# VU Discrete Mathematics

# Exercises for 21st October 2025

**7)** Let $M = (E, S)$ be a matroid and $M_0 = (E_0, S_0)$ where $E_0 \subseteq E$ and $S_0 = \{X \cap E_0 \mid X \in S\}$. Prove that $M_0$ is a matroid.

**8)** Prove that an independence system $(E, S)$ is a matroid if and only if for every $A \subseteq E$, all maximal independent subsets of $A$ have the same cardinality.

**9)** Let $G = (V, E)$ be the graph with

$$V = \{a, b, c, d, e, v, w, x, y, z\}$$

and

$$E = \{ab, bc, cd, de, ea, av, bw, cx, dy, ez, vx, vy, wy, wz, xz\}.$$

Prove that it does not have a subgraph which is a subdivision of $K_5$, but there is a subgraph being a subdivision of $K_{3,3}$.

**10)** Let $G$ be a connected planar graph. Suppose that every vertex has the same degree $d$ and that the boundary of every face is a cycle of length $\ell$.

a) Prove that $2\alpha_1(G) = \ell \cdot \alpha_2(G)$.

b) Find an equation relating $\alpha_0(G)$, $\alpha_1(G)$ and $d$ and prove it.

c) Use Euler's polyhedron formula and what you proved so far to show that $\ell \geq 6$ implies $d = \alpha_2(G) = 2$.

**11)** Use a suitable graph model to reformulate the following exercise as a graph theoretical problem and solve it:
Given a subset $A \subseteq \mathbb{R}^2$ which has area $a$ and two decompositions of $A$ into $m$ pairwise disjoint subsets $A_1, A_2 \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ such that all the $A_i$'s and all the $B_i$'s have the same area $a/m$. Prove that there exists a permutation $\pi$ of $\{1, 2, \ldots, m\}$ such that for all $i = 1, \ldots, m$ we have $A_i \cap B_{\pi(i)} \neq \varnothing$.

**12)** Follow the hint below to construct a schedule for the matches in a league of $2n$ teams which meets the following constraints:

(a) In each round each team plays exactly one match.

(b) In the end each team must have played against each of the other teams exactly once.

Hint: Consider the graph $K_{2n}$ on the vertex set $\{1, 2, \ldots, 2n\}$ and show that each of the sets $M_i = \{1i\} \cup \{xy \mid x + y \equiv 2i \mod 2n - 1 \text{ and } x \neq y,\ x \neq 1,\ y \neq 1\}$ is a perfect matching (for $i = 2, \ldots, 2n$).