30.)

- obere Schranke: 3
- untere Schranke: $\sqrt{6}$
- Monotonie: streng monoton fallend

$\rightarrow$ konvergent

n	a_n
0	3
1	2.5
2	2.45
3	2.449489795918368
3	2.449489742783179
5	2.449489742783178
6	2.449489742783178
7	2.449489742783178
8	2.449489742783178
9	2.449489742783178
10	2.449489742783178

$$a_{n+1} - a_n = \frac{1}{2}\underbrace{\left(a_n + \frac{6}{a_n}\right)}_{>0} \underbrace{- a_n}_{<0} = \frac{6}{2a_n} - \frac{a_n}{2} = \frac{6 - a_n^2}{2a_n}$$

$$= \frac{(\sqrt{6}+a_n)(\sqrt{6}-a_n)}{\underbrace{2a_n}_{>0}} < 0$$

Behauptung: $\sqrt{6} < a_n \leq 3$ (Beschränktheit)

- I.A. $n = 0$: $a_0 = 3$ ✓
- I.S. z.z.: $\sqrt{6} < a_n \leq 3 \Rightarrow \sqrt{6} < a_{n+1} \leq 3$

Beweis:

$$a_{n+1} - \sqrt{6} = \frac{a_n}{2} + \frac{6}{2a_n} - \frac{\sqrt{6}\cdot 2a_n}{2a_n} = \frac{a_n^2 - 2\sqrt{6}a_n + 6}{2a_n} = \frac{(\underbrace{a_n}_{>\sqrt{6}} - \sqrt{6})^2}{\underbrace{2a_n}_{>\sqrt{6}}} \overset{\text{I.V.}}{>} 0 \quad \blacksquare$$

$$a_{n+1} - 3 = \frac{a_n}{2} + \frac{6}{2a_n} - \frac{3\cdot 2a_n}{2a_n} = \frac{a_n^2 - 6a_n + 6}{2a_n} = \frac{\overbrace{(a_n - 3 + \sqrt{3})}^{>0}\overbrace{(a_n - 3 - \sqrt{3})}^{\leq -\sqrt{3}}}{\underbrace{2a_n}_{>0}} \overset{\text{I.V.}}{<} 0 \quad \blacksquare$$

Sei $\lim\limits_{n\to\infty} a_n = a$

Dann: $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n$

$$\frac{1}{2}\left(a + \frac{6}{a}\right) = a$$

$$a + \frac{6}{a} = 2a$$

$$a = \frac{6}{a}$$

$$a^2 = 6$$

$$a = \pm\sqrt{6}$$

$$(\sqrt{6} < a_n \leq 3)$$

$$a = \sqrt{6}$$

$$\lim_{n\to\infty} a_n = \sqrt{6}$$

50.)

$$a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$$

Seien:

$$b_n := \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+n}} = \frac{n}{\sqrt{n^2+n}} \cdot \frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2}}} = \frac{1}{\sqrt{1+\frac{1}{n}}} \Rightarrow \lim_{n\to\infty} b_n = 1 \wedge b_n \leq a_n$$

$$c_n := \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}} \cdot \frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2}}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}} \Rightarrow \lim_{n\to\infty} c_n = 1 \wedge c_n \geq a_n$$

Daher laut Sandwich-Theorem:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 1.$$

55.)

$$a_n = \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2} \cdot \sum_{k=1}^{n} k =$$

$$= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{n^2+n}{2n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} =$$

$$= \frac{1+\frac{1}{n}}{2} = \frac{1}{2} + \frac{1}{2n}$$

$$\lim_{n\to\infty} a_n = \frac{1}{2}.$$

63.)

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) =$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{7}} - + \cancel{\ldots} =$$

$$= 1$$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$A(n+1) + Bn = 1$$

$$(A+B)n + A = 1$$

$$\Rightarrow A = 1$$

$$B = -1$$

68.) Moivresche Formel: $(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$

$$\sin \frac{n\pi}{3} = \operatorname{Im}\left(\cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3}\right) =$$
$$= \operatorname{Im}\left(\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)^n\right) =$$
$$= \operatorname{Im}\left(\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^n\right) =$$
$$= \operatorname{Im}\left(\left(\frac{1+i\sqrt{3}}{2}\right)^n\right)$$

$$\frac{\sin \frac{n\pi}{3}}{2^n} = \frac{\operatorname{Im}\left(\left(\frac{1+i\sqrt{3}}{2}\right)^n\right)}{2^n} = \operatorname{Im}\left(\frac{\left(\frac{1+i\sqrt{3}}{2}\right)^n}{2^n}\right) = \operatorname{Im}\left(\left(\frac{1+i\sqrt{3}}{4}\right)^n\right)$$

Sei $q = \frac{1+i\sqrt{3}}{4}$

Dann $$\sum_{n \geq 0} q^n = \frac{1}{1-q} = \frac{1}{\frac{3-i\sqrt{3}}{4}} = \frac{4}{3-i\sqrt{3}} = \frac{\cancel{4}^{1}(3+i\sqrt{3})}{\cancel{9+3}_{3}} = \frac{3+i\sqrt{3}}{3} =$$
$$= 1 + \frac{\sqrt{3}}{3} i$$

$$\sum_{n \geq 0} \frac{\sin \frac{n\pi}{3}}{2^n} = \sum_{n \geq 0} \operatorname{Im}(q^n) = \operatorname{Im}\left(\sum_{n \geq 0} q^n\right) = \operatorname{Im}\left(1 + \frac{\sqrt{3}}{3} i\right) = \frac{\sqrt{3}}{3}.$$

73.) Seien:

$$a_n := \frac{n-2}{2n^3+5n-3} \leq \frac{2n}{2n^3-2n^2} = \frac{1}{n^2-n} = \frac{1}{n(n-1)} =: b_n \text{ (Majorante)}$$

$$\sum_{n\geq 2} b_n = \sum_{n\geq 2} \left(\frac{1}{n(n-1)}\right) = \sum_{n\geq 2} \left(-\frac{1}{n} + \frac{1}{n-1}\right) =$$

$$= -\cancel{\tfrac{1}{2}} + 1 - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{4}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}} + \cancel{\tfrac{1}{4}} - \frac{1}{6} + \cancel{\tfrac{1}{5}} - + \cancel{\ldots} =$$

$$= \underline{1}$$

Majorantenkriterium:

$$\sum_{n\geq 2} b_n \text{ konvergent } \wedge |a_n| = \left|\frac{n-2}{2n^3+5n-3}\right| \leq b_n \text{ für fast alle } n \geq 0$$

$$\Rightarrow \sum_{n\geq 0} a_n = \sum_{n\geq 0} \frac{n-2}{2n^3+5n-3} \text{ konvergent.}$$

84.)

Sei:

$$a_n = (-1)^n \frac{1}{\sqrt[3]{n+2}} \longrightarrow$$ $\sum_{n\geq 0} a_n$ ist eine alternierende Reihe

$$|a_n| = \frac{1}{\sqrt[3]{n+2}} \longrightarrow$$ streng monoton fallende Nullfolge

Leibnitz-Kriterium:

$\Rightarrow \sum_{n\geq 0} a_n$ ist konvergent

Aber:

$$\sum_{n\geq 0} |a_n| = \sum_{n\geq 0} \frac{1}{\sqrt[3]{n+2}} \geqslant \sum_{n\geq 1} \frac{1}{\sqrt[3]{n}} \geqslant \sum_{n\geq 1} \frac{1}{n} = +\infty$$

(bekannte Reihe)

Minorantenkriterium: Sei $d_n = |a_n|$.

$\sum_{n\geq 0} \frac{1}{n}$ divergent $\wedge$ $|d_n| \geqslant \frac{1}{n}$ für alle $n \geq 0$

$\Rightarrow \sum_{n\geq 0} d_n$ divergent

Daher:

$\sum_{n\geq 0} a_n = \sum_{n\geq 0} \frac{(-1)^n}{\sqrt[3]{n+2}}$ bestimmt konvergent.