Homework 1 for the course Advanced Mathematical Logic

Deadline: May 17, 2021

May 3, 2021

1. Let $\mathcal{L}$ be any finite language and let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Show that there is an $\mathcal{L}$-sentence φ such that $\mathcal{N} \models \varphi$ if and only if $\mathcal{N} \cong \mathcal{M}$.

2. Show that if T is an unsatisfiable $\mathcal{L}$-theory then every $\mathcal{L}$-sentence φ is a logical consequence of T.

3. Suppose that a theory T has arbitrarily large finite models. Show that T has an infinite model.

   For the next exercise we will need several natural definitions. Let $\mathcal{A}$ be an $\mathcal{L}$-structure, for some language $\mathcal{L}$. Define the language $\mathcal{L}_A = L \cup \{c_a : a \in A\}$, where A is the domain of $\mathcal{A}$. In other words, we augment $\mathcal{L}$ by constant symbols for each element of $\mathcal{A}$. We interpret each c_a naturally as a in $\mathcal{A}$.

   Define the *complete* (or *elementary*) *diagram* of $\mathcal{A}$ to be the set of all first-order $\mathcal{L}_A$-sentences true in $\mathcal{A}$, also written as $D^c(\mathcal{A}) = \mathrm{Th}(\mathcal{A}, a)_{a \in A} = \mathrm{Th}_A(\mathcal{A})$.

   Define the *atomic diagram* $D(\mathcal{A})$ of $\mathcal{A}$ to be the set of all atomic $\mathcal{L}_A$-sentences true in $\mathcal{A}$ and negations of atomic $\mathcal{L}_A$-sentences false in $\mathcal{A}$.

   If $\mathcal{M}, \mathcal{N}$ are $\mathcal{L}$-structures, then an $\mathcal{L}$-embedding $j : \mathcal{M} \to \mathcal{N}$ is an *elementary embedding* if

   $$\mathcal{M} \models \phi(a_1, \dots, a_n) \iff \mathcal{N} \models \phi(j(a_1), \dots, j(a_n))$$

   for all $\mathcal{L}$-formulas $\phi(v_1, \dots, v_n)$ and all $a_1, \dots, a_n \in M$.

   If $\mathcal{M}$ is a substructure of $\mathcal{N}$, we say that it is an *elementary substructure* and write $\mathcal{M} \prec \mathcal{N}$, if the inclusion map is elementary. We also call $\mathcal{N}$ an *elementary extension* of $\mathcal{M}$.

4. Let $\mathcal{M}, \mathcal{N}$ be structures. Prove that:

a) If $\mathcal{N} \models D(\mathcal{M})$ then $\mathcal{M}$ is a substructure of $\mathcal{N}$.

b) If $\mathcal{N} \models D^c(\mathcal{M})$ then $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$.

5. Let $\mathcal{L} = \{E\}$ where E is a binary relation symbol. Let T be the $\mathcal{L}$-theory of an equivalence relation with infinitely many infinite classes and no finite classes.

   a) Write axioms for T.

   b) How many models of T are there of cardinality $\aleph_0$? $\aleph_1$?

   c) Is T complete?

6. Let $\mathcal{L} = \{E\}$ where E is a binary relation symbol. Which of the following theories (each satisfying the axioms stating that E is an equivalence relation) have quantifier elimination? Explain your answer.

   a) E has infinitely many equivalence classes all of size 2 and no other equivalence classes.

   b) E has infinitely many equivalence classes classes all of which are infinite and no other classes.

   c) E has infinitely many equivalence classes of size 2, infinitely many classes of size 3, and every class has size 2 or 3.

   d) E has exactly one equivalence class of size n for each $n < \omega$.