Discrete Mathematics (104.697)

Alttest Collection

Repository

Contents

1 2024-11-08 (vowi)

https://vowi.fsinf.at/wiki/TU_Wien:Discrete_Mathematics_VO_(Gittenberger)/Written_Exam_2024-11-08

1.1

Is $\mathbb{Z}_5[x]/(x^4 + x^2 + 1)$ a field? List all members of it.

Is $x + 2$ a unit in it? If so, find its multiplicative inverse

1.2

Determine the number of spanning trees for a given graph (consisting of two connected components and 8 total vertices) using the matrix-tree-theorem.

The *Matrix-Tree-Theorem (Kirchhoff)* cannot be used without having the adjacency matrix A and the degree matrix D. Further calculating the det of a 7×7 matrix would be way too hard.

1.3

Let φ be a ring homomorphism and R, S two rings. Let I be the ideal of S. Prove that $\varphi^{-1} = \{x \in R \mid \varphi(x) \in I\}$ is an ideal as well.

1.4

A Hasse diagram of a poset was given. Determine the value of the Möbius function $\mu(0, 1)$

> ⚠️ **Not Relevant**
>
> Not in the 6 ECTS version of *Discrete Math*

2 2024-10-04 (vowi)

https://vowi.fsinf.at/wiki/TU_Wien:Discrete_Mathematics_VO_(Gittenberger)/Written_exam_2024-10-04

2.1

1. How many multisets of $\{a, b\}$ are there of size 12?
2. How many multisets which are subsets of $M = \{a, a, a, a, b, b, b, b, b, b, c, c, c, c, c\}$ are there of size 12? Answer this question by
   1. Determining the number of multisets which are not subsets of M because they contain too many a's
   2. Using the principle of inclusion-exclusion

2.2

Let R be an integral domain, and $a \in R$. Prove that $(a) = \{ar \mid r \in R\}$

> ⚠️ **Missing Details**
>
> Maybe some details in the tasks are missing: problem could be too easy or unsolvable!

The proof is just the definition of the principal ideal

2.3

Let $G = (V, E)$ be a simple, undirected graph of $n = |V|$ vertices and $m = |E|$ edges. Also let $A \in \{0, 1\}^{n \times n}$ be the adjacency matrix, $B \in \{0, 1\}^{n \times m}$ the incidence matrix and D the degree matrix of G, where $a_{ij} = 1 \iff (v_i, v_j) \in E$ and $b_{ij} = 1 \iff v_i$ incident to e_j. Prove that $A + D = BB^T$

2.4

Solve the following two systems of linear congruences or prove that there is no solution:

1. $\begin{cases} 2x \equiv 4 \bmod 8 \\ 5x \equiv 2 \bmod 11 \\ 4x \equiv 5 \bmod 15 \end{cases}$

2. $\begin{cases} 4x \equiv 4 \bmod 2 \\ 5x \equiv 2 \bmod 2 \\ 4x \equiv 5 \bmod 5 \end{cases}$

3 2024-07-02 (vowi)

https://vowi.fsinf.at/wiki/TU_Wien:Discrete_Mathematics_VO_(Gittenberger)/Written_Exam_2024-07-02

3.1

Is the polynomial $p(x) = x^6 + x^5 + x^2 + x + 1$ a primitive in $\mathbb{Z}_2$?

3.2

Compute the number of integers x with $1 \leq x \leq 1000000$ where x is neither a square nor a 3rd, 4th or 5th power of some positive integer y.

3.3

For any simple, undirected graph G, prove via induction over $\alpha_0(G)$, that the following holds: $\chi(G) \leq 1 + \max_{v \in V} d(v)$.

3.4

Use the Chinese Remainder Theorem to solve the following system of congruence relations:

$$3x \equiv 12 \bmod 13$$
$$5x \equiv 7 \bmod 22$$
$$2x \equiv 3 \bmod 7$$

4 2023-04-12 (vowi)

https://vowi.fsinf.at/wiki/TU_Wien:Discrete_Mathematics_VO_(Gittenberger)/Written_Exam_2023-04-12

4.1

In a room there are m chairs and n people $(n \leq m)$. The people take a break and leave the room. How many ways can the n people sit on the m chairs such that no one sits on the same chair as before the break.

4.2

Let R be an integral domain. Two elements a, b of R are called associated if $a = b * r$ with r being a unit (so element of R^*). Prove that two elements x, y of R are associated if and only if $x|y$ and $y|x$.

4.3

Let I be an integral domain. Define $a \sim b$ the equivalence relation as $a - b \in I$. Prove that $\sim$ is an equivalence relation. Furthermore show that $[x]$ (which is the set of all elements related to x) is equal to $x + I$ (or something similar to this)

> ⚠ **Missing Details**
>
> Maybe some details in the tasks are missing: problem could be too easy or unsolvable!

4.4

Show that $\sqrt{3} + i$ is algebraic over $\mathbb{Q}$ and determine its primitive polynomial.

5 2023-01 (vowi)

https://vowi.fsinf.at/wiki/TU_Wien:Discrete_Mathematics_VO_(Gittenberger)/Written_Exam_2023-01

5.1

Solve:

$$x^2 \equiv 1 \bmod 4$$
$$3x \equiv 4 \bmod 5$$
$$6x \equiv 3 \bmod 9$$

5.2

Prove (E, S) with $|E| = n$ and $S = \{X \subseteq E : |X| \leq m\}$ is a matroid

5.3

Show that for a set of distinct numbers $A \subset \{1 ... 15\}$ of size 8, there always exist two numbers which sum up to 16

5.4

Prove if $\mathbb{Z}_3/(x^2 + x + 1)$ is a field, list all elements.
Is $x + 1$ a unit? If yes, give its multiplicative inverse.

6 2022-11-25 (vowi)

https://vowi.fsinf.at/wiki/TU_Wien:Discrete_Mathematics_VO_(Gittenberger)/Written_Exam_2022-11-25

6.1

Compute the number of spanning trees of the given Graph $G = (V, E), V = \{1...8\}$ and $E = \{(1, 2), (1, 3), (2, 4), (3, 4), (5, 8), (5, 6), (6, 8), (6, 7), (7, 8)\}$.

6.2

Compute the value of $\mu(0, 1)$ for the given Hasse diagram

⚠️ Not Relevant

Not in the 6 ECTS version of *Discrete Math*

6.3

Solve the following recurrence relation using generating functions (*something like this*)
$a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 1$ with $a_0 = 1, a_1 = 3, a_2 = 0$

Comment by arch user

Error in the given formula from vowi, but fixed here

6.4

Show the identity:

$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{n+m+1}{m+1}$$

using combinatorial interpretation.

⚠️ **Hint:** Consider $\{0, 1\}$ sequences and group them according to the position of the last 1.

7 2022-09-30 (vowi)

https://vowi.fsinf.at/wiki/TU_Wien:Discrete_Mathematics_VO_(Gittenberger)/Written_Exam_2022-09-30

7.1

Use the Chinese remainder theorem to solve the following system of congruence relations:

$$3x \equiv 12 \bmod 13,$$
$$5x \equiv 7 \bmod 22,$$
$$2x \equiv 3 \bmod 7$$

i Duplicate

This exercise is equivalent to Section 3.4

7.2

Compute the number of integers x with $1 \leq x \leq 1000000$ where x is neither a square nor a 3rd, 4th or 5th power of some positive integer y.

i Duplicate

This exercise is equivalent to Section 3.2

7.3

List all elements of $\mathbb{Z}_3[x]/(x^2 + x + 1)$. Examine whether or not this is a field and whether or not $x + 1$ is a unit in it. If $x + 1$ is a unit, compute its multiplicative inverse.

i Duplicate

This exercise is equivalent to Section 5.4

7.4

List all sets S such that $(\{1, 2, 3\}, S)$ is a matroid.

8 2022-05-06 (vowi)

https://vowi.fsinf.at/wiki/TU_Wien:Discrete_Mathematics_VO_(Gittenberger)/Written_Exam_2022-05-06

8.1

Given $A = \{a, b, c\}$

1. What is the number of multisets of cardinality 12 built up from A?
2. How many of these sets are subsets of $B = \{\dots\}$ (where B was a specific multiset) Answer by ...
   1. restricting the number of multisets found in 1?
   2. using the principle of inclusion-exclusion (*some more hints were given*)

⚠️ Missing Details

If you want to try it anyways, $B = \{a, a, a, a, a, b, b, b, c, c, c, c, c, c\}$ should work.

8.2

Let R be an integral domain. Prove that $(a) = \{ra \mid r \in R\}$

⚠ Missing Details

That's the definition of a principal ideal...
The task ("prove that *this* is an ideal" or something similar) is missing.

8.3

Given an undirected Graph G. Let D be the degree matrix and A the adjacency matrix. Define B the incidence matrix with nodes as rows and edges as columns and $B[i][e] = 1$ iff node i is incident to edge e and $B[i][e] = 0$ otherwise. Prove that $A + D = B \cdot B^T$ where B^T is the transpose of B.

✏️ **Assumption (by us)**

undirected, *simple* graph is important, else we could have loops or double-edges

i **Duplicate**

This exercise is equivalent to Section 2.3

8.4

Given the following two systems of congruences. Either give the solutions using CRT or prove that there is none:

1. $2x \equiv 2 \bmod 8$
2. $4x \equiv 2 \bmod 8$

⚠ Missing Details

Maybe some details in the tasks are missing: problem could be too easy or unsolvable!

9 2022-03-11 (vowi)

https://vowi.fsinf.at/wiki/TU_Wien:Discrete_Mathematics_VO_(Gittenberger)/Written_Exam_2022-03-11

9.1

Let A, B be bases of a matroid. prove that $|A| = |B|$

9.2

Is $\sqrt{2} + \sqrt{3}$ algebraic over $\mathbb{Q}$, determine its minimal polynomial.

9.3

Determine $\mu(0, 1)$ of the following partial order:

```
   1
   |
   d
/    \
b      c
\    /
   a
   |
   0
```

⚠️ **Not Relevant**

Not in the 6 ECTS version of *Discrete Math*

9.4

Let A, B be two finite sets with $|A| = n$ and $|B| = k$.

1. How many injective mappings: $f : A \mapsto B$ are there?
2. Show that the number of surjective mappings $f : A \to B = k! S_{n,k}$

10 2022-01-26 (vowi)

https://vowi.fsinf.at/wiki/TU_Wien:Discrete_Mathematics_VO_(Gittenberger)/Written_Exam_2022-01-26

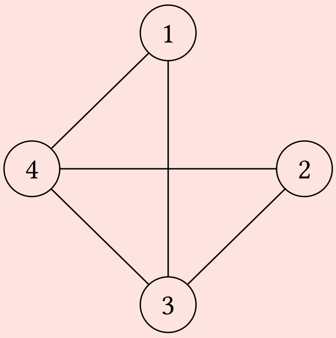
10.1

Calculate the number of spanning forests of a given graph

⚠️ Missing Details

❗ As you can see the graph isn't given 🤡
Maybe take a look at Section 6.1 (page 24)

✏️ Assumption (by us)



10.2

$\{a, b\}^* = \{a\}^* \times (\{b\} \times \{b\}^* \times \{a\} \times \{a\}^*)^* \times \{b\}^*$

Explain why this equality holds. Furthermore, state the generating function counting all elements made of $\{a, b\}$ which contain neither aaa or bbb.

10.3

$$4x \equiv 2 \pmod{11},$$
$$x^2 \equiv 1 \pmod{6},$$
$$12x \equiv 8 \pmod{20}$$

10.4

$R = \mathbb{Z}_5[x]/(x^2 + 3x + 1)$ List all elements of R.

1. Is R a field?
2. Is $x + 3$ a unit in R?

11 2021-02-05 (vowi, Drmota)

https://vowi.fsinf.at/images/a/af/TU_Wien-Discrete_Mathematics_VO_(Gittenberger)_-_Written_Exam_2021-02-05.pdf

11.1

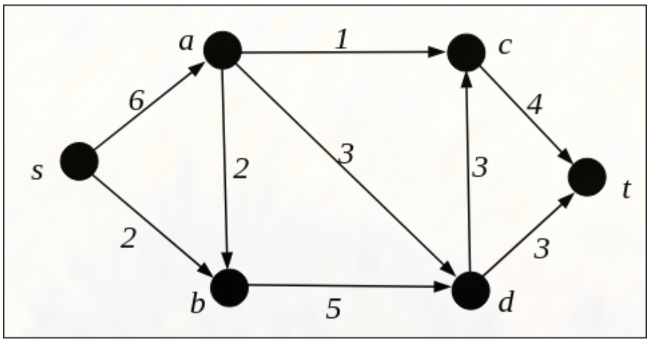
Determine explicitly the coefficient a_n of the power series

$$f(z) = \frac{1}{(1-z)(1+3z)} + \frac{3}{\sqrt{1+2z}}$$

11.2

Determine (with the help of the principle of inclusion and exclusion) the number of natural numbers n with $1 \leq n \leq 2000$ such that $\gcd(n, 140) = 5$.
Hint: Reformulate the condition $\gcd(n, 140) = 5$ into several condition of the form $p_i | n$ and p_j does not divide n (for some primes p_i, p_j).

11.3



Determine a maximum flow from s to t on the above network by starting with the zero flow and by determining a series of augmenting paths. Does this maximum flow change if the edge (a, c) (with capacity 1) is cut? Explain your answer

⚠️ **Not Relevant**

Gittenberger skipped *Max Flow/Min Cut* theorem

11.4

1. Which polynomial is irreducible over $\mathbb{Z}_3$
   1. $f(x) = x^3 + x^2 - 1$
   2. $g(x) = x^4 - x + 1$

   - Hint: Be careful with $g(x)$. It has no zeros but ...
2. Furthermore determine all solutions of the following set of congruences
   - Hint: Reduce the given system into a system of the form $x \equiv a_i \bmod m_i$

$$x^2 \equiv 1 \bmod 2,$$
$$28x \equiv 24 \bmod 44$$

12 2025-06-27 (Stufler)

12.1 Number Theory

1. For which integers m is the group $(\mathbb{Z}/\langle m \rangle)^*$ cyclic?
2. Compute with Euler's totient function the number of elements in $(\mathbb{Z}/\langle 100 \rangle)^*$
3. Compute $7^{100} \bmod 40$

12.2 Combinatorics

Show that each set of integers with 5 elements has 3 elements whose sum is divisible by 3.

12.3 Finite Fields

Consider the field $\mathbb{F}_2/\langle f\rangle$ with $f = x^4 + x + 1$

1. show that f is a field
2. using euclid's algorithm, determine the inverse of $x^3 + x + 1$

12.4 Graph Theory

Show that every simple graph with $\deg(v) \geq 3$ for all vertices has a cycle with length at least 4.

13 2025-03-01 (Stufler)

13.1 Graphs

1. prove that for a planar graph with at least 10 vertices, the average degree of vertices must be less than 10
2. prove for a matroid (E, S), if A and B are bases of the matroid, then $|A| = |B|$

13.2 Generating Functions

Determine explicitly the coefficient a_n of the power series:

$$f(x) = \frac{1}{(1-x)(1+3x)} + \frac{3}{\sqrt{1+2x}}$$

13.3 Abstract Algebra

For $\mathbb{Z}_3$, show which is irreducible

1. $f(x) = x^3 + x^2 - 1$
2. $g(x) = x^4 - x + 1$

13.4 Pigeonhole Principle

Given a Set A, which is a subset of $\{1, ..., 41\}$, and $|A| = 21$, prove that there must exist $x, y \in A$ such that $x + y = 42$

14 2025-01-24

14.1 Number Theory (10pt)

Let p denote a prime number, k a positive integer and $\mathbb{F}_{p^k}$ the field with p^k elements.

1. What is the characteristic $\operatorname{char}(\mathbb{F}_{p^k})$ [1pt]
2. Prove that if $p = 2$ then $a = -a$ for all $a \in \mathbb{F}_{2^k}$ (Hint: $a = 1 \cdot a$) [1pt]
3. How many elements does the group of units $\mathbb{F}_{p^k}^{\times}$ have? [1pt]
4. How many elements does the group of units $\mathbb{Z}/\langle p^k \rangle^{\times}$ have? [1pt]
5. How many elements does the group of units $\mathbb{Z}/\langle 90 \rangle^{\times}$ have? [1pt]

14.2 Polynomials in finite fields (10pt)

Let $\mathbb{F}_8$ denote the field with 8 elements. Let $f, g \in \mathbb{F}_8[x]$ with $f = x^4 + x^3 + x + 1$ and $g = x^3 + x + 1$.

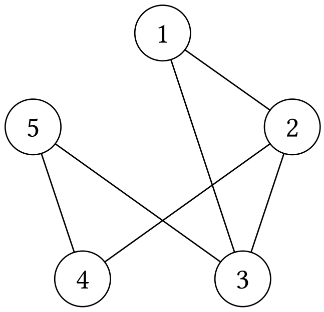
1. Determine $\gcd(f, g)$ [4pt]
2. Find $\lambda, \mu \in \mathbb{F}_8[x]$ with $\lambda f - \mu g = \gcd(f, g)$ [4pt]
3. Is g invertible in $\mathbb{F}_8[x]/\langle f \rangle$? If yes, determine its inverse [2pt]

14.3 Combinatorics (10pt)

1. let $A = \{1, 2, ..., 6\}$. How many ways are there to partition A into 2-element subsets? (For example $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ is one such partition) [5pt]
2. Show that for any positive integer n the number of positive divisors of n^2 is always odd. (Hint: use prime factor decompositions) [5pt]

14.4 Graph Theory (10pt)

Calculate the number of spanning trees of the following graph:



15 2025-12-18

15.1 Polynomials in finite fields (10pt)

Let K be the field $\mathbb{Z}_7$. Examine whether the quotient ring $K[x]/x^2 + 5$ is a field or not. Is $x + 2$ a unit in it?

15.2 Matroids (10pt)

Let E be a set, $1 \leq k \leq |E|$ an integer, and let S denote the set of all subsets $X \subseteq E$ with cardinality at most k. Examine wether $M = (E, S)$ is a matroid.

15.3 Combinatorics (10pt)

Given $A = \{a, b, c\}$

1. What is the number of multisets of cardinality 12 built up from A?
2. How many of these sets are sub-multisets of $M = \{a, a, a, a, b, b, b, b, b, b, c, c, c, c, c\}$? Answer by ...
   1. Showing how many multisets cannot be sub-multisets due to containing too many a’s
   2. Using the principle of inclusion-exclusion

15.4 Theory (10pt)

What is an unlabeled combinatorial structure?

What counting problem regarding it did we formulate in the lecture? How did we approach solving this problem? Why is this solution method appropriate?

To show this, show both basic constructions we used during the lecture, and how this method applies to them. Prove every step of your explanation!

16 Random Exercises

16.1 Matroid

https://vowi.fsinf.at/images/a/a1/TU_Wien-Discrete_Mathematics_UE_(diverse)_-_Ue3.pdf

Let E be a set, $1 \leq k \leq |E|$ an integer, and let S denote the set of all subsets $X \subseteq E$ with cardinality at most k. Examine wether $M = (E, S)$ is a matroid.

16.2 Binom

https://vowi.fsinf.at/images/3/36/TU_Wien-Discrete_Mathematics_UE_(diverse)_-_2023W_Ue7_Lsg.pdf

Prove for all complex numbers x and $k \in \mathbb{N}$ we have

$$\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}$$

16.3 Squared Fibonacci Generating Function

https://vowi.fsinf.at/images/3/36/TU_Wien-Discrete_Mathematics_UE_(diverse)_-_2023W_Ue7_Lsg.pdf

Prove that the squares of the Fibonacci numbers satisfy the recurrence relation

$$a_{n+3} - 2a_{n+2} - 2a_{n+1} + a_n = 0$$

and solve this recurrence relation with the correct initial conditions

16.4 Generating Function

https://vowi.fsinf.at/images/3/36/TU_Wien-Discrete_Mathematics_UE_(diverse)_-_2023W_Ue7_Lsg.pdf

Solve the following recurrence using generating functions:

$$a_{n+1} = 3a_n - 2 \quad \text{for} \quad n \geq 0, a_0 = 2$$

16.5 Ring Family of Ideals

https://sh1.fsinf.at/~prawn/sheet_10.html

Let R be a ring and $\left(I_j\right)_{j \in J}$ be a family of ideals of R. Prove that $\bigcap_{j \in J} I_j$ is an ideal of R.

16.6 Extended Euclidean

https://sh1.fsinf.at/~prawn/sheet_7.html

Use the Euclidean Algorithm to find all the greatest common divisors of $x^3 + 5x^2 + 7x + 3$ and $x^3 + x^2 - 5x + 3$ in $\mathbb{Q}[x]$

16.7 Ideal of $\mathbb{Z}_6$

https://github.com/VeryMilkyJoe/discrete-maths-collection/blob/main/src/sheet_10.md

Let $U = \{\overline{0}, \overline{2}, \overline{4}\} \subseteq \mathbb{Z}_6$. Show that U is an ideal of $\mathbb{Z}_6$

16.8 Primitive Polynomials over $\mathbb{Z}_3$

Which of the following polynomials are primitive over $\mathbb{Z}_3$?

1. $x^3 + x^2 + x + 1$
2. $x^3 + x^2 + x + 2$
3. $x^3 + 2x + 1$

Comment by arch user

Übungsbeispiel 55