MathProg

ILP notation: $\max\{c'x : Ax \le b,\ x \in \mathbb{Z}^n_+\}$ or $\max\{c'x : x \in X\}$, $X = \{x : Ax \le b,\ x \in \mathbb{Z}^n_+\}$

n variables, m constraints

1. Formulations

1.1 Sequential (SEQ) (closed TSP)

$$u_i + x_{ij} \le u_j + M\cdot(1 - x_{ij}) \qquad \forall i,j \in V\setminus\{1\},\ i \ne j$$

$$1 \le u_i \le n-1 \qquad \forall i \in V\setminus\{1\}$$

$u_i \in \mathbb{R}_+$: ordering of visited vertices

$M = n-2$ (or similar)

1.2 Single-commodity flow (SCF)

$$\sum_{j\in V\setminus\{1\}} f_{1j} = n-1$$

$$\sum_{i\in V\setminus\{k\}} f_{ik} - \sum_{j\in V\setminus\{k\}} f_{kj} = 1 \qquad \forall k \in V\setminus\{1\}$$

$$0 \le f_{ij} \le (n-1)\cdot x_{ij} \qquad \forall i,j \in V,\ i \ne j$$

$f_{ij} \in \mathbb{R}_+$: flow from vertex i to vertex j

1.3 Multi-commodity flow (MCF)

$$\sum_{j\in V\setminus\{1\}} f^k_{1j} = 1 \qquad \forall k \in V\setminus\{1\}$$

$$\sum_{j\in V\setminus\{1\}} f^k_{ik} = 1 \qquad \forall k \in V\setminus\{1\}$$

$$\sum_{i\in V\setminus\{j\}} f^k_{ij} - \sum_{j\in V\setminus\{j\}} f^k_{ji} = 1 \qquad \forall j,k \in V\setminus\{1\},\ j \ne k$$

$$0 \le f^k_{ij} \le x_{ij} \qquad \forall i,j \in V,\ \forall k \in V\setminus\{1\},\ i \ne j$$

$f^k_{ij} \in \mathbb{R}_+$: flow from vertex i to vertex j with label k.

1.4 Subtour-elimination constr. (SUB/SEC)

$$\sum_{\substack{e\in E(S)\\ (i,j)\in A(S)}} x_e \le |S| - 1 \qquad \forall S \subseteq V,\ S \ne \emptyset,\ |S| \ge 2$$

exclude full for closed tour!

$E(S)$ ... edges fully contained in set S

$$\sum_{(i,j)\in \delta^-(j)} x_{ij} = 1 \qquad \forall j \in V\setminus\{1\}$$

1.5 Cycle-elimination constr. (CEC)

$$\sum_{e\in C} x_e \le |C| - 1 \qquad \forall C \subseteq E,\ |C| \ge 2,\ C \text{ forms a cycle}$$

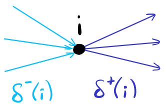
1.6 Cutset Constraints (CUT)

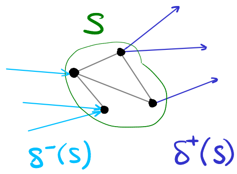
$$\sum_{\substack{e\in \delta(S)\\ (i,j)\in \delta^+(S)}} x_e \ge 1 \qquad \forall S \subset V,\ S \ne \emptyset,\ r \in S$$

Leave every subset at least once

... directed formulation

Cutsets: $\delta(\cdot)$: set of ars (i,j)





1.7 General closed TSP

$$\min \sum_{i\in C}\sum_{j\in C\setminus\{i\}} c_{ij}x_{ij} \qquad x_{ij}\in\{0,1\} \quad \forall i,j\in C$$

$$\text{s.t.} \quad \sum_{j\in C\setminus\{i\}} x_{ij} = 1 \qquad \forall i\in C$$

$$\sum_{j\in C\setminus\{i\}} x_{ji} = 1 \qquad \forall i\in C$$

$C=\{1,\dots,n\}$: cities
c_{ij} : cost to travel from i to j

$x_{ij}\in\{0,1\}$: travels from i to j

+ Subtour tackling methods (1.1 – 1.6)

1.8 Uncapacitated Facility Location (UFL)

$$\min \sum_{i\in C}\sum_{j\in F} c_{ij}x_{ij} + \sum_{j\in F} f_j y_j$$

$$\text{s.t.} \quad \sum_{j\in F} x_{ij} = 1 \qquad \forall i\in C$$

option 1: $\sum_{i\in C} x_{ij} \le |C|\cdot y_j \qquad \forall j\in F$

or

option 2: $x_{ij} \le y_j \qquad \forall i\in C, \forall j\in F$

$$x_{ij}\in\{0,1\} \qquad \forall i\in C, \forall j\in F$$
$$y_j\in\{0,1\} \qquad \forall j\in F$$

$C=\{1,\dots,m\}$: customers
$F=\{1,\dots,n\}$: facilities

c_{ij} : cost to serve customer i from facility j
f_j : opening cost for facility j

x_{ij} : customer i is served from facility j
y_j : facility j is opened

1.9 0-1 Knapsack

$$\max \sum_{i\in I} p_i x_i$$

$$\text{s.t.} \quad \sum_{i\in I} w_i x_i \le W$$

$$x_i\in\{0,1\} \qquad \forall i\in I$$

$I=\{1,\dots,n\}$: items , W : weight cap.

p_i : prize (profit) of item i
w_i : weight of item i

x_i : item i is taken

Disjunctive constraints:

a) ≥ 1 of two constraints
1 decision variable $y\in\{0,1\}$

$$a'x \ge y\cdot b$$
$$c'x \ge (1-y)\cdot d$$

b) $\ge k$ of m constraints
m decision variables $y_i\in\{0,1\}$, $\forall i\in\{1,\dots,m\}$

$$a_i'x \ge y_i\cdot b_i \qquad \forall i\in\{1,\dots,m\}$$
$$\sum_{i=1}^{m} y_i \ge k$$

Finite set of discrete values

Let x take exactly one discrete value $x\in\{a_1,\dots,a_m\}$

$$\sum_{j=1}^{m} a_j y_j = x$$
$$\sum_{j=1}^{m} y_j = 1$$

$$y_j\in\{0,1\} \qquad \forall j\in\{1,\dots,m\}$$

2. Total Unimodularity (TU)

Useful because: A is TU $\Longrightarrow$ integral optimal solution at a vertex $\Longrightarrow$ just solve LP!
↖ AND b-vector is integral AND optimal solution is finite (bounded region)

Definition:

A is TU $\Longleftrightarrow$ every square submatrix S of A has $\det(S) \in \{-1, 0, 1\}$.

Positive test:

A fulfills a)–c) $\Longrightarrow$ A is TU: (conjunction)

a) every element $a_{ij} \in \{-1, 0, +1\}$ $\forall i,j$

b) ≤ 2 non-zero elements in each column $\left(\sum_{i=1}^{m} |a_{ij}| \leq 2,\ \forall j \in \{1,\dots,n\}\right)$

c) $\exists$ partition of rows M into potentially empty subsets M_1, M_2
s.t. each column j with exactly 2 non-zero elements satisfies

$$\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0$$

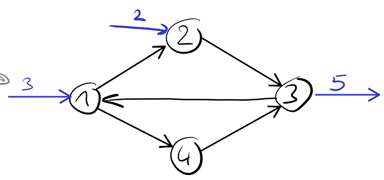
Quick negative result: If any $a_{ij} \notin \{-1, 0, 1\}$ $\Longrightarrow$ not TU

Other equivalences:

A is TU $\Longleftrightarrow$ X is TU, with X: "line" = row/column

1) Transpose A'
2) (A, I) and $\binom{A}{I}$ are TU
3) multiplying any line by -1
4) swapping lines
5) duplicating lines
6) deleting a line with $=1$ entry

Classic example of TU-problem:
Minimum cost network flow: $\left(\sum_i b_i = 0!\right)$



i	x_{12}	x_{14}	x_{23}	x_{24}	x_{31}		b_i
1	1	1	0	0	−1	=	3
2	−1	0	1	1	0	=	2
3	0	0	−1	0	1	=	−5
4	0	−1	0	−1	0	=	0
5	1	0	0	0	0	≤	h_{12}
6	0	1	0	0	0	≤	h_{23}

domain constraints $x_{12} \leq h_{12}$

$\Longleftarrow$ matrix A is of the form $\binom{C}{I}$ and C is TU $\Longrightarrow$ A is TU

Subgradient algorithm for Lagr. dual:

$u^0 = 0$, $k = 0$
while not stop criteria met do
  $x(u^k) \longleftarrow$ solve $z(u^k)$
  $u^{k+1} = \max\{u^k - \mu_k(d - Dx(u^k));\ 0\}$
  $k = k + 1$

For a given LD:

$$z(u) = \max_{x \in X} c'x + u'(d - Dx)$$

Vector $(d - Dx(u^k))$ is a subgradient of $z(u)$ at u^k.

Choose step size μ_k:
- $\sum_k \mu_k \to \infty$, $\mu_k \to 0$, e.g. $\mu_k = \frac{1}{k}$
- $\mu_k = \mu_0 \rho^k$ for some $\rho < 1$ and μ_0

3. Valid Inequalities

Valid inequality: $ax \le b$ s.t. all $x \in X$ satisfy this (X = feasible set)

Domination: $ax \le b$ dominates $cx \le d$ if $\exists u > 0$: $a \ge uc$ and $b \le ud$ and $(a,b) \ne (uc, ud)$

$\Rightarrow \{x \in \mathbb{R}^n_+ : ax \le b\} \subseteq \{x \in \mathbb{R}^n_+ : cx \le d\}$

Redundancy: $cx \le d$ redundant if $\left(\sum_{i=1}^{k} u_i a_i\right) x \le \sum_{i=1}^{k} u_i b_i$ dominates it (for some $k \ge 1$)

Knapsack: $X = \{x \in \{0,1\}^3 \mid 2x_1 - 3x_2 + 3x_3 \ge 2\}$

$\Rightarrow$ $x_1 + x_3 \ge 1$ (need to choose at least one of them)
$x_2 \le x_3$ (if I choose x_2, x_3 will have to compensate)

Bounds: Constraints $x_{ij} \le b_j y_j$, $x \le a_i$ ($x \ge 0$, $y \in \{0,1\}$)

$\Rightarrow$ $x_{ij} \le \min\{a_i, b_j\} \cdot y_j$

3.1 Chvatal-Gomory

$$\sum_{j=1}^{n} \underbrace{\lfloor u' A_j \rfloor}_{\text{column } j} \underbrace{x_j}_{\text{variable } j} \le \lfloor u' b \rfloor \quad \text{for } u \in \mathbb{R}_+^{\overbrace{m}^{|\text{constraints}|}}$$

1. Choose $u = (u_1, \dots, u_m)$
2. Muliply each constraint i by u_i
3. Sum constraints

4a. Floor LHS
4b. Floor RHS

$\frac{1}{3} \cdot (2x_1 + 3x_2) \le 6 \cdot \frac{1}{3}$

$\frac{1}{2} \cdot (x_1 - 4x_2) \le 2 \cdot \frac{1}{2}$ $\quad u = (\frac{1}{3}, \frac{1}{2})$

$\lfloor \frac{7}{6} \rfloor x_1 - \lfloor 1 \rfloor x_2 \le \lfloor 3 \rfloor$

$x_1 - x_2 \le 3$ ✓

3.2 Cover Inequalities

for a knapsack constraint $\sum_{j=1}^{n} a_j x_j \le b$ $\Rightarrow$ C is a cover if $\sum_{j \in C} a_j > b$

$\Rightarrow$ $\boxed{\sum_{j \in C} x_j \le |C| - 1}$ is valid. (also valid for $E(C)$)

minimal: removing any item $\rightarrow$ not a cover anymore

extended cover $E(C)$: $C \cup \{j \mid a_j \ge a_i, \forall i \in C\}$

Lifting cover inequality for $X = \{x \in \{0,1\}^5 : 12x_1 + 7x_2 + 5x_3 + 5x_4 + 3x_5 \le 14\}$

1. min. $C = \{2, 3, 4\}$
$\Rightarrow$ $x_2 + x_3 + x_4 \le 2$

2. $E(C) = C \cup \{1\}$
$\Rightarrow$ $x_1 + x_2 + x_3 + x_4 \le \underbrace{2}_{|C|}$

3. Compute lifting coefficients for all x_i s.t. $i \notin C$
$\Rightarrow$ x_1, x_5
(choose ordering)

3a) Lift x_1: $\zeta_1 = \max\{x_2 + x_3 + x_4 : 7x_2 + 5x_3 + 5x_4 \le 14 - 12 = 2\}$
$\zeta_1 = 0$ $\quad (x^* = (0,0,0,0))$

$\alpha_1 = |C| - 1 - \zeta_1 = 3 - 1 - 0 = \underline{\underline{2}}$

3b) Lift x_5: $\zeta_2 = \max\{2x_1 + x_2 + x_3 + x_4 : 12x_1 + 7x_2 + 5x_3 + 5x_4 \le 14 - 3 = 11\}$
$\zeta_2 = 2$ $\quad (x^* = (0,0,1,1))$

$\alpha_5 = |C| - 1 - \zeta_2 = 3 - 1 - 2 = \underline{\underline{0}}$

$\Rightarrow$ final inequ.: $\underline{\underline{\overbrace{2}^{\alpha_1}x_1 + x_2 + x_3 + x_4 + \overbrace{0}^{\alpha_5}x_5 \le 2}}$

4. Lagrange

Putting some constraints into obj. function as a penalty

Take IP:

$$\begin{aligned} z = \max\ & c'x \\ & Ax \le b \\ & Dx \le d \\ & x \in \mathbb{Z}^n_+ \end{aligned} \quad \Longrightarrow \quad z(u) = \max_{x \in X}\ c'x + u'(d - \underline{Dx})$$

multiply Dx out, then subtract from d once

$X = \{x \mid Ax \le b,\ x \in \mathbb{Z}^n_+\}$

Lagrangian relaxation $IP(u)$ for some u-multiplier

Find best u-vector: $w_{LD} = \min\{z(u) : u \ge 0\}$

($\Rightarrow$ best dual bound)

IP max $\Rightarrow$ LD min

IP min $\Rightarrow$ LD max

$Dx \le d \Rightarrow u \ge 0$

$Dx \ge d \Rightarrow u \le 0$

$Dx = d \Rightarrow u \in \mathbb{R}^m$

Lagr. relaxation $IP(u)$ gives optimal solution for IP if: $(u \ge 0)$

i) $x(u)$ is optimal for $IP(u)$

ii) $Dx(u) \le d$ — x-vector sats orig. D-constraints

iii) $(Dx(u))_i = d_i$ if $u_i > 0$ — In a row i, if component $u_i > 0$, then $x(u)$ must satisfy constraint to equality.

Ex. Dualizing a BILP

Given:

$$z = \max\ 4x_1 + 9x_2 + 5x_3 \qquad (1)$$

$$\text{s.t.} \quad 3x_1 + 7x_2 + 3x_3 \le 8 \qquad (2)$$

$$x_1 + x_2 + x_3 = 2 \qquad (3)$$

$$x_1, x_2, x_3 \in \{0,1\} \qquad (4)$$

a) Identify optimal value and solution: $z^* = 9 \qquad x^* = (1, 0, 1)$

b) Relax constraints (2) and (3) using $u = (u_2, u_3)$ and write down $z((u_2, u_3))$.

$$z((u_2,u_3)) = \max\ 4x_1 + 9x_2 + 5x_3 + u_2(d - Dx) + u_3(e - Ex)$$

$$x_1, x_2, x_3 \in \{0,1\}$$
$$u_2 \ge 0$$
$$u_3 \in \mathbb{R}$$

with $d = 8$, $e = 2$, $D = (3, 7, 3)$, $E = (1, 1, 1)$

c) Compute $z(u)$ for $u = (1,1)$ and $(1,-2)$.
Which gives the best bound? Are the optimal solutions unique?

$$z((u_2,u_3)) = \max\{4x_1 + 9x_2 + 5x_3 + u_2(8 - 3x_1 - 7x_2 - 3x_3) + u_3(2 - x_1 - x_2 - x_3) : x \in \{0,1\}^3\}$$

$$= \max\{8u_2 + 2u_3 + (4 - 3u_2 - 1u_3)x_1 + (9 - 7u_2 - 1u_3)x_2 + (5 - 3u_2 - 1u_3)x_3 : x \in \{0,1\}^3\}$$

Thus:

$$z((1,1)) = \max_{x \in \{0,1\}^3}(10 + 0x_1 + x_2 + x_3) = \underline{\underline{12}}$$

$$z((1,-2)) = \max_{x \in \{0,1\}^3}(8 - 4 + 3x_1 + 4x_2 + 4x_3) = \underline{\underline{15}}$$

$\rightarrow$ best bound: $\min\{z((1,1)), z((1,-2))\} = \underline{\underline{12}}$

$\rightarrow$ unique? No, we get 12 for $x = (0,1,1)$ and $(1,1,1)$

5. Uncertainty

5.1 Stochastic Programming

Given scenarios $\{1,\dots,K\}$ with probility p_k. $\sum_{k=1}^{K} p_k = 1$

uniform distribution: $D \in [l,h] \Rightarrow F(z) = \begin{cases} 0 & \text{, if } z<l \\ \frac{z-l}{h-l} & \text{, else} \\ 1 & \text{, if } z>h \end{cases}$

K scenarios: $p_k = \frac{1}{K}$

We calculate an optimal solution for x given a fixed value d_k for the random variable $D \Rightarrow G(x,d_k)$

Then we optimize the expected value $\mathbb{E}[G(x,D)]$ by approximation:

$$\boxed{\mathbb{E}[G(x,D)] \approx \sum_{k=1}^{K} p_k \cdot G(x,d_k)}$$

$\Rightarrow G(x,d_k)$ can be calculated since d_k is fixed!

Ex.

Take a merchant buying newspaper for cost c with d being today's demand.

↳ If $d>x \Rightarrow$ penalty b per missing unit

↳ If $d<x \Rightarrow$ holding cost h per missing unit

Goal: minimize total cost!

$$\min_{x\geq 0} G(x,d) = cx + \max\{b\cdot(d-x); 0\} + \max\{h\cdot(x-d); 0\} =$$

move cx into max cases

$$= \max\{(c-b)\cdot x + bd;\ (c+h)x - hd\}$$

$\Rightarrow$ for $K=2$, $p_k = \frac{1}{2}$ with $d_1 = 20$, $d_2 = 80$; $c=1$, $b=1{,}5$, $h=0{,}1$

$$\mathbb{E}[G(x,d)] \approx \sum_{k=1}^{K} p_k \cdot G(x,d_k) =$$

p_1

already evaluated

$$0{,}5 \cdot \max\{(1-1{,}5)x + 1{,}5\cdot 20;\ (1+0{,}1)x - 0{,}1\cdot 20\} + 0{,}5\cdot \max\{-0{,}5x + 120;\ 1{,}1x - 8\}$$

max{...} is non-linear! $\Rightarrow$ Turn $\min \sum_{k=1}^{K} p_k \cdot G(x,d_k)$ into LP:

$$\min \sum_{k=1}^{K} p_k t_k$$

$$\begin{aligned} \text{s.t.}\quad & t_k \geq (c-b)\cdot x + bd_k && \forall k = 1,\dots,K \\ & t_k \geq (c+h)\cdot x - hd_k && \forall k = 1,\dots,K \\ & x \geq 0 \end{aligned}$$

new variable t_k: max total cost in scenario k

Two-stage lin. stoch. P.

$$\boxed{\min_{x\in X} c'x + \sum_{k=1}^{K} p_k \cdot Q(x,\xi)}$$

optimal obj. value of 2nd stage

$$Q(x,\xi) = \min q'y$$

$$\text{s.t.}\quad Tx + Wy \leq h$$

x ... 1st stage decision vector

y ... 2nd stage decision vector

$\xi = (q,T,W,h)$ uncertain data

$\Rightarrow$ 2-stage LP:

$$\begin{aligned} \min\quad & c'x + \sum_{k=1}^{K} p_k \cdot q_k' y \\ \text{s.t.}\quad & T_k x + W_k y_k \leq h_k \\ & x \in X && \forall k = 1,\dots,K \\ & y_k \in \mathbb{R}^m && \forall k = 1,\dots,K \end{aligned}$$

$\Rightarrow x^*$ ... optimal first-stage solution

y_k^* ... optimal second-stage solution for scenario k

5.2 Robust Formulation

feasible in the worst case

W.l.o.g. only matrix A includes uncertain values. $\max\{c'x \mid Ax \le b,\ l \le x \le h\}$

↳ If costs c uncertain: $\max\{z \mid z - c'x \le 0,\ Ax \le b,\ l \le x \le h\}$

↳ If budgets b uncertain: $\max\{c'x \mid Ax - by \le 0,\ l \le x \le h,\ 1 \le y \le 1\}$

J_i ... set of uncertain coefficients in row $i \in \{1, \dots, m\}$

For every $\tilde{a}_{ij} \in J_i$: bounded uncertain in range $[a_{ij} - \hat{a}_{ij},\ a_{ij} + \hat{a}_{ij}]$

a_{ij} ... midpoint
$\hat{a}_{ij}$ ... Schwankungsbreite
$\tilde{a}_{ij}$ ... actual value

Normalized random deviation $\boxed{\eta_{ij} = \frac{\tilde{a}_{ij} - a_{ij}}{\hat{a}_{ij}}} \in [-1, 1]$

5.2 a) Soyster Formulation

protection term, ensures safety gap

$$\begin{aligned}
\max\quad & c'x \\
\text{s.t.}\quad & \sum_j a_{ij} x_j + \sum_{j \in J_i} \hat{a}_{ij} y_j \le b_i && \forall i \\
& -y_j \le x_j \le y_j && \forall j \\
& y_j \ge 0 && \forall j \\
& l_j \le x_j \le h_j && \forall j
\end{aligned}$$

$\forall i$ ... "for all LP rows"

y_j represents $|x_j|$

At optimality: $y_j = |x_j^*|$

5.2 b) Bertsimas & Sim

Don't account for all coefficients reaching the worst case at the same time.
⟹ Choose degree of conservatism $\Gamma_i \in \{0, \dots, |J_i|\}$ for row i.
e.g. $\Gamma_i = 2.8$ ... two coefficient may differ by their $\hat{a}_{ij}$ and one by $0{,}8 \cdot \hat{a}_{ij}$

$\Gamma_i = 0 \Rightarrow$ deterministic
$\Gamma_i = |J_i| \Rightarrow$ Soyster's model

Idea:

non-linear

alternative sub-problem

$$\begin{aligned}
\max\quad & c'x \\
\text{s.t.}\quad & \sum_j a_{ij} x_j + \max_{S_i \subseteq J_i:\ |S_i| = \Gamma_i} \sum_{j \in S_i} \hat{a}_{ij} y_j \le b_i && \forall i \\
& -y_j \le x_j \le y_j && \forall j \\
& y_j \ge 0 && \forall j \\
& l_j \le x_j \le h_j && \forall j
\end{aligned}$$

(fixed x^*!) takes current optimum

absolute value of j-th component of this constant

$$\beta_i(x^*, \Gamma_i) = \max_{S_i \subseteq J_i:\ |S_i| = \Gamma_i} \sum_{j \in S_i} \hat{a}_{ij} |x_j^*|$$

Or as an LP:

$$\begin{aligned}
\beta_i(x^*, \Gamma_i) = \max\quad & \sum_{j \in J_i} \hat{a}_{ij} |x_j^*| \cdot z_{ij} \\
\text{s.t.}\quad & \sum_{j \in J_i} z_{ij} \le \Gamma_i \\
& 0 \le z_{ij} \le 1 && \forall j \in J_i, \forall i
\end{aligned}$$

β takes "max" since the subset S_i with the biggest sum value is the hardest case to satisfy.
⟹ biggest safety gap needed

Resulting in: (by taking the dual of β_i with dual var p_{ij})

$$\begin{aligned}
\max\quad & \sum_j c_j x_j \\
\text{s.t.}\quad & \sum_j a_{ij} x_j + \sum_{j \in J_i} p_{ij} + \Gamma_i z_i \le b_i && \forall i \\
& z_i + p_{ij} \ge \hat{a}_{ij} y_j && \forall i,\ \forall j \in J_i \\
& -y_j \le x_j \le y_j && \forall j \\
& l_j \le x_j \le h_j && \forall j \\
& y_j \ge 0 && \forall j \\
& p_{ij} \ge 0 && \forall i,\ \forall j \in J_i \\
& z_i \ge 0 && \forall i
\end{aligned}$$

Ex. Bertsimas' model of a knapsack:

Deterministic KS: $\max \left\{ \sum_{i \in I} c_i x_i : \sum_{i \in I} w_i x_i \leq W, \ x \in \{0,1\}^{|I|} \right\}$

Now with uncertain weights: $w_i \in [w_i - \hat{w}, \ w_i + \hat{w}_i] \quad \forall i \in I$

$$\Rightarrow \max \left\{ \sum_{i \in I} c_i x_i : \sum_{i \in I} w_i x_i + \beta(x, \Gamma) \leq W, \ x \in \{0,1\}^{|I|} \right\}$$

$$\text{with} \quad \beta(x, \Gamma) = \max_{S \subseteq I : |S| = \Gamma} \sum_{j \in S} \hat{w}_j x_j$$

$J_i \mathrel{\hat{=}} I$ here, because all items are uncertain.

no $\forall i \in I$!

i ... number of constraints that encounter uncertainty and here we just have one.

Addendum: Reference Formulations

A.1 Minimum Spanning Tree (SCF)

$A = \{(i,j),(j,i) \mid \{i,j\} \in E\}$

$$\min \sum_{(i,j)\in A} w_{ij} x_{ij} \qquad n = |V|$$

$$\text{s.t.} \quad \sum_{(i,k)\in\delta^-(k)} f_{ik} - \sum_{(k,j)\in\delta^+(k)} f_{kj} = 1 \qquad \forall k \in V \setminus \{1\} \qquad \text{consume one unit}$$

$$\sum_{(1,j)\in\delta^+(1)} f_{1j} = n-1 \qquad \text{source flow}$$

$$f_{ij} \leq (n-1)\cdot x_{ij} \qquad \forall (i,j)\in A$$

$$\sum_{(i,j)\in A} x_{ij} = n-1 \qquad \text{tree} \rightarrow \text{take enough edges}$$

$$f_{ij} \geq 0 \qquad \forall (i,j)\in A \qquad \text{flow across arc } (i,j)$$

$$x_{ij} \in \{0,1\} \qquad \forall (i,j)\in A \qquad \text{arc } (i,j) \text{ included}$$

A.2 k-MST (SCF)

We add an auxiliary root: $D_0 = (V_0, A_0)$: $V_0 = V \cup \{0\}$ $\quad A_0 = A \cup \{(0,j) \mid j \in V\}$

$$\min \sum_{(i,j)\in A} w_{ij} x_{ij} \qquad n = |V|$$

$$\text{s.t.} \quad \sum_{(i,v)\in A_0} f_{iv} - \sum_{(v,j)\in A_0} f_{vj} = y_v \qquad \forall v \in V \qquad \text{consume one unit}$$

$$\sum_{j\in V} f_{0j} = k \qquad \text{source flow}$$

$$f_{ij} \leq k \cdot x_{ij} \qquad \forall (i,j)\in A_0$$

$$y_v \geq x_{iv} \qquad \forall i,v:\ v\in V,\ (i,v)\in A_0$$

$$y_v \leq \sum_{(i,v)\in A_0} x_{iv} \qquad \forall v \in V$$

$$f_{ij} \geq 0 \qquad \forall (i,j)\in A_0 \qquad \text{flow across arc } (i,j)$$

$$x_{ij} \in \{0,1\} \qquad \forall (i,j)\in A_0 \qquad \text{arc } (i,j) \text{ included in MST}$$

$$y_i \in \{0,1\} \qquad \forall i \in V \qquad \text{vertex } v \text{ included in MST}$$

alternative: SEQ

[obj.]

$$\text{s.t.} \quad u_i + x_{ij} \leq u_j + k\cdot(1 - x_{ij}) \qquad \forall (i,j)\in A_0 \qquad \text{ordering}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij} = y_j \qquad \forall j \in V$$

$$y_i + y_j \geq 2\cdot x_{ij} \qquad \forall (i,j)\in A$$

$$\sum_{j\in V} x_{0j} = 1$$

$$\sum_{(i,j)\in A} x_{ij} = k-1$$

$$u_0 = 0$$

$$y_i \leq u_i \leq k\cdot y_i \qquad \forall i \in V$$

$$x_{ij} \in \{0,1\} \quad \forall (i,j)\in A_0$$

$$y_i \in \{0,1\} \quad \forall i \in V$$