{y}}(y)$.

c) Calculate the mean and the variance of x and y.

d) Calculate the covariance of x and y.

e) Calculate the probability $\mathrm{P}\{\mathsf{x}+\mathsf{y} \leq 1/2\}$.

Problem 5.2 Consider the following joint pdf of two random variables x_1 and x_2:

$$f_{\mathsf{x}_1,\mathsf{x}_2}(x_1,x_2) = \begin{cases} a\, e^{-x_1} e^{-x_2}, & 0 < x_1 < x_2 < \infty \\ 0, & \text{otherwise} \end{cases}.$$

a) Find the constant a so that $f_{\mathsf{x}_1,\mathsf{x}_2}(x_1,x_2)$ is a valid joint pdf.

b) Calculate the correlation $R_{\mathsf{x}_1,\mathsf{x}_2}$.

c) Calculate the joint pdf $f_{\mathsf{y}_1,\mathsf{y}_2}(y_1,y_2)$ for $\mathsf{y}_1 = 2\mathsf{x}_1$ and $\mathsf{y}_2 = \mathsf{x}_2 - \mathsf{x}_1$.

d) Are x_1 and x_2 statistically independent? Justify your answer.

e) Are y_1 and y_2 statistically independent? Justify your answer.

Problem 5.3 A complex random variable z is defined by

$$\mathsf{z} = \cos(\mathsf{x}) + j \sin(\mathsf{y}),$$

where x and y are statistically independent random variables uniformly distributed in $[-\pi, \pi]$.

a) Calculate the mean value of z.

b) Calculate the variance of z.

Problem 5.4 Let $\mathsf{x}_i, i = 1, 2, \ldots, N$, be *real-valued* independent and identically distributed (iid) $\mathcal{N}(0, \sigma^2)$ random variables. Let the random variable y be defined as

$$\mathsf{y} = \sum_{i=1}^{N} |\mathsf{x}_i|^2.$$

a) Calculate the distribution of $|\mathsf{x}_i|^2$.

b) Calculate the characteristic function of $|\mathsf{x}_i|^2$.

c) Use the characteristic function obtained in b) to find the distribution of y.

d) Repeat subtasks a) – c) for the case that the x_i are statistically independent circularly symmetric *complex* Gaussian random variables with zero mean and identical variance σ^2.

Hint: $\displaystyle\int_0^\infty x^y e^{-ax} dx = \frac{\Gamma(y+1)}{a^{y+1}},\ y > -1, a > 0.$