VO Geschichte der Logik

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Part I

Logic and Syllogisms of Aristotle

Introduction

What is Logic?

- The meaning of logic changed during the course of history.
- The name “logic” derives from the Greek “logos”, which has several meanings, among them:
  - word, sense, meaning, thought, speech, reason, principle, standard

What is Logic? (Ctd.)

- In general, one may distinguish between a more *narrower* and a more *wider* sense of logic.
- In a more narrower sense, logic deals with the *correctness of reasoning*
  - i.e., it deals with the question which *conclusions* follow from a given set of premisses.
  - ➥ From a modern point of view, logic in this sense is the *formal study of inference relations*.
- In a more wider sense, logic is understood as *the analysis of correct reasoning, its requirements, and its applications*.

What is Logic? (Ctd.)

This includes the following areas:

Formal logic: most important subjects are propositional and predicate logic.
Methodology: is the study of logical methods common to all branches of science, comprising the theory of *definitions*, *distinctions*, *divisions*, and *proofs*.
Semiotics: the theory of *signs* (also referred to as *metalogic*). It includes the study of

- *syntax* (relation of signs with each other),
- *semantics* (relation of signs with their denotations), and
- *pragmatics* (the relation of signs with their users).

Philosophy of logic: deals with the relation of logic to other philosophical disciplines, like *ontology*.
Applied logic: e.g., logic in computer science, mathematics, law, etc.

What is Logic? (Ctd.)

☞ We are interested in the development of *formal logic*, i.e., logic in the narrower sense, which is the logic arguably most relevant for computer science.

Roots of Logic

- The history of logic encompasses more than two and a half millennia!
- For a long time, any student at a university had to attend a course on logic—the *Collegium Logicum*.
- Thus wrote Goethe in Faust I (1790):

Mein teurer Freund, ich rat Euch drum
Zuerst Collegium Logicum.
Da wird der Geist Euch wohl dressiert,
In spanische Stiefeln eingeschnürt,
Daß er bedächtiger so fortan
Hinschleiche die Gedankenbahn . . .

Roots of Logic (ctd.)

- Formal logic originated in precisely two cultural circles:
  - Greek antiquity, at the time of the classic Greek philosophy in Athens (at about the 4th century B.C.);
  - India (at about the 1st century A.D.).
- In Greek philosophy, the term *knowledge*, closely associated with logic, provoked philosophical controversies early on.
  - Some of the deepest controversies originated with Socrates in the 5th century B.C., by claiming to know very little, if anything.
- Plato (427–349 B.C.), Socrates' student, established the subject of *epistemology*, the study of the nature of knowledge and its justification.
  - However, no writings of Plato about logic itself survived, although some logical methods were studied in his academy, besides geometry and philosophical problems.

Roots of Logic (ctd.)

- Aristotle (384–322), Plato's student, is considered as the founder of western logic.
- He shifted the emphasis of philosophy from the *nature of knowledge* to the less controversial problem of *representing knowledge*.
- He could build upon the teachings of his master Plato, but he was the first to *systematically lay down a theory of logic*
  - this theory includes general laws of valid arguments and methods for performing deductions.

Roots of Logic (ctd.)

- Aristotle's logic (later referred to as the "old logic " or the "traditional logic") was the predominant logical method for the next two thousand years.
- This lead Immanuel Kant (1724–1804) to write in the introduction of his *Kritik der reinen Vernunft*:

  Aristotle invented logic so perfect, that it made no step forward or backward since then.

  This was of course far from true.
- Also, the philosopher and historian Carl von Prantl (1820 - 1888) expressed the following belief:

  any logician after Aristotle who said anything new was confused, stupid, or perverse.

  ☞ Prantl wrote an important treatise on the history of logic: *Geschichte der Logik im Abendlande*, 1855–70, four volumes.

Periods of Logic

- The history of western logic can be classified into five periods:
  1. ancient Greece (until the 6th century A.D.);
  2. Middle Ages (from the 7th to the 11th century);
  3. scholasticism (from the 11th to the 15th century);
  4. modern "classical" logic (16th to 19th century);
  5. mathematical logic (since the mid 19th century).

- Two of these periods (the Middle Ages and modern "classical" logic) contain, with some notable exceptions, no real innovation.
  - In fact, modern "classical" logic is a degenerated form of logic, being a mixture of fragments of scholastic and ancient Greek logic.

Periods of Logic (ctd.)

- The logics of the scholastics first followed Greek antiquity (namely mainly Aristotle's logic) ...
- ... but since the end of the 12th century, something completely new was created.
  - This logic was predominately formulated as a metalanguage, with a precise and complex semantics.
  - Formulas of this language are words from natural language (i.e., Latin) with none or only few variables.
  - Scholastic logic can be seen as an ambitious attempt to capture formal laws, expressed in natural language, in terms of syntactic rules and semantic functions.
  - The logic of the scholastics was very sophisticated but eventually most of their achievements became forgotten and resulted in the decadent modern "classical" logic.

Periods of Logic (ctd.)

- The revival of logic, yielding the powerful mathematical form of logic, was motivated by questions concerning the foundations of mathematics.
- This new logic is the outcome of the joining of four ideas and methods:
  1. Aristotle's logic;
  2. the idea of a complete and automatic language for reasoning;
  3. the new developments in algebra and geometry after 1825; and
  4. the idea of the parts of mathematics as being systems of *deductions*.

Some Notation

- In what follows, we assume familiarity with modern predicate logic.
- Formulas of predicate logic are built in the usual way, employing the following operators:
  - $\wedge$ (conjunction),
  - $\vee$ (disjunction),
  - $\neg$ (negation),
  - $\supset$ (material implication),
  - $\equiv$ (material equivalence),
  - $\forall$ (universal quantification),
  - $\exists$ (existential quantification).

Some Notation (ctd.)

- "$\vdash$" denotes the derivability relation of predicate logic.
  - "$\vdash A$" means that A is provable, or that A is a *theorem*.
  - "$T \vdash A$" is called a *sequent* and expresses that A is derivable from the set T of premisses.
- "$\models$" is the semantic consequence relation of predicate logic.
  - "$\models A$" means that A is *valid*.

- It holds that $T \vdash A$ iff $T \models A$ ("soundness and completeness of predicate logic").

The Logic of Aristotle

Overview

Aristotle: A Short Biography

Portrait of Aristotle; Imperial Roman copy (1st or 2nd century); Louvre

Aristotle: A Short Biography (ctd.)

- Aristotle was born in 384 B.C. at Stageira, a colony of Andros on the Macedonian peninsula of Chalkidiki.
  - His father, Nicomachus, was court physician to King Amyntas III of Macedon.
- From the age of 18 to 37, Aristotle remained in Athens as a pupil of Plato and distinguished himself at the Academy.
- After the death of Plato (347 B.C.), Aristotle was considered as the next head of the Academy, but the post was eventually awarded to Plato's nephew.
- Aristotle then went to the court of Hermias, ruler of Atarneus in Asia Minor, and married the niece and adopted daughter of Hermias, Pythia.

Aristotle: A Short Biography (ctd.)

- In 343 B.C., he became the tutor of Alexander the Great, who was then 13.
- In about 335 B.C., Alexander departed for his Asiatic campaign, and Aristotle, who had served as an informal adviser since Alexander ascended the Macedonian throne, returned to Athens and opened his own school of philosophy, the *Lyceum*.
  - The school was active until 40 B.C., from which the Peripatetics emerged.
- After the death of Alexander, Aristotle faced anti-Macedonian hostilities and left Athens.
- He fled to Chalcis, the birthplace of his mother, and died there the following year, in 322 B.C.

The Organon

Aristotle's treatises on logic were grouped together by ancient commentators under the title *Organon* ("Instrument"), comprising the following parts:

Categories: introduces a hierarchical classification based on ten basic categories; ambiguities of natural language; and different forms of opposites.

On Interpretation: discusses different forms of propositions and their relations, including modalised statements.

Prior Analytics: contains his deductive logical system, the *syllogisms*.

Posterior Analytics: deals with the first theory of science.

Topics: probably the oldest part of the Organon, collecting an early, informal logic for practical, rhetoric purposes.

On Sophistical Refutations: discusses *fallacies*, i.e., invalid arguments.

The Organon (ctd.)

☞ The title "Organon" reflects the later controversy whether logic is
- a part of philosophy (as the Stoics maintained), or
- merely a tool used by philosophy (as the later Peripatetics thought).

$\Longrightarrow$ Using the title "Instrument" leans toward the latter viewpoint.

Aristotle's Work

➤ Aristotle not only invented logic, but he established also the terminology and the scope of
- physics,
- metaphysics,
- biology,
- psychology,
- linguistics,
- politics,
- ethics,
- rhetoric, and
- economics.

➥ In fact, his life's work resulted *in an encyclopedic compilation of the knowledge of his day.*

Syllogisms

Deductions

➤ Central for Aristotle's logic is the notion of a *deduction* ("sullogismos").

➤ According to the Prior Analytics I.2, 24b18-20:

> "A deduction is speech (logos) in which, certain things having been supposed, something different from those supposed results of necessity because of their being so."

- Each of the "things supposed" is a *premiss* ("protasis") of the argument, and what "results of necessity" is the *conclusion* ("sumperasma").
- The phrase "results of necessity" corresponds to the modern notion of logical consequence:
  - A results of necessity from B and C iff it is impossible that A is false but B and C are true.

Deductions (ctd.)

Aristotle's definition of a deduction bears some important differences to its modern-day sibling:

1. A conclusion must be different from the premisses in Aristotle's definition $\Longrightarrow$ this is not required in modern logic.
2. The plural "certain things having been supposed" was understood by some ancient commentators to rule out arguments with only one premiss.
3. The expression "because of their being so" has sometimes been understood as excluding arguments in which the conclusion is not "relevant" to the premisses, e.g.,
   - ➤ arguments with inconsistent premisses,
   - ➤ arguments with conclusions that would follow from any premisses whatsoever, or
   - ➤ arguments with superfluous premisses.

Assertions

➤ In accord to the definition of a "sullogismos", Aristotle considers only arguments of a *specific form*, involving just three assertions.
$\Longrightarrow$ In later usage, such arguments are referred to as *syllogisms*.

➤ By an *assertion* Aristotle understands a sentence which is either true or false having the following structure:

- it must contain a *subject* and a *predicate* and must either affirm or deny the predicate of the subject.

➤ An assertion is either *universal* or *particular*

- "all" and "no" added to the subject refer to universality,
- while "some" and "some not" or "not all" refer to particularity.

Assertions (ctd.)

➤ Assertions confirm to the following table:

	Affirmations	Denials
Universal	P is affirmed of all S Every S is P All S is P	P is denied of all S No S is P
Particular	P is affirmed of some S Some S is P	P is denied of some S Some S is not P Not every S is P

➤ Note: The above table uses modern-day formulations.

- Aristotle never writes, e.g., "all S is P", rather he uses for this "P is predicated of all S" or "P belongs to all S".

Terms

➤ Aristotle calls subjects and predicates of assertions *terms*.

➤ The Prior Analytics says nothing about terms, but On Interpretation contains a definition of *individual* and *universal* terms:

- a term is universal if it can be predicated of many subjects, like "man", otherwise a term is singular, e.g., Callias.

☞ Aristotle ignores thereby that a non-universal term need not to be singular, as it may be empty.

➤ In formulating his logic, Aristotle does not use singular or empty terms.
$\Longrightarrow$ Only universal terms are mentioned in The Prior Analytics.

- E.g., Aristotle would not accept an expression like "All Calliases are men" if there is only one Callias.

A Mnemonic Convention

➤ During the middle ages, letters a, e, i, o have been assigned to refer to assertions (from the Latin words *a*ff*i*rmo and n*e*g*o*):

- a: universal affirmation;
- e: universal denial;
- i: particular affirmation;
- o: particular denial.

➥ We write assertions in the form $S\omega P$, for $\omega \in \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}\}$.

Assertions: Basic Relations

Aristotle identifies the following basic relations between these four forms of assertions:

➤ $S\mathrm{a}P$ (every S is P) and $S\mathrm{e}P$ (no S is P) *cannot both be true but can both be false*.

➥ $S\mathrm{a}P$ and $S\mathrm{e}P$ are said to be *contrary* to each other.

➤ $S\mathrm{a}P$ (every S is P) and $S\mathrm{o}P$ (some S is not P) *cannot both be true and cannot both be false*.

➥ $S\mathrm{a}P$ and $S\mathrm{o}P$ are said to be *contradictory*.

➤ Likewise, $S\mathrm{e}P$ (no S is P) and $S\mathrm{i}P$ (some S is P) are contradictory.

Assertions: Basic Relations (ctd.)

➤ $S\mathrm{i}P$ (some S is P) and $S\mathrm{o}P$ (some S is not P) *cannot both be false but can both be true*.

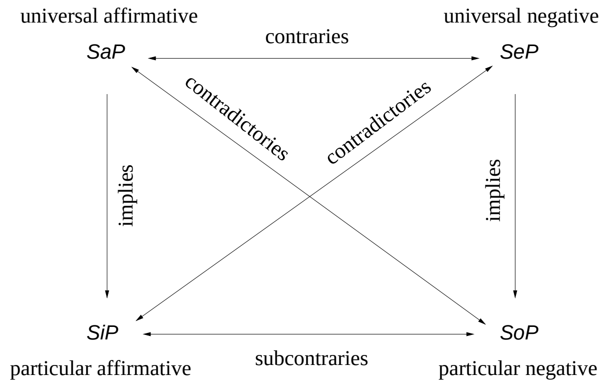
➥ $S\mathrm{i}P$ and $S\mathrm{o}P$ are said to be *subcontrary* to each other.
(The term "subcontrary" derives from the fact that later logicians called the particular statements *subaltern* to the universal ones.)

➤ Lastly, Aristotle assumes that each universal statement entails its subaltern.

☞ This property is referred to by saying that universal assertions have *existential import*.

The Square of Opposition

The previous basic relations are traditionally set out in a diagram (although not by Aristotle himself), called *the square of opposition*:



Syllogisms

A *syllogism* is an inference pattern of the form

if α and β, then γ

where:

- α, β, γ are distinct assertions of the form a, e, i, o
  - α, β are the premisses and γ is the conclusion.
- The assertions involve three terms s.t. α and β share exactly one term in common, called the *middle term*, and γ contains the other two terms not shared by the premisses.
  - The first term in γ is the *minor term* and the second one the *major term*.
  - α contains the major and middle term and β contains the minor and middle term.
    - Accordingly, α is called the *major premiss* whilst β is the *minor premiss*.

Example

- For instance,
  if every F is H and some H is G, then no G is F
  is a syllogism.
- In particular,
  - F is the major term, since it appears in the first premiss,
  - G is the minor term, since it appears in the second premiss,
  - H is the middle term, since it appears in both premisses but not in the conclusion.

A Common Misconception

- Often, textbooks discussing Aristotelian syllogisms use examples like the following:
  All men are mortal. Socrates is a man. Therefore, Socrates is mortal.
- However, this is not a syllogism in Aristotle's original sense, for two reasons:
  1. he does not use singular terms ("Socrates");
  2. syllogisms are not inferences (indicated by "therefore") but *implications*.

Patterns of Syllogisms

➤ Although there are infinitely many terms, there are only finitely many different *patterns* of syllogisms.

➤ Since the order of terms in the conclusion is fixed, there are just four possible ways of permuting the two terms in the two premisses:

I. if $M\omega_1 P$ and $S\omega_2 M$, then $S\omega_3 P$;

II. if $P\omega_1 M$ and $S\omega_2 M$, then $S\omega_3 P$;

III. if $M\omega_1 P$ and $M\omega_2 S$, then $S\omega_3 P$;

IV. if $P\omega_1 M$ and $M\omega_2 S$, then $S\omega_3 P$.

($\omega_i \in \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}\}$, $i = \{1, 2, 3\}$.)

- These arrangements are called *figures*.

Patterns of Syllogisms (ctd.)

➤ Instantiating concrete values for each $\omega_i \in \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}\}$, $i = \{1, 2, 3\}$, results in particular syllogisms.

➤ The specific distribution of values for $\omega_1, \omega_2, \omega_3$ for a given syllogism is called the *mood* of this syllogism.
➥ A syllogism is thus uniquely characterised by its mood and its figure.

➤ We use the string $\omega_1\omega_2\omega_3$ to refer to the mood of a syllogism.
- That is, the major premiss is of form ω_1, the minor premiss is of form ω_2, and the conclusion is of form ω_3.

Patterns of Syllogisms (ctd.)

Example:

➤ Consider the following two syllogisms:
1. If no bird is a mammal and every horse is a mammal, then no horse is a bird.
2. If no mammal is a bird and every mammal is an animal, then no animal is a bird.

➤ These are of the following form:
1. If no P is M and all S is M, then no S is P.
2. If no M is P and all M is S, then no S is P.

➤ Both have the mood eae, but the first one belongs to Figure II whilst the second one belongs to Figure III.

Remarks

☞ Aristotle considered only the first three figures because he mistakenly claims that all syllogisms must be of the first three forms.
- the fourth was generally accepted only since the 1500s.

☞ Important aside: Aristotle introduced the use of *variables* in specifying syllogisms!

Mnemonic Convention—The Second

➤ During the middle ages, words involving the sequence of letters denoting the mood of a syllogism are introduced to refer to specific syllogisms.

➤ Examples:
Barbara: mood aaa of Figure I: If $M\mathrm{a}P$ and $S\mathrm{a}M$, then $S\mathrm{a}P$.
- I.e., if all M is P, and all S is M, then all S is P.

- E.g., if all humans are mortal and all Greeks are humans, then all Greeks are mortal.

Celarent: mood eae of Figure I: If MeP and SaM, then SeP.

- I.e., if no M is P, and all S is M, then no S is P.
- E.g., if no politician is sincere and all members of the congress are politicians, then no member of the congress is sincere.

Valid Inference Patterns

- There are only finitely many different syllogistic patterns:
  - In Figure I, there are only
    - four possible major premisses (a, e, i, or o),
    - four possible minor premisses, and
    - four possible conclusions.

    Hence, Figure I comprises only $4 \cdot 4 \cdot 4 = 64$ possible syllogisms.
  - The same holds for the other figures, so in total we have $4 \cdot 64 = 256$ different possible syllogistic patterns.
- The main burden of traditional logic is to distinguish, among the 256 patterns, those which are valid from those which are invalid.

Valid Inference Patterns (ctd.)

- Aristotle's method to determine valid patterns is to accept some as "evidently true",
  - these are the *perfect syllogisms*, or *axioms* in modern terminology,

  and then using certain principles to *deduce* the valid patterns from the remaining figures.
- This method is known as *reduction to the first figure*, and takes two forms, *direct* and *indirect* reduction.

Valid Inference Patterns (ctd.)

- Indirect reduction is basically an indirect proof, using one of the following lines of reasoning:
  - if α and β entail γ, then α and not γ entail not β, or
  - if α and β entail γ, then not γ and β entail not α.
- A direct reduction is a series of steps leading from the premisses to the conclusion
  - each of which is either a *conversion* of a previous step or an inference from two previous steps relying on a perfect syllogism.

Valid Inference Patterns (ctd.)

- Conversion, in turn, is inferring from a proposition another which has the subject and predicate interchanged.
  - Aristotle uses the following conversions:
    - $MeP \Rightarrow PeM$ (no M is P entails that no P is M) ["simple conversion of e"];
    - $MiP \Rightarrow PiM$ (some M is P entails that some P is M) ["simple conversion of i"];
    - $MaP \Rightarrow PiM$ (all M are P entails that some P is M) ["conversion per accidens"].

☞ The soundness of the last conversion depends on Aristotle's underlying assumption that terms are non-empty.

- For otherwise we could obtain "some liars are angels" from "all angels are liars".

Example

We give Aristotle's proof of *Camestres*, which is the pattern

if $P\mathrm{a}M$ and $S\mathrm{e}M$, then $S\mathrm{e}P$.

```
(1) SeM     % premiss
(2) MeS     % conversion, from (1)
(3) PaM     % premiss
(4) PeS     % Celarent, from (2) and (3)
(5) SeP     % conversion, from (4)
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Syllogisms and Modern Logic

Translating Assertions into Modern Logic

- Assertions can be translated into modern predicate logic as follows:

$S\mathrm{a}P$	Every S is P	$\forall x(S(x) \supset P(x))$
$S\mathrm{e}P$	No S is P	$\forall x(S(x) \supset \neg P(x))$
$S\mathrm{i}P$	Some S is P	$\exists x(S(x) \wedge P(x))$
$S\mathrm{o}P$	Some S is not P	$\exists x(S(x) \wedge \neg P(x))$

- Question:
  - If we translate the basic relations between the four forms of assertions (as expressed by the square of opposition) using the above translations, do we obtain theorems of the predicate calculus or not?
- Answer:
  - in some cases yes, in others, some additional provisos are required, reflecting Aristotle's underlying assumptions!

Syllogisms in Modern Logic

Let us first consider $S\mathrm{i}P$ and $S\mathrm{o}P$:

- As maintained in traditional logic, $S\mathrm{i}P$ and $S\mathrm{o}P$ are subcontraries, i.e.,

  $(*)$ $S\mathrm{i}P$ and $S\mathrm{o}P$ cannot both be false but can both be true.

- A direct translation of $(*)$ yields the formula

$$\exists x(S(x) \wedge P(x)) \vee \exists x(S(x) \wedge \neg P(x)). \tag{1}$$

  - However, this formula is not provable in predicate logic, as there may be no objects satisfying S. Indeed, it holds that

$$\vdash \exists x S(x) \equiv \big(\exists x(S(x) \wedge P(x)) \vee \exists x(S(x) \wedge \neg P(x))\big).$$

  - So, if (1) were a theorem, then $\exists x S(x)$ would be a theorem as well, *for any property S*, which clearly cannot hold.

Syllogisms in Modern Logic

- In accepting (∗) as a principle of logic, traditional logic (and thus Aristotle) makes therefore the assumption that *every term S is non-empty*, i.e., there are objects satisfying S.

- Hence, the correct rendering of (∗) in predicate logic is

$$\exists x S(x) \vdash \exists x(S(x) \wedge P(x)) \vee \exists x(S(x) \wedge \neg P(x)).$$

- Concerning the contradictory pairs $S\mathrm{a}P$ and $S\mathrm{o}P$, as well as $S\mathrm{e}P$ and $S\mathrm{i}P$, here the following holds:

$$\vdash \neg(\forall x(S(x) \supset P(x)) \equiv \exists x(S(x) \wedge \neg P(x)),$$
$$\vdash \neg(\forall x(S(x) \supset \neg P(x)) \equiv \exists x(S(x) \wedge P(x)).$$

Syllogisms in Modern Logic

- The remaining items, i.e.,
  - the contrary assertions $S\mathrm{a}P$ and $S\mathrm{e}P$ (which cannot both be true but may both be false) and
  - the implicational relations of universal assertions and its subalterns,

  require again existential assumptions:

$$\exists x S(x) \vdash \neg\big(\forall x(S(x) \supset P(x)) \wedge \forall x(S(x) \supset \neg P(x))\big),$$
$$\exists x S(x) \vdash \forall x(S(x) \supset P(x)) \supset \exists x(S(x) \wedge P(x)),$$
$$\exists x S(x) \vdash \forall x(S(x) \supset \neg P(x)) \supset \exists x(S(x) \wedge \neg P(x)).$$

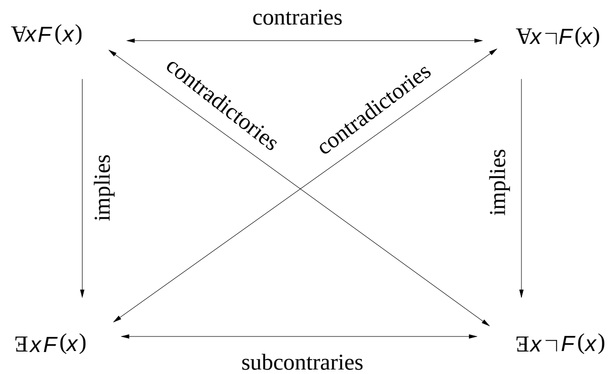
The Square of Opposition: Modern Rendering

We can formulate a square of opposition in modern predicate logic as well:

- Instead of the a-, e-, i-, and o-assertions, formulas $\forall x F(x)$, $\forall x \neg F(x)$, $\exists x F(x)$, and $\exists x \neg F(x)$ are used.
- The square of opposition represents then the following theorems:

$$\vdash \big(\neg \forall x F(x)\big) \equiv \exists x \neg F(x),$$
$$\vdash \big(\neg \forall x \neg F(x)\big) \equiv \exists x F(x),$$
$$\vdash \neg\big(\forall x F(x) \wedge \forall x \neg F(x)\big),$$
$$\vdash \exists x F(x) \vee \exists x \neg F(x),$$
$$\vdash \forall x F(x) \supset \exists x F(x),$$
$$\vdash \forall x \neg F(x) \supset \exists x \neg F(x).$$

The Square of Opposition: Modern Rendering (ctd.)



Some Remarks

➤ $\vdash \exists x F(x) \vee \exists x \neg F(x)$ does not entail that something has F, rather that *there is something*, viz.

- that something either has F or does not.

➤ Consequently, the formula reflects the fact that predicate logic assumes a *non-empty universe of discourse*.

☞ If there were nothing at all, $\exists x F(x) \vee \exists x \neg F(x)$ would be false.

➥ But Aristotle's theory of syllogisms makes the stronger assumption *that any term is non-empty, i.e., that every property has instances*.

Translating Syllogisms into Modern Logic

Using the translations of assertions, we can construct translations of syllogisms.

➤ By the *corresponding predicate calculus sequent* to a syllogism we understand the sequent in which

- the two premisses of the syllogism appear translated as assumptions and
- the conclusion of the syllogism appears translated as the conclusion.

➤ For example, the corresponding predicate calculus sequent to *Celarent*, which is the pattern

if MeP and SaM, then SeP,

is the sequent

$$\forall x(M(x) \supset \neg P(x)), \forall x(S(x) \supset M(x)) \vdash \forall x(S(x) \supset \neg P(x)).$$

Translating Syllogisms into Modern Logic

➤ Some translations again require additional existential assumptions.

- For example, translating the syllogism *Bamalip*, which is the pattern

  if PaM and MaS, then SiP

  (mood aai of Figure IV), we need the additional premiss $\exists x P(x)$ (otherwise the sequent would be unprovable):

$$\exists x P(x), \forall x(P(x) \supset M(x)), \forall x(M(x) \supset S(x)) \vdash \exists x(S(x) \wedge P(x)).$$

➤ In general, additional existential premisses are required whenever a particular conclusion is drawn from two universal premisses.

➤ The general picture is as follows:

- Among the 256 possible syllogistic patterns, 24 are valid.
- Among the latter, nine require additional existential assumptions.

Valid Syllogisms

All valid syllogisms are summarised in the table below.

- The shorthand $\omega_1\omega_2\omega_3$, for $\omega_i \in \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}\}$, refers to the mood of the considered syllogism under Figure $\alpha \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$.
- The table gives also the corresponding Latin mnemonic name.

Figure I		Figure II		Figure III		Figure IV	
aaa	*Barbara*	eae	*Cesare*	aai†	*Darapti*	aai‡	*Bamalip*
aii	*Darii*	aee	*Camestres*	aii	*Datisi*	aee	*Camenes*
eae	*Celarent*	aoo	*Baroco*	iai	*Disamis*	eao†	*Fesapo*
eio	*Ferio*	eio	*Festino*	eao†	*Felapton*	eio	*Fresison*
aai*	*Barbari*	eao*	*Cesaro*	eio	*Ferison*	iai	*Dimatis*
eao*	*Celaront*	aeo*	*Camestrop*	oao	*Bocardo*	aeo*	*Camenop*

* requires additional premiss $\exists x S(x)$

† requires additional premiss $\exists x M(x)$

‡ requires additional premiss $\exists x P(x)$

(Here, S is the minor term, M is the middle term, and P is the major term.)

Discussion

➤ In summary: The principles underlying

- the square of opposition,
- the laws of conversion, and
- the 24 valid syllogisms

are all derivable as theorems or sequents of modern predicate logic.

➤ In some cases, special existential assumptions are required.

➥ The theory of syllogisms can therefore be viewed as

- *that fragment of predicate logic in which four forms of proposition are selected for special study,*
- *assuming that the terms appearing in these forms are not empty.*

Part II

Gottfried Leibniz

Leibniz

Leibniz: Life and Work

Portrait and signature of Gottfried Wilhelm Leibniz (1646-1716)

Leibniz: Life and Work (ctd.)

- Leibniz lead an extraordinarily active life, and was not only an important scholar but also a man of action.
- He was uninterested in an academic career and had several professions, including
  - professional diplomat,
  - mining engineer,
  - inventor,
  - historian, and
  - librarian.
- He was a famous philosopher and mathematician, as well as a trained lawyer.

Leibniz: Life and Work (ctd.)

- He had an enormous output, but mostly in form of private communications.
  - The systematic publication of his complete writings, the *Akademie Ausgabe* (published by the German Academy of science), started in 1923 and is still unfinished.
- He had an exceptional breadth of interests but actually completed little in comparison
  - he described himself as "a man of thousand and one distractions".

Leibniz: Life and Work (ctd.)

- His interests included the following topics:
  - theology,
  - philosophy and science,

- mathematics,
- Chinese history and philosophy,
- diplomatic works, and
- philology and etymology.

➤ To summarise:
- Leibniz was arguably one of the most brilliant intellectuals of the seventeenth century
- as well as the most knowledgeable person of that period.

A Short Biography

➤ Leibniz was born 1646 in Leipzig as the son of a professor of philosophy at the University of Leipzig.

➤ His father died when he was 6 years old.

➤ At the age of 8, he studied two books on Latin he found at his home.

➤ He then studied Greek and Scholastic philosophy, and became interested in Aristotelian logic during the last years in school.

➤ He entered the university in Leipzig at the age of 15 studying philosophy and jurisprudence and finished it within five years
- his bachelor thesis is entitled *Metaphysical Disputation on the Principle of Individuation*.
- Here, he already laid down the principle that all individuals are individuated by their totality, which he retained throughout the rest of his life.

Leibniz: A Short Biography (ctd.)

➤ Leipzig declined to award him a doctorate in law because of his age
  ➥ he moved to the University of Altdorf in Nuremberg, where he was awarded the doctorate the following year.

➤ He declined an offer for a professorship at Altdorf and accepted a position as assistant to the legal advisor of the Elector of Mainz.
  ➥ For the rest of his life, he was occupied with various posts for princes and dukes.

➤ He travelled much during his life and, among other occupations, spent time in Paris from 1672 to 1676 trying to convince Louis XIV to invade Egypt in order to distract France from Germany.

Leibniz: A Short Biography (ctd.)

➤ After the death of the Elector of Mainz, he worked for Duke Friedrich of Hanover as head librarian.

➤ There, he also submitted many plans for technological projects like using wind power to drain the silver mines in the Harz mountains.

➤ After the death of Duke Friedrich, the Duke's younger brother Ernst Augusto took over the succession
- he asked Leibniz to write a history of the whole Guelf family from earliest times to the present.

➤ Leibniz gathered a huge amount of data but eventually was never able to complete the project
- surely also due to the fact that Leibniz accepted many part-time assignments, because he liked to be paid well.

Leibniz: A Short Biography (ctd.)

- His last employer, George Ludwig, the son of Ernst Augusto, had little sympathy for Leibniz's interest.
- In 1712, Leibniz began a two year residence in Vienna, where he was appointed Imperial Court Councillor to the Habsburgs.
- In 1714, George Ludwig moved to England and became King George I but Leibniz had to stay in Hanover to finish his history on the Guelf family.
- Leibniz wrote his most important philosophical correspondences after that.
- Suffering from a severe gout, which almost paralysed him and hindered him from writing, he died in 1716.
  - Being fallen out of favor, only his personal secretary attended the funeral.

The Main Interests of Leibniz

- As a young adult, Leibniz had three fundamental projects:
  1. An alphabet of human thought—building a universal language that would make all reasoning transparently clear; the *characteristica universalis*.
  2. Work on physics—he was unhappy with the then current schools of physics, rejecting Descartes' and Newton's physics.
     - He built his own theory of forces and matter, and invented the calculus of differentiation and integration.
  3. Work on trying to unify the Catholic and Protestant churches, motivated in part by the brutalities of the Thirty-Years' War which left Europe utterly ravaged.
     - He thought that the religious divisions where based on philosophical misunderstandings, so he sought to clarify theology.
- ➥ We focus in the following on his work on logic!

The Mathematics of Thought

- His plans for a mathematics of thought basically amounts to a *mechanisation of reasoning*.
- This should be achieved by an *arithmetisation of language*, allowing to perform deductions just by calculating numbers.
- Hence, the rules of deduction would be reduced to the *manipulation of symbols*—referred to by Leibniz as a *calculus ratiocinator*, i.e., a symbolic logic in modern terminology.
- If we could discover the simple elements (vocabulary) and the rules according to which they are combined (syntax), then we would understand the composition of human thought.
- This way, we might be able to explain the human mind, and even the universe, as a machine.

The Mathematics of Thought (ctd.)

- The design of his characteristica universalis required to have all human knowledge in an encyclopedic form, and Leibniz envisioned such an encyclopedia—the *Plus ultra*.

- He realised that he could not do this alone so he worked on forming academies, being involved in the founding of academic societies in
  - Berlin,
  - Dresden,
  - Saint Petersburg, and
  - Vienna.

The Arithmetisation of Language

- The idea of an alphabet of human thought rests on two ideas:
  1. all concepts are either simple or complex;
  2. the latter are composed out of the former.
- The idea here is thus:
  - Just like any natural number can be represented as a product of prime numbers, any complex concept should be thought of as the product of simple concepts.
  - The simple concepts are represented by primes; the complex ones by the product of the numbers representing the simple ones.
    - Example: if we assign "rational" $= 2$ and "animal" $= 3$, then the concept human, which is rational animal, is represented by $2 \cdot 3 = 6$.
  - To test whether category A is a subcategory of B, Leibniz would divide b by a, where a, b represent A, B, respectively.

The Arithmetisation of Language (ctd.)

- Leibniz never achieved more than a rough design of his characteristica universalis—basically because his goals where too far reaching.
- Human knowledge is never complete, so a complete vocabulary is illusionistic.
- However, he made important discoveries but these were not taken seriously until very much later
  - in large part because they were not put in print or offered to the public in Leibniz's time or long thereafter.

Approaches to Formal Reasoning

- He had in mind a general theory of structures that could provide the syntax for his *characteristica universalis*.
- He developed only two kinds of calculi in detail:
  - one of identity and inclusion, and
  - the other a geometrical calculus of similarity and congruence.
- The first anticipated Boole's *Mathematical Analysis of Logic* of 1847; the second aimed at treating geometrical structure without the use of coordinates.
- We look at his work on a calculus for inclusion.

Approaches to Formal Reasoning (ctd.)

- Leibniz had many different projects for a calculus of inclusion.
- He worked on the subject in 1679, 1689, and 1690.
- He intended at the beginning most likely a system more general than traditional logic.
- He was able to describe some fragments in an abstract fashion, admitting different interpretations of the calculus.
- He realised quite early an isomorphism between some assertions in logic and assertions made in geometry, using geometrical illustrations later put forth by Euler and Venn.
- He also realised that his desired calculus was distinguished by two special principles: $AB = BA$ and $AA = A$.

An Arithmetical Calculus of the Syllogistic

- As an integral part of his overall program, Leibniz looked for numerical methods for representing logical inferences.
- He experimented over the years with several different methods trying to make logical deduction into a numerical calculation.
- The following approach, dating from 1679, is superior to his earlier efforts in that direction, providing an arithmetical interpretation of the theory of syllogism.

An Arithmetical Calculus of the Syllogistic (ctd.)

- The basic idea of the approach is to correlate variables of the syllogistic with ordered pairs of natural numbers prime to each other
  (n and m are prime to each other iff $gcd(n, m) = 1$.)
- Suppose to S corresponds the pair (s_1, s_2) and to P the pair (p_1, p_2).
  Then:
  - $S\mathrm{a}P$ is true iff $p_1|s_1$ and $p_2|s_2$, i.e., s_1 is divisible by p_1 and s_2 is divisible by p_2.
  - $S\mathrm{i}P$ is true iff $gcd(s_1, p_2) = 1$ and $gcd(s_2, p_1) = 1$, i.e., s_1 is prime to p_2 and s_2 is prime to p_1.
  - conditions for $S\mathrm{e}P$ and $S\mathrm{o}P$ follow from these by the observation that $S\mathrm{o}P$ is true iff $S\mathrm{a}P$ is false and $S\mathrm{e}P$ is true iff $S\mathrm{i}P$ is false.

Examples

- Modus Barbara
  if $M\mathrm{a}P$ and $S\mathrm{a}M$, then $S\mathrm{a}P$
  is verified by this interpretation:
  - Let (m_1, m_2), (p_1, p_2), (s_1, s_2) be the associated pairs of M, P, and S.
  - Assume that $M\mathrm{a}P$ and $S\mathrm{a}M$ are true, i.e., $p_i|m_i$ and $m_i|s_i$, for $i = 1, 2$.
  - By the transitivity of divisibility, we obtain $p_i|s_i$, for $i = 1, 2$, i.e., $S\mathrm{a}P$ is true.

Examples (ctd.)

- Modus Datisi
  if $M\mathrm{a}P$ and $M\mathrm{i}S$, then $S\mathrm{i}P$

is verified by this interpretation:

- Consider associated pairs (m_1, m_2), (p_1, p_2), (s_1, s_2) as before.
- Assume that $M\mathrm{a}P$ and $M\mathrm{i}S$ are true, i.e.,
  - $p_i | m_i$, for $i = 1, 2$, and
  - $gcd(m_1, s_2) = gcd(m_2, s_1) = 1$.
- We show that $S\mathrm{i}P$ is true, i.e., $gcd(s_1, p_2) = gcd(s_2, p_1) = 1$.
  - Let $gcd(s_1, p_2) = k$ and assume that $k > 1$.
  - Then, by definition, $k|s_1$ and $k|p_2$. But $p_2|m_2$ by hypothesis. Hence, $k|m_2$, and therefore $gcd(m_2, s_1) > 1$.
  - This contradicts our assumption that $gcd(m_2, s_1) = 1$.
    $\Longrightarrow gcd(s_1, p_2) = 1$.
  - Similar arguments show that $gcd(s_2, p_1) = 1$.

Examples (ctd.)

➤ The syllogism
  if $M\mathrm{a}P$ and $S\mathrm{a}P$, then $S\mathrm{i}M$
is rejected.
(Example of such a pattern: if all birds are animals and all reptiles are animals, then some reptiles are birds).

- Take for instance the following pairs:

$$\begin{aligned}(s_1, s_2) &= (15, 14),\\ (p_1, p_2) &= (3, 7),\\ (m_1, m_2) &= (12, 35).\end{aligned}$$

- Then:
  - $M\mathrm{a}P$ is true as $p_1|m_1$ and $p_2|m_2$;
  - $S\mathrm{a}P$ is true as $p_1|s_1$ and $p_2|s_2$;
  - but $S\mathrm{i}M$ is false as s_1 and m_2 are not prime to each other.

Remarks

➤ Leibniz once said that scientific and philosophic controversies should always be settled by calculus ("calculemus!").

➥ It seems that this statement is connected with this arithmetic interpretation of the syllogistic.

Approaches to Formal Reasoning: 2nd Version

In his second attempt, discussed in his *Specimen Calculi Universalis* and its *Addenda*, Leibniz proposes the following:

➤ Propositions true of themselves:

- *A is A*;
- *AB is A*;
- *A is not not-A*;
- *not-A is not A*;
- *whatever is not A is not-A*;

- *whatever is not not-A is A.*

- *Consequentia* true of itself:
  - *A is B and B is C therefore A is C.*

Approaches to Formal Reasoning: 2nd Version (ctd.)

Principles of the calculus:

- Anything included in certain undetermined letters is to be understood as concluded in any others whatsoever which are subject to the same conditions.
  - E.g., since AB *is* A is true, BC *is* B is also true.
- Transposition of letters in the same term makes no difference,
  - i.e., AB is equivalent to BA.
- Repetition of the same letter in the same term is useless
  - e.g., B *is* AA.

Approaches to Formal Reasoning: 2nd Version (ctd.)

Principles of the calculus (ctd.):

- A proposition can be made from any number of propositions by combining all the subjects into one subject and all the predicates into one predicate,
  - i.e., from A *is* B and C *is* D and E *is* F we can get ACE *is* BDF.
- From any proposition whose predicate is composed from several terms it is possible to make several propositions each of which has the same subject as the original but a predicate which is part of the predicate of the original,
  - i.e., given A *is* BCD we get A *is* B and A *is* C and A *is* D.

Approaches to Formal Reasoning: 2nd Version

For the further development of his calculus, Leibniz expressed Aristotle's four types of propositions in terms of two schemas:

Type	*First Schema*	*Second Schema*
Every A is B	A non-B est non-ens	$AB = A$
Some A is not B	A non-B est ens	$AB \neq A$
No A is B	AB est non-ens	$AB \neq AB$ ens
Some A is B	AB est ens	$AB = AB$ ens

- Here, *ens* and *non-ens* may be rendered as "something" and "nothing".
- However, the use of *ens* as a term involved in all others is problematic!

Approaches to Formal Reasoning: 2nd Version (ctd.)

- When he returned to the subject some years later, in his paper *Non Inelegans Specimen Demonstrandi in Abstractis*, Leibniz abandoned all reference to existential propositions and made a small change of notation:
  - instead of juxtaposition, he used "+" to refer to the combination of concepts.
- This suggests the possibility of introducing the inverse operation of *subtraction*:
  - He stipulates $A - B = C$ iff $A = B + C$ and B and C have nothing in common.

- The second part is important because otherwise subtraction would not give a unique result.
  - E.g., in such a case, we would have not only $A - (L + M) = N$ but also $A - (L + M) = M + N$
  - since $L + M + N = L + M + M + N$, as repetition of a letter in a complex term makes no difference.

Approaches to Formal Reasoning: 2nd Version (ctd.)

➤ Subtraction is not the same as negation:
- E.g., Man not Rational = Impossible, but Man − Rational = Brute.

➤ Using + and − in a calculus where $A + A = A$ holds proved unfruitful, so Leibniz abandoned it for his last and most highly developed calculus, taking instead the symbol $\oplus$.

➥ We describe this calculus in some detail next, reproducing parts of Leibniz's original formulation.

Approaches to Formal Reasoning: 3rd Version

Def. 1. Terms are the same or coincident which can be substituted one for another wherever we please without altering the truth of any statement. $A = B$ signifies that A and B are the same.
[N.B. This is Leibniz's famous substitution principle "salva veritate".]

Def. 2. Terms which are not the same, i.e., terms which cannot always be substituted one for another, are different. $A \neq B$ signifies that A and B are different.
[N.B. Leibniz used "∞" and "non ∞" for "=" and "$\neq$, respectively.]

Prop. 1. If $A = B$, then also $B = A$.

For since $A = B$ (by hyp.), it follows (by Def. 1) that in the statement $A = B$ (true by hyp.) B can be substituted for A and A for B; hence, we have $B = A$.

Approaches to Formal Reasoning: 3rd Version (ctd.)

Prop. 2. If $A \neq B$, then also $B \neq A$.

Otherwise we should have $B = A$, and in consequence (by the preceding prop.) $A = B$, which is contrary to hypothesis.

Prop. 3. If $A = B$ and $B = C$, then $A = C$.

For if in the statement $A = B$ (true by hyp.) C be substituted for B (by Def.1, since $B = C$), the resulting statement will be true.

Prop. 4. If $A = B$ and $B \neq C$, then $A \neq C$.

For if in the proposition $B \neq C$ (true by hyp.) A be substituted for B, we have (by Def.1, since $A = B$) the true proposition $A \neq C$.

Def. 3. A is in L, or L contains A, is the same as to say that L can be made to coincide with a plurality of terms taken together of which A is one. $B \oplus N = L$ signifies that B is in L and that B and N together compose or constitute L. The same thing holds for a large number of terms.

Approaches to Formal Reasoning: 3rd Version (ctd.)

Axiom 1. $B \oplus N = N \oplus B$.

Postulate. Any plurality of terms, as A and B, can be added to compose as single term $A \oplus B$.

Axiom 2. $A \oplus A = A$.

Prop. 5. If A is in B and $A = C$, then C is in B.

For in the proposition A is in B (true by hyp.) the substitution of C for A (by Def. 1, since by hyp. $A = C$) gives C is in B.

Prop. 6. If C is in B and $A = B$, then C is in A.

For in the proposition C is in B the substitution of A for B (since $A = B$) gives C is in A.

Prop. 7. A is in A.

For A is in $A \oplus A$ (by Def. 3) and $A \oplus A = A$ (by Ax. 2). Therefore (by Prop. 6) A is in A.

Approaches to Formal Reasoning: 3rd Version (ctd.)

Leibniz continues with further results which we omit here, except for the following:

Prop. 15. If A is in B and B is in C, then A is in C.

For A is in B (by hyp.), hence $A \oplus L = B$ (by Def. 3). Similarly, since B is in C, $B \oplus M = C$, and putting $A \oplus L$ for B in this statement (since we have shown that they are coincident), we have $A \oplus L \oplus M = C$. Therefore (by Def. 3) A is in C.

Approaches to Formal Reasoning: 3rd Version (ctd.)

- What is important to note is that Leibniz showed interest in the *abstractness of his calculus*, i.e., in the possibility of interpreting it in various ways.
- He thought primarily of the combination of *attributes*, but he illustrated many of this results in terms of *classes*.

Approaches to Formal Reasoning: 3rd Version (ctd.)

- The interpretation of the symbol $\oplus$ changes according to the kind of interpretation used for the letters A, B, etc.
  - If the symbols refer to attributes, then he understands $\oplus$ as a symbol of conjunction
  - but when Leibniz thought of classes, $\oplus$ means disjunction.
  - For if we say that the class of men is contained in the class of animals, we mean that anything which is an animal is *either* a man *or* a horse *or* a dog etc.
- The sense of “is in” alters similarly.
  - To say an attribute is contained in another means that the first is entailed by the second,
  - but a class is contained in another means that the first is a subclass of the second.

Approaches to Formal Reasoning: 3rd Version (ctd.)

- These two interpretations correspond to an *extensional* and an *intensional* interpretation of his letters:
  - an extensional interpretation of $A \oplus B = L$ implies that class A includes class L,
  - while in an intensional interpretation, $A \oplus B = L$ means that the concept of L includes the concept of A.
  - For instance, the class of animals includes the class of humans, while the concept of a human includes the concept of an animal.

- In accord to being mainly interested in attributes, he favours the intensional interpretation.

Summary

- With his last calculus, Leibniz certainly gained much less than he hoped for.
- In particular, it lacks the provision for negation and having conjunction and disjunction used together.
- Also, he could demonstrate only one syllogistic principle (Modus Barbara; Prop. 15).
- For the most part, the deficiency of his calculus comes from Leibniz's preoccupation with the conjunction of attributes and his inability to give a satisfactory account of existence.
- Nevertheless, it was the first attempt of an abstract mathematics, i.e., a mathematics not directly concerned with space or numbers.

Part III

Abstraction and George Boole

Boole and the Algebra of Logic

Prolegomenon: Changes in Algebra

Algebra and Geometry

➤ Before 1825, algebra was a theory of equations concerning *numbers*.

- I.e., the symbols $+$, $-$, $\times$, and $\div$ were used in their standard arithmetical meaning, and variables stand for numbers.
- The goal of the theory was to get knowledge how to solve such equations.
- The operations were, however, done on an intuitive basis without explicit stating of rules governing moves.

➤ In contrast, *geometry* was already early in the history of mathematics casted into an axiomatic system, originating with Euclid's *Elements*

- explicit postulates allow deductions from a set of axioms.

Symbolic Algebra

➤ George Peacock (1791-1858), British mathematician and professor in Cambridge, put forth the idea of viewing algebra *as a science of deductions*, like geometry.

- The seminal work in this direction is his book *Treatise in Algebra* (1830).

➤ Peacock had two main points:

1. all operations are based on a complete body of specific laws
2. the signs of operations have no other senses other than those given by the laws.

➥ Peacock thus defined *symbolic algebra*.

Symbolic Algebra (ctd.)

Further support of symbolic algebra:

➤ development of *group theory*; starting with work by

- Niels Henrik Abel (1802-1829; Norwegian mathematician; died on lung tuberculosis) and
- Évariste Galois (1811-1832; died in a duel in Paris)

➤ development of

- *vector algebra* (Hamilton and Grassmann; during the 1840s) and
- *matrix algebra* (Cayley; 1850s).

Symbolic Algebra (ctd.)

➤ In both vector and matrix algebra, the commutative law, $a \cdot b = b \cdot a$, is not generally true, e.g.,

$$(1 \quad 2)\begin{pmatrix}3\\4\end{pmatrix} = (1 \cdot 3 + 2 \cdot 4) = (11) \neq \begin{pmatrix}3\\4\end{pmatrix}(1 \quad 2) = \begin{pmatrix}3 & 6\\4 & 8\end{pmatrix}.$$

➤ This was a severe shock and lead to a rising consciousness of the need of having a clear axiomatic basis of the different fields of algebra.

Non-Euclidean Geometry

- A further shock was the discovery of non-Euclidean geometry by Lobachevsky in 1825.
  - In Lobachevsky's geometry, Euclid's *Parallel Postulate* no longer holds,
    - i.e., more than one straight line goes through a point not on a given straight line and parallel to it.
  - Also, the *Law of Pythagoras* is not valid in non-Euclidean geometry.
- This lead to discussions about the status of mathematics:
  - In view of Euclid's axiomatic treatment, geometry was considered so far as the most complete branch of mathematics.
  - However, the belief that mathematics provides *completely certain knowledge about nature* had to be abandoned.
  - The new belief was then that it is just a *work of art*
    - being of help in the natural sciences and limited only by its possible uses and by the rules of logic.

Non-Euclidean Geometry (ctd.)

- Non-Euclidean geometry shows that the Parallel Axiom is not a theorem deducible from the other axioms.
- This observation is related to questions of the following kind concerning axiomatic systems in general:
  - Can one axiom be derived from others?
  - Is the system *consistent*, i.e., is it impossible to derive a contradiction?
- Such questions where not seriously discussed until the 1890s, when Peano and his group started investigations along these lines.
- This lead eventually to a new branch of mathematics:

  Metamathematics
- But prior to this, George Boole laid the foundation of a new theory of logic, namely *mathematical logic*.

The Logic of Boole

Boole: A Short Biography

Portrait of George Boole (1815-1864)

Boole: A Short Biography (ctd.)

- Born 1815 in Lincoln, England, as son of a cobbler.
- He had very little formal education, but taught by his ambitious father and was mostly self-educated.
- Boole became assistant teacher at the age of 16 to support his family.
- He founded his own school at the age of 20 and started to study mathematics.
- In 1849, he was appointed professor of mathematics at Queen's College in Cork, Ireland, where he stayed for the rest of his life.

Boole: A Short Biography (ctd.)

- He was involved in activities for social improvement, being the founder and trustee of a home for "female penitents" in Lincoln.
- He married Mary Everest (17 years younger than Boole), daughter of Lieutenant-Colonel Sir George Everest, who's name was given to the world's tallest mountain.
- He died in 1864 of pneumonia, partly as a consequence to a mistreatment of his wife.
- The Booles had five daughters.
  - The youngest, Ethel Lilian, became involved with Russian revolutionaries and married the Polish revolutionary Wilfrid Michael Voynich.
  - She is the author of the novel *The Gadfly*, which was mandatory reading in schools in the Soviet Union and was adapted to film several times.

Towards Boolean Logic

- Boole was influenced by the works of Sir William Hamilton (1788-1856) and Augustus De Morgan (1806-1871).
- Two points of their development are important:
  1. They went beyond the traditional a, e, i, and o statements, having statements like
     - "all S is all P" (= all S is P and all P is S) and
     - "all S is some P" (= all S is P and some P is not S).
  2. A change from the *intentional* to the *extensional* interpretation of letters in an argument took place.
     - As done by Leibniz, variables were traditionally interpreted as *attributes*, i.e., signs of *qualities*,
     - but now they were interpreted as being *signs of the things themselves having the qualities*.

Towards Boolean Logic (ctd.)

- So, assertions like "all S is P" were turned into "all S's are P's".
  - E.g., "all leaf is green" in the intensional sense means that the quality of green is a part of the quality of leaf,
  - but in the extensional interpretation it means that all things being leaves are things having the property green.
- After all, mathematics is a science *about things having certain qualities but not about qualities themselves!*

- E.g., geometry is a theory about things which are points, lines, and planes, and not about the qualities of being a point, a line, or a plane.

➤ Consequently:
- the outcome of the Hamilton-De Morgan theory was *to make it possible to view logic as an algebra of classes!*

Boole's Logical Works

➤ Boole's work on logic is laid down in two books:
- *Mathematical Analysis of Logic* (1847)
- *An Investigation of the Laws of Thought on which are founded The Mathematical Theories of Logic and Probabilities* (1854)

➤ The second book is considered as Boole's masterpiece but it does not extend the logical formalism presented in his first book.
- The chief innovation of the *Laws of Thought* was an application of his ideas to the calculus of probabilities.
- It was paid at the expense of Boole and a friend and did not sell very well.

Boole's Formalism

➤ Boole uses lowercase letters like x, y to stand for classes, called *elective symbols* (because they "elect", i.e., *select*, certain things).

➤ The juxtaposition, xy, of two letters x and y represents their intersection.
- The notation is suggested by the way in which we string adjectives together when defining narrower classes.
- E.g., referrring to the class of large, red, square things.

➤ Furthermore, two special classes are introduced:
- The *universe class* 1 and the *null class* 0.
  $\Longrightarrow$ it holds that $1x = x$ and $0x = 0$ as in ordinary algebra.
- The universe class signifies what De Morgan calls the *universe of discourse*,
  - i.e., some definினite category of things which are under discussion, but not all objects per se.

Boole's Formalism (ctd.)

➤ Although intersection of classes can be thought of as their *logical product*, there is one important peculiarity in the system of classes, viz. *idempotency*: $xx = x$.

$\Longrightarrow$ Boole notes that this is the distinguishing feature of his calculus.

➤ Given the parallel to ordinary product, what about division?
- Boole says that it is not interpretable in logic.
- He argues that from $xz = yz$ we cannot pass to $x = y$.
- Indeed, if the class of bachelor archdeacons coincides with the class of red-haired archdeacons, it does not follow that bachelors are red-haired.
- However, he allows some limited use of division (discussed later).

Boole's Formalism (ctd.)

- The addition $x + y$ of classes signifies the class of things belonging to one of x or y, *providing x and y are mutually exclusive, i.e., if $xy = 0$.*
  - ☞ The proviso of disjoint classes has been removed by successors of Boole, giving rise to the assertion that $x + x = x$, and thus to a principle of duality for union and intersection.
- Subtraction $x - y$ is defined as the inverse of addition, i.e., from $x = y + z$ we can derive $x - y = z$.
  - Here, the assumption that addition is defined only for mutually exclusive classes is crucial.
  - Consequently, $x - y$ is defined only if y is a subclass of x.
    - ☞ This is similar to the ordinary algebra of natural numbers where $x - y$ is undefined if $y > x$.

Boole's Formalism (ctd.)

- Boole writes $\overline{x}$ for $1 - x$, denoting the *complement of x*.
- This allows him to assert

$$x(1 - x) = 0,$$

representing *the principle of non-contradiction*, described by Aristotle and Leibniz as the most fundamental of all principles.

☞ Sometimes, he derives this principle from the equation $x^2 = x$.

Expressing Categorical Assertions

- The current system of notation is sufficient to express the traditional a, e, i, and o propositions:

SaP	Every S is P	$x(1 - y) = 0$
SeP	No S is P	$xy = 0$
SiP	Some S is P	$xy \neq 0$
SoP	Some S is not P	$x(1 - y) \neq 0$

  - Here, x stands for the class of all things being S and y for the class of all things being P.
- Boole prefers to express all categorical propositions by *equations*, thus he writes:

SiP	Some S is P	$xy = v$
SoP	Some S is not P	$x(1 - y) = v$

  - v is a special letter representing a class "indefinite in all respects but one", viz. that it contains a member or members.

Expressing Categorical Assertions (ctd.)

- This usage of the letter v is unsatisfactory and resembles a similar unfortunate technique by Leibniz.
  - If some S is P, the class denoted by xy does indeed contain at least one member.
  - But we cannot assert this by equating xy with a class whose sole defining characteristic is that of containing a member or members—for there is no such class!
- Boole uses v in some respects like a class symbol, but this is a defect rather than a merit, because it suggests mistaken inferences.

- We may be tempted that $ab = v$ and $cd = v$ yield $ab = cd$, but this is wrong and Boole actually does not fall into such traps!

Basic Principles

(1) $xy = yx$	(5) If $x = y$, then $xz = yz$
(2) $x + y = y + x$	(6) If $x = y$, then $x + z = y + z$
(3) $x(y + z) = xy + xz$	(7) If $x = y$, then $x - z = y - z$
(4) $x(y - z) = xy - xz$	(8) $x(1 - x) = 0$

➤ Propositions (1)-(7) are similar to rules of ordinary numerical algebra, while (8) distinguishes Boole's system.

➤ (8) is a valid equation for numbers if x is either 0 or 1!

➤ Hence, the algebra of logic coincides with the algebra of numbers if only 0 and 1 are allowed as values!

Basic Principles (ctd.)

➤ This lead some commentators to describe Boole's system as *two valued*, but this is a mistake:
  - Treating the system as a calculus of classes, it does not hold that every class is either universal or null.

➤ We can turn Boole's system into a two-valued system by adding the principle

$$(9) \text{ Either } x = 1 \text{ or } x = 0.$$

  - This extended system can be interpreted numerically but not in terms of classes!
  - However, Boole does not distinguish sharply between these two interpretations of his system!

Truth Values

➤ The narrower system (satisfying (9)) can also be interpreted as a system of *propositions*:
  - Boole follows this by suggesting the convention of taking the equation $x = 1$ to represent that the proposition x is true while $x = 0$ means that the proposition x is false.

➤ However, Boole never uses the term "truth value" as such, which was only later introduced by Frege.

Development

A fundamental process in the formal elaboration of Boole's system is what he calls *development*.

➤ Suppose $f(x)$ is an abbreviation for an expression involving the letter x and possibly other elective symbols.

➤ $f(x)$ can be thought of a class of objects.
  - These objects either satisfy the property x or not.
  - Hence, the elements comprising $f(x)$ can be divided into those satisfying x and those which don't.

  ➥ $f(x)$ can be represented, for suitable coefficients a, b, as
$$f(x) = ax + b(1 - x).$$

➤ For determining the values of a and b, we need only suppose x to take values 1 and 0

$$\Longrightarrow f(1) = a \text{ and } f(0) = b.$$
$$\Longrightarrow f(x) = f(1)x + f(0)(1-x).$$

The last formula is *the development of $f(x)$ with respect to x.*

Development (ctd.)

- More generally, suppose now that $\phi(x, y)$ is an expression involving x, y and further algebraic signs.
- Developing $\phi(x, y)$ with respect to x we get

$$\phi(x, y) = \phi(1, y)x + \phi(0, y)(1-x).$$

- Developing the result in turn with respect to y, we get

$$\begin{aligned} \phi(x, y) &= \phi(1, 1)xy + \phi(1, 0)x(1-y) \\ &+ \phi(0, 1)(1-x)y + \phi(0, 0)(1-x)(1-y). \end{aligned}$$

Solution

- Another kind of operation is *solution*.
- Suppose we have the equation $f(x) = 0$ and we want to find an equation of the form $x = \phi(y, z, \ldots)$.
- Developing our original equation with respect to x we get

$$\begin{aligned} & f(1)x + f(0)(1-x) = 0. \\ \Longrightarrow\ & (f(1) - f(0))x + f(0) = 0. \\ \Longrightarrow\ & x = \frac{f(0)}{f(0) - f(1)}. \end{aligned}$$

- Here, Boole allows division but he gives special rules which values such coefficient can take, viz. they can take only four forms:

$$\frac{1}{1}, \frac{1}{0}, \frac{0}{1}, \frac{0}{0}.$$

He again uses v to represent indeterminate symbols like $\frac{0}{0}$.

Elimination

- Finally, Boole lays down rules for *elimination*.
- Suppose again an equation $f(x) = 0$ and we want to find what relations, if any, hold independently of x between the other classes occurring in $f(x)$.
- By the process of solution, we get

$$x = \frac{f(0)}{f(0) - f(1)}.$$

- From this, we obtain

$$1 - x = -\frac{f(1)}{f(0) - f(1)}.$$

- Since $x(1-x)=0$, it follows that

$$-\frac{f(0)f(1)}{(f(0)-f(1))^2}=0, \text{ and hence } f(0)f(1)=0.$$

Elimination (ctd.)

- If the left-hand side of the last equation yields only $0=0$, the original equation $f(x)=0$ covers no relations independent of the class denoted by x.
- If, however, it yields something of the form $\phi(y,z,\ldots)=0$, we have established some relation independent of the class denoted by x.

Example

- By various combinations of these (and some other) processes of calculation, Boole can give algebraic representations of all the kinds of reasoning in traditional logic.
- Consider, e.g., an application of Modus Barbara:
  If every animal is mortal and every human is an animal, then every human is mortal.
- Let us use the variables h, a, and m standing for the respective classes of humans, animals, and mortals.
- The premisses can be represented by the equations

$$a(1-m)=0.$$
$$h(1-a)=0,$$

Example (ctd.)

- Reducing the two equations into one, we have $h-ha+a-am=0$.
- By development with respect to a, we get:

$$(h-h1+1-1m)a+(h-h0+0-0m)(1-a)=0.$$

$\Longrightarrow$ $(1-m)a+h(1-a)=0.$

- Identifying $f(1)=(1-m)$ and $f(0)=h$, by elimination (i.e., using $f(0)f(1)=0$), we can conclude

$$h(1-m)=0.$$

$\Longrightarrow$ This equation represents the truth of our conclusion "every human is mortal".

Part IV

Georg Cantor

1 Georg Cantor

Portrait of Georg Ferdinand Ludwig Phillipp Cantor

- Born March 3rd, 1845 in St Petersburg, Russia
  - son of Marie and Georg Waldemar (a stock broker).
  - Father became ill, moved to Wiesbaden, then Frankfurt.
  - Proved to be very talented in mathematics already in school.
  - 1860 graduation with distinction, Realschule Darmstadt.
- 1862 started studying at Federal Polytechnic Zürich.
  - 1863 death of father; moved studies in Mathematics to Berlin.
  - Lectres by Kronecker, Weierstrass, and Kummer.
  - Dissertation 1867 on number theory.
- Worked breifly as a school teacher in Berlin.
- 1869 appointed to the university of Halle (Privatdozent).
  - Research turned from numer theory to analysis in eraly 70's.
  - 1872 promoted to Extraordinary Professor.
  - 1874 marriage with Vally Guttmann, friendship with Dedekind.
- 1879 became full professor.
- Desired a chair in Berlin, impeded by controversies with Kronecker .
- 1881 Dedekind declined a chair in Halle after Cantor's suggestion.
- From 1884 on: attacks of depression.
- 1890 instrumental in founding Deutsche Mathematiker-Vereinigung, first president.
- 1899 second hospitalization, mental illness interfered with his work.
- Recognition only at the end of his life.
  - 1904 Sylvester Medal of the Royal Society,
  - 1911 distinguished scholar 500th anniversary University of St. Andrews (Scotland)
- Retired in 1913, suffering from poverty during WW I.
- Died Jan. 6th, 1918 in sanatorium.

1.1 Significant Works

- 1874: beginning of set theory; different kinds of infinity: "*On a Characteristic Property of All Real Algebraic Numbers*"
- 1878: paper on dimension making the notion of *power* and *equivalence* of sets precise;*Continuum Hypothesis*
- 1879–1884: series of six articles in Mathematische Annalen as introduction to his set theory; most important fifth paper 1883: "*Foundations of a General Theory of Aggregates*"; *transfinite numbers* as a systematic extension of natural numbers, well-ordered sets, ordinal numbers, *arithmetics* for both.
- 1888: philosophical correspondence with Husserl
- 1890/91: "*Über eine elementare Frage der Mannigfaltikeitslehre*",
- 1895-1897: Last significant papers on set theory, surveys of transfinite arithmetics.

2 Diagonal Argument

- Motivation: Work on fourier series and *transcendental* (non algebraic) *numbers*, i.e., numbers which are not solutions of polynomial equations with integer coefficients.
- Question: How rare are transcendental numbers?
- Cantor: uncountably many.
  - Cantor (before): The union of denumerable sets is denumerable.
  - The real numbers are the union of the algebraic numbers and the transcendental numbers.
  - The algebraic numbers are denumerable.
  - Now proof that the real numbers are uncountable.
  - transcendental numbers are uncountable *follows logically*.

- *Uncountability of the real numbers* proved before (letter to Dedekind informing about the discovery in 1873) "*On a Characteristic Property of All Real Algebraic Numbers*", 1874

- first *published proof* of the (second) *diagonal argument*: "*Über eine elementare Frage der Mannigfaltikeitslehre*", Journal der Deutschen Mathematiker-Vereinigung Bd. 1:(1890-1), pp. 75-78 .

- Prior to this, proof of countability of the set of rational numbers.

2.1 The Proof

- From the proposition proved in 1874 it follows that:
  The totality (Gesamtheit) of all real numbers of an arbitrary interval $(\alpha \ldots \beta)$ cannot be arranged in the series $\omega_1, \omega_2, \ldots, \omega_\nu, \ldots$.
- There is a proof that is much *simpler*, and *does not depend on the irrational numbers*.
- Let m and w be two different characters, and consider a set (Inbegriff) M of elements $E = (x_1, x_2, \ldots, x_\nu, \ldots)$
  - which depend on infinitely many coordinates $x_1, x_2, \ldots, x_\nu, \ldots$, and
  - each of the coordinates is either m or w.
  - M is the totality (Gesamtheit) of all elements E.
  - E.g., $E' = (m, m, m, m, \ldots)$, $E'' = (w, w, w, w, \ldots)$, and $E''' = (m, w, m, w, \ldots)$ belong to M.

- *Claim*: Such a manifold (Mannigfaltigkeit) M does not have the power of $1, 2, 3, \ldots, \nu, \ldots$.
- This follows from the following proposition:
  If $E_1, E_2, \ldots, E_\nu, \ldots$ is any simply infinite (einfach unendliche) series of elements of the manifold M, then there always exists an element E_0 of M, which cannot be connected with any element E_ν.
- For proof, let there be

$$
\begin{aligned}
E_1 &= (a_{1,1}, a_{1,2}, \ldots, a_{1,\nu}, \ldots) \\
E_2 &= (a_{2,1}, a_{2,2}, \ldots, a_{2,\nu}, \ldots) \\
E_\mu &= (a_{\mu,1}, a_{\mu,2}, \ldots, a_{\mu,\nu}, \ldots) \\
&\ldots
\end{aligned}
$$

  where the characters $a_{\mu,\nu}$ are either m or w.

- Then there is a series $b_1, b_2, \ldots, b_\nu, \ldots$, defined so that
  - b_ν is also equal to m or w but is *different from* $a_{\nu,\nu}$.
  - Thus, if $a_{\nu,\nu} = m$, then $b_\nu = w$.
- Then consider the element $E_0 = (b_1, b_2, b_3, \ldots)$ of M.

- One sees straight away, that the equation $E_0 = E_\mu$ cannot be satisfied by any positive integer μ,
- otherwise for that μ and for all values of ν:

$$b_\nu = a_{\mu,\nu}$$

and so we would in particular have $b_\mu = a_{\mu,\mu}$ which through the definition of b_ν is impossible.

➥ From this proposition it follows immediately that the totality of all elements of M cannot be put into the sequence (Reihenform): $E_1, E_2, \ldots, E_\nu, \ldots$, otherwise we would have the contradiction, that a thing (Ding) E_0 would be both an element of M, but also not an element of M.

3 Theory of Sets

Cantor's discovery of uncountable sets (1874) led him to the subsequent development of new mathematical discipline called the *Theory of Sets*:

➤ devlopment of *cardinal numbers* and *ordinal numbers*

➤ with their underlying *arithmetic*,

➤ leading to numerous fundamental questions, e.g., the *Continuum Hypothesis*.

➥ Debates about the theory are very closely *related to important questions about logic*.

➤ Cantor's definition of a set (Menge)
"*A set is a collection into one whole of definite, distinct objects of our perception or our thought, which are called the elements of the set*"

➤ In earlier works the term '*manifold*' (Mannigfaltigkeit) is used.

➥ A set is said to contain its elements (or members), and they in turn are said to belong to it.

➤ A subset of a given set S is one whose elements are all elements of S; and this, in distinction from the elements, may be said to be a part of S.

➤ A set may be indicated either by the *listing of its elements*, or by the giving of some *general description* appropriate to all its elements and to nothing else.

Important for their comparison: *one-to-one correspondence*.

➤ Two sets S and T are said to be equivalent if there exists a one-to-one correspondence between them, i.e.,
  - if there is some *relation*, such that
  - each element of S is correlated by the relation with one and only one element of T, and
  - each element of T has one and only one element of S correlated with it by the relation.

➤ The *power* (Mächtigkeit), or cardinal number $\overline{\overline{S}}$, of a set S can then be introduced as *that which it has in common with all equivalent sets but with no others*.

➥ According to Cantor: "*the general concept which [...] results from a set when we abstract from the nature of its various elements and from the order of their being given*".

If S is finite, then $\overline{\overline{S}}$ is some natural number.

➤ But the theory is intended to cover also *infinite sets*.

➥ New definition of infinity: A set is infinite if and only if it can be put in one-to-one correspondence with a *proper* subset.

➤ First formulated explicitly by Dedekind; before C.S. Peirce defined a finite set as one which does not have this property.

➤ As finite sets may be compared w.r.t. their cardinal numbers, so too for infinite sets (*transfinite cardinal numbers*):
- Any set S has a smaller cardinal number than a set T if
- S is equivalent to some subset of T
- but not to T itself.

➥ Thus, $\mathfrak{a}$ the cardinal number of $\mathbb{N}$ is smaller than $\mathfrak{c}$ the cardinal number of $\mathbb{R}$.

Operations by which *sets* may be *constructed out of sets* (finite or infinite):

1. $S + T$, called the *logical sum* (or union or join) of S and T.
2. ST, called the *inner product* (or intersection or meet) of S and T.
3. $S \times T$, called the *outer* (or Cartesian or cross) *product* of S and T.
4. T^S, sometimes called the *insertion set* of T into S is the set of all single valued functions by which elements of T may be assigned to all the various elements of S, when
   (a) S is not null,
   (b) one element of T may be assigned to more than one element of S, and
   (c) two functions are different if there is at least one element of S to which they assign different elements of T.

A particular set constituted by insertion is of special importance.

➤ Let $\mathfrak{U}S$ be the insertion set of $\{0, 1\}$ into S.

➥ $\mathfrak{U}S$ is the *power set* (Potenzmenge) of S, i.e., the set of all subsets of S (including the null set).

☞ A subset of S is determined as soon as it has been decided for each element of S, whether or not it should be included.

➤ Cantor proves that $\mathfrak{U}S$ *has a higher cardinal number* than S, whatever S may be.

➥ So important for his theory that this proposition is called '*Cantor's Theorem*'.

➤ Proof in "*Über eine elementare Frage der Mannigfaltikeitslehre*", 1891.

4 Cantor's Theorem

The proof [diagonal argument] seems to be remarkable not only because of its simplicity, but also because without further ado the *principle can be extended* to the *general proposition* that the powers of well-defined manifolds do not have a maximum or, what is the same, that for any given manifold L one can give another M, which is of higher power.

➤ E.g., let L be a linear continuum, for instance the set of reals z, such that $z \geq 0$ and $z \leq 1$.

➤ Consider the set M of all unique functions $f(x)$, that take on the values 0 or 1, while x runs through all real values ≥ 0 and ≤ 1.

➥ That *M does not have smaller power than L* follows from the fact that there are subsets of M, which correspond one-to-one L.
- E.g., the subset of unique functions of x which assign the value 1 to a single value x_0, and 0 to all other values of x.

➥ Also, *M does not have the same power as L*,
- otherwise M could be brought in a one-to-one correspondence with the variable z, and M could be thought of being of the form of a unique function $\varphi(x, z)$,
- such that every specialization (Spezialisierung) of z gives one element $f(x) = \varphi(x, z)$ of M, and

- every element $f(x)$ of M is obtained from $\varphi(x, z)$ by a single specialization of z.

➤ This leads to contradiction.
  - Consider $g(x)$ as the function of x taking on the values 0 or 1, such that for all values of x its value is different form $\varphi(x, x)$.
  - Then, on the one hand $g(x)$ is an element of M,
  - on the other hand $g(x)$ cannot be obtained by a specialization $z = z_0$ from $\varphi(x, z)$, because $\varphi(z_0, z_0)$ is different from $g(z_0)$.

➤ Since the power of M is neither smaller nor equal to the power of L, *it follows that it must be greater than the power of L*.

5 Continuum Hypothesis

The operations by which sets can be constituted out of sets suggest a *general arithmetic for cardinal numbers*.

➤ If S and T are mutually exclusive, then

➥ $\overline{\overline{S+T}} = \overline{\overline{S}} + \overline{\overline{T}}$, $\overline{\overline{S \times T}} = \overline{\overline{S}} \times \overline{\overline{T}}$, and $\overline{\overline{T^S}} = \overline{\overline{T}}^{\overline{\overline{S}}}$.

➤ Analogously to cardinal numbers, Cantor developed ordinal numbers and an arithmetic for them.

➤ No peculiarities for finite sets.

➤ However, for transfinite cardinal numbers different from the arithmetic of natural numbers.
  - E.g., $\mathfrak{a} = \mathfrak{a} + n = n\,\mathfrak{a} = \mathfrak{a}^n$, for a natural number $n > 0$.

Well-known paradox for illustration: "*Hilbert's Hotel*"

➤ A hypothetical hotel with infinitely many rooms, all of which are occupied.

➤ A new guest arrives.

➥ Move each guest in the next room, i.e. 1 to 2, 2 to 3, etc.

➤ A countably infinite number of new guests arrive.

➥ Move each guest from room n to room $2n$, and each new guest into room $2n - 1$.

➤ All these sets have the same cardinality.

An *immediate corollary of Cantor's Theorem* is the following:

➤ $2^c > c$ for any cardinal c (finite or transfinite).

➥ Since $\overline{\overline{\mathfrak{U}S}} > \overline{\overline{S}}$ and $\overline{\overline{\mathfrak{U}S}} = 2^{\overline{\overline{S}}}$, whatever S may be.

➤ We have an unending succession of ever greater transfinite cardinals $\mathfrak{a}, 2^{\mathfrak{a}} = \mathfrak{c}, 2^{2^{\mathfrak{a}}}$, etc.

To every ordinal number α belongs a transfinite cardinal number $\aleph_\alpha$

➤ α is the ordinal number of the set of the preceeding transfinite cardinal numbers.

➤ $\aleph_0 = \mathfrak{a}$.

➤ For every $\aleph_\alpha$ there exists a greater cardinal number, namely $\aleph_{\alpha+1}$.

Continuum Hypothesis (CH): Cantor conjectured that $\aleph_1 = 2^{\aleph_0}$, respectively that $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$ (generalized CH).

In other words is there a cardinality between $\mathfrak{a}$ and $\mathfrak{c}$?

- Is there a set S with a one-to-one correspondence of a proper subset of S to the natural numbers, such that no one-to-one correspondence of S to the reals exists?

- The *first of Hilbert's 23 open problems*.

- 1940: Gödel proved its consistency with axiomatised set theory (Zermelo-Fraenkel) together with the axiom of choice.

- 1963 Paul Cohen showed that it can neither be proved nor disproved within this system.

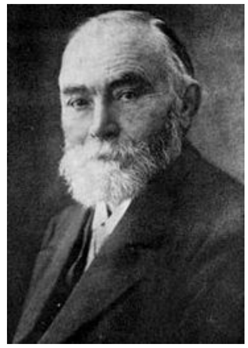
6 Cantor's Influence

- His Theory of Sets is a milestone in mathematics.
  - The first to realize that set theory has *nontrivial* content,
  - before 'the infinite' was considered a topic of philosophical, rather than mathematical, discussion
  - Set theory is a fundamental part in modern mathematics and many concepts and symbols introduced by Cantor are still in use.
- The diagonal argument is fundamental
  - in the solution of the *halting problem*, and
  - the proof of Gödel's (first) *incompleteness theorem*.
- The next great advance in logic (after Boole) was made by *Frege*
  - who wished to exhibit that arithmetic is identical with logic.
  - Cantor was the first to put forward a 'logical' theory of arithmetic.

Part V

Gottlob Frege

1 Gottlob Frege



Portrait of Friedrich Ludwig Gottlob Frege

- Born Nov. 8th, 1848 in Wismar, Mecklenburg-Schwerin
  - son of Auguste and Karl Alexander Frege (headmaster).
  - Attended gymnasium in Wismar, taught by Leo Sachse,
  - graduated at the age of 15.
  - 1866 his father died.
- 1869 matriculated at the University of Jena
  - 4 semesters, approx. 20 courses, most on math and physics;
  - most important teacher Ernst Abbe (also director of Zeiss).
- 1871 continued studies in Göttingen;
  - Dissertation 1873 on geometry.
- 1874 Habilitation in Jena (supported by Abbe)
  - on abelian groups and invariant theory.
  - Appointed Privatdozent in mathematics.

- From 1874 on: University of Jena
  - minimal contacts with students and colleagues,
  - scientific contacts with Rudolf Eucken (philosopher).
- 1887 marriage with Magarete Lieseberg, no children but adopted son Paul Otto Alfred.
- 1896 full professor.
- 1902 Russell's Paradox; Magarete died in 1904.
- ⇒ Depression, no publications. Political situation also distressed him:
  - disliked the move towards democracy in German Empire of 1871, even more since socialists gained in power.
  - Attacked fellow mathematicians beyond professional criticism.
- 1917 retirement, started publishing again afterwards.
- 1923 view: whole of mathematics should be based on geometry.
- died July 26th, 1925 in Bad Kleinen.

1.1 Significant Works

- 1879: *Begriffsschrift*: attempting to show that all of the basic truths of arithmetic could be *derived from purely logical axioms*.

- 1884: *Grundlagen der Arithmetik*: informal defence of his logicist views of mathematics, logic, and language.

- ➤ 1891, 1892: *Funktion und Begriff*, *Über Sinn und Bedeutung*, *Über Begriff und Gegenstand*: new theories about the nature of language, functions and concepts, philosophical logic, and meaning.
- ➤ 1893: *Grundgesetze der Arithmetik I*: Frege's new logical language and its usage to *define the natural numbers* and their properties.
- ➤ 1902: *Grundgesetze der Arithmetik II*
- ➤ 1903–1906: *Über die Grundlagen der Geometrie*: series of articles.

2 Begriffsschrift

Frege's logicist view: *arithmetic is an elaboration of logic*.

- ➤ Put forward in the preface of the Begriffsschrift.
- ➤ Suggestions had been made by other writers before.
- ➥ How to develop logic into a system which captures arithmetic?

Improvements had to be made in the *presentation of logic*:

1. traditional and newer contributions by Leibniz and Boole must be
   - ➤ organized to make clear the structure of the science and
   - ➤ the forms considered in general logic.
2. deduction must be reduced to a small number of standard moves.

- ➥ Construction of a *formalized language of pure thought* ("Eine der artihmetischen nachgebildete Formelsprache des reinen Denkens").
- ➤ I.e., a system of symbolism more regular than ordinary language.

2.1 Judgement

The first special sign is intended to express *judgement* (Urteil):

$$\vdash$$

- ➤ Vertical stroke: '*judgement stroke*' (Urteilsstrich);
- ➤ Horizontal stroke: '*content stroke*' (Inhaltsstrich);
- ➤ Left to a (complex of) sign(s) giving the *permissible content* (beurteilbarer Inhalt) of judgement, abbrev. by Greek capital letters.
- ➤ Missing judgement stroke: content without affirmation or denial.

- ➥ E.g., Let $\vdash \mathit{\Gamma}$ abbreviate 'Unlike magnetic poles attract each other', then $- \mathit{\Gamma}$ just conveys the thought of mutual attraction of unlike magnetic poles, without any judgement of correctness.

Frege rejects distinction between subject and predicate.

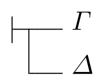
- ➤ There may be linguistic differences.
  - "The Greeks defeated the Persians."
  - "The Persians were defeated by the Greeks."
  - ➥ the *conceptual content* (begrifflicher Inhalt) is the same.
- ➤ Imagine a language where the whole content of a judgement is a subject phrase, e.g.,
  - "The violent death of Archimedes at the capture of Syracuse",
- ➤ and a single predicate phrase for all statements: 'is a fact'.
- ➥ Intuitive readings:

$- \mathit{\Gamma}$	"The circumstance that $\mathit{\Gamma}$."
$\vdash \mathit{\Gamma}$	"The circumstance that $\mathit{\Gamma}$ is a fact."

2.2 Conditionality

Given two permissible contents of judgement Γ and Δ,

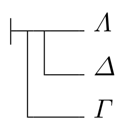
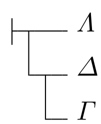
- there are four possibilities:
  1. Γ is affirmed (bejaht) and Δ is affirmed.
  2. Γ is affirmed and Δ is denied (verneint).
  3. Γ is denied and Δ is affirmed.
  4. Γ is denied and Δ is denied.
- The complex sign



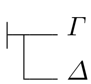
  stands for the *judgement that the third possibility does not occur*, but one of the others is the case.
- Called material implication (or Philonian conditional) today: $\Delta \rightarrow \Gamma$.

The second vertical stroke is called '*condition stroke*';

- left to condition stroke: content stroke of content of the whole sign;
- right to it: content strokes of Γ and Δ, respectively.
- *Composition* by vertical alignment and horizontal arrangement:

 and 

Interest in Philonian junction because of deductive rigour:

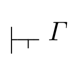
- The judgements  and $\vdash \Delta$ entail $\vdash \Gamma$.
- If n is the number of the first judgement, he expresses inference by

$$(n) : \frac{\vdash \Delta}{\vdash \Gamma}$$

- Called *modus ponens* (or detachment) nowadays.
- Single form of inference (however also uses substitution).

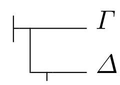
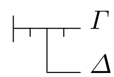
2.3 Negation

Negation is introduced by a small vertical stroke at the lower side of a content stroke:



which means 'It is not the case that Γ'.

- By combination various notions of logic can be expressed, e.g.,

 and 

Frege also introduces a symbol for identity of content:

- Taking 'Γ' and 'Δ' as names of any kind,

$$\vdash (\Gamma \equiv \Delta)$$

  means that the *names have the same conceptual content*,
- thus Γ can always be replaced by Δ and vice versa.
- Can be used for definitions.
- ➡ This judgement is about names!

2.4 Function

Important: Notion of *function*.

- If a simple or complex symbol occurs in one or more places in an expression and it is *replaceable* by another at one or more occurrences,
- then the *invariant part* of the expression is called the function,
- and the replaceable part, the argument of the function.

- ➡ The complex signs $\Phi(\Gamma)$ and $\Psi(\Gamma, \Delta)$ express indeterminate functions of the arguments Γ, respectively Γ and Δ.

- If, when completed by arguments, they express permissible contents of judgement, then we may read $\vdash \Phi(\Gamma)$ as 'Γ has the property Φ', respectively $\vdash \Psi(\Gamma, \Delta)$ as 'Γ stands in the relation Ψ to Δ'.

2.5 The Universal

- Since the sign Φ is replacable in $\Phi(\Gamma)$,
- we may view $\Phi(\Gamma)$ also as a function with argument Φ.
- ➡ Script allows for second (and higher) order functions.

In a judgement, a complex symbol to the right can always be regarded as a function of an occurring sign.

- Using a gothic letter as its argument

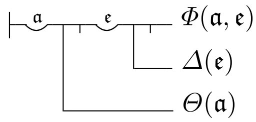
$$\vdash\!\!\smile^{\mathfrak{a}}\!\!-\ \Phi(\mathfrak{a})$$

  signifies that the function is a fact *whatever we take as argument*.
- We can derive any judgments with less general content by
  - omitting concavity and substituting a definite symbol for $\mathfrak{a}$.
- ➡ 'Everthing is Φ'; implicit condition: admissible substitution yields permissible content for judgement.

The universal quantifier, as later called, may appear elsewhere.

- For instance, $\vdash\!\top\!\!\smile^{\mathfrak{a}}\!\!\top\ \Phi(\mathfrak{a})$.

- This allows for many interesting kinds of statements, e.g.,



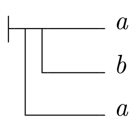
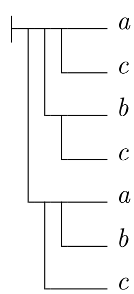
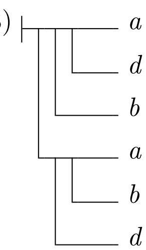
  for “Every cat fears some dog.”
- Cannot be classified in the Aristotelian scheme.

- *Caution* in choice of gothic letters: When replacing, differences have to be preserved.
- Substitution only if quantifier immediatly after judgement stroke.
- Simplified notation for the latter: roman letter, no quantification.
- Instead of $\vdash\!\overset{\mathfrak{a}}{\smile}\!-\, \Phi(\mathfrak{a})$ one may write $\vdash\, \Phi(a)$.

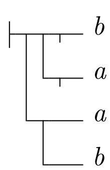
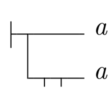
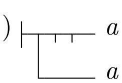
2.6 Logical Principles

Frege identifies *logical principles* which together posess the force of all others.

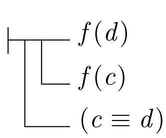
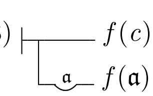
- Various arrangements are possible, selects *nine axioms*.
- The first three concern the condition stroke:

(1)  (2)  (8) 

- Three principles for negation:

(28)  (31)  (41) 

- Two axioms concerning identity and one for general logic:

(52)  (54) $\vdash (c \equiv c)$ (58) 

For *deriving theorems* from these axioms:

- Modus ponens and the principle of substitution.
- Additionally, replacement of roman letter (’free variable’) by quantified gothic letter (which does not occur), and

- from $\begin{array}{l} \vdash\!\!\!\lrcorner\!\!-\, \Phi(a) \\ \quad\ \ \Gamma \end{array}$ we may deduce $\begin{array}{l} \vdash\!\!\!\lrcorner\!\!-\overset{\mathfrak{a}}{\smile}\!-\, \Phi(\mathfrak{a}) \\ \quad\ \ \Gamma \end{array}$ if
  - a does not occur in Γ, and
  - only occupies argument positions in $\Phi(a)$.

- Can be shown to be complete (not at Frege's time).

- Also, the third axiom (8) can be inferred from the first two.

3 Frege on Sense and Reference, Objects, and Functions

Working on his *Die Grundgesetze der Arithmetik*, Frege modified some of his earlier notions and philosophical conceptions.

- Some results published in his works 1891 and 1892.
- We briefly review relevant novelties.

3.1 Sense and Reference

Kept all symbols of *Begriffsschrift* except '$\equiv$' replaced by *equality*.

- Change of his views about meaning.
- Is equality a relation between objects or between names and signs?
- The latter has been assumed in the *Begriffsschrift*.
- Consequence: $a = a$ has the same meaning as $a = b$ if 'a' and 'b' have the same content.

Example: The Morning Star is identical with the Evening Star (Venus).

- "Morning Star" and "Evening Star" have equal conceptual content.
- No difference between "Morning Star = Morning Star" and "Morning Star = Evening Star".
- However, the former is a priori, the latter conveys further insight.

Instead of content: distinguish *sense* (Sinn) and *reference* (Bedeutung).

- A *proper name* (Eigenname)
  - expresses its sense ("containing the mode of being given").
  - and designates its reference (the object which is denoted).
- "Morning Star" and "Evening Star" refer to the same object but do so in different ways, hence differ in sense.
- Expressions may lack reference: "The current king of France".

Frege extends the distinction between sense and reference to *sentences*.

- An intelligible sentence has a sense, the *thought* (Gedanke) it expresses.
- This cannot be its reference: If a part of the sentence is replaced by another with the same reference, the reference of the entire sentence must not change (but the sense may change).
- "The Morning Star is a planet" vs. "The Evening Star is a planet".

- Sentences may lack reference: "The current king of France is bald".
- For determining the truth of a sentence, we require reference.

- The *truth value* of a sentence is its reference.
- It depends on the references of its parts and stays unaltered when replaced by an expression with the same reference.
- All true/false sentences have the same reference: the True/False.

Frege rejects his initial interpretation of judgement: “The circumstance that ... is true.” just expresses (the same) thought.

- The relation of thought to the True is not given by that between subject and predicate.
- Judgement is the *advancement from thought to truth value*.
- Thus, if ‘a’ and and ‘b’ differ in sense, then so do ‘$a = a$’ and ‘$a = b$’, although they might have the same reference (truth value).
- Therefore, also the judgements will differ.

The above considerations on sense and reference of sentences just apply to complete sentences (direct speech).

- Frege additionally considers subordinate clauses (indirect speech).

3.2 Objects and Functions

Distinction between proper names and signs of functions.

- Name of a function is what he formerly called function, i.e.,
  - an *incomplete expression* which can be completed by proper names to make a new proper name.
  - Gaps are indicated by small Greek letters.
  - Marks, not abbreviations or notation for universality.
- ‘$\tan\xi$’ designates the function as distinct from its values.

Objects are all things that are not functions.

- A function is a certain connexion between the objects which are its arguments and those which are its values.
- Special terminology for functions whose values are truth values:
  - *Concept*: function with one argument, e.g., ‘ξ is a man’.
  - *Relation*: function with two arguments, e.g., ‘ξ is father of ζ’.

- To designate what a function expresses:
  - *Range of values* (Werteverlauf) $\acute{\alpha}\Phi(\alpha)$.
  - The first letter makes the whole a name for an object!
  - $\acute{\alpha}\Phi(\alpha) = \acute{\varepsilon}\Psi(\varepsilon)$ has same reference as $\overset{\mathfrak{a}}{\smile}\ \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$.
  - For a concept, its range of values is its *extension* (Umfang).
- Range of values (as an object) not to be confused with the set of values the function can take on.

- As an argument *every* object is admissible.
- Every proper name has to have a reference.
- Particular function $\backslash\xi$ for concepts with a single object as extension:
  - for any argument identical with $\acute{\varepsilon}(\varepsilon = \Delta)$, where Δ is some object, its value is Δ.
  - for any other argument its value is just the argument.
- If $\Phi(\xi)$ is such a concept, then $\backslash\acute{\alpha}\Phi(\alpha) = \Delta$, else $\backslash\acute{\alpha}\Phi(\alpha) = \acute{\alpha}\Phi(\alpha)$.

4 Grundgesetze

In *Grundgesetze*, *function becomes basic notion* in account of Logic.

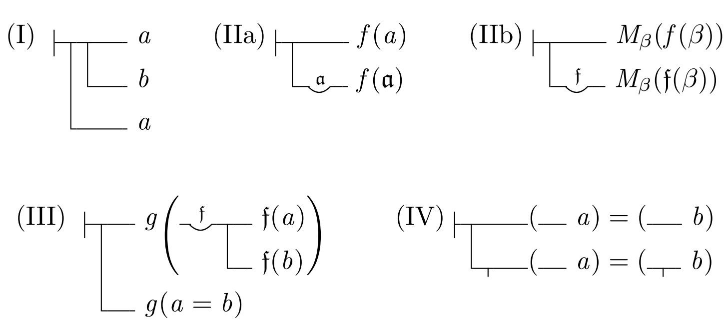
- Some reinterpretation of signs.
- Judgement is the *acceptance of the truth of a thought*.
- The *content stroke* is considered to be a *functional sign*:
  - $— \xi$ is a function with a truth value as its value (concept).
  - $— \Phi(\xi)$ is a concept ($— \Psi(\xi, \zeta)$ is a relation), independent of whether $\Phi(\xi)$ is a concept (whether $\Psi(\xi, \zeta)$ is a relation).
- Negation: The value of $\top \xi$ is the False for every argument such that $— \xi$ is the True, and vice versa.
- Like negation, the condition stroke can only occur in connexion with horizontals and thus is a function from truth values to truth values (later called truth-function).

Notation for talking in general about *functions of the second level*.

- A second level function with one argument is denoted as $M_\beta(\phi(\beta))$, where
  - M signifies the function name (free variable),
  - $\phi(\ )$ indicates the argument place,
  - and β fills the argument place of the function occurring as argument.

Seven assertions constitute the axioms or *basic laws* of the *Grundgesetze*.

- Each basic sign (except horizontal stroke) has a single axiom.
- The order of introduction is of increasing complexity.
- The only axiom with lengthier comment is V:
  - Essential for the system and cannot be derived from others;
  - must be accepted as distinct law of logic.



$$\text{(V)} \vdash (\acute{\varepsilon}f(\varepsilon) = \acute{\alpha}g(\alpha)) = (\overset{\mathfrak{a}}{\smile} f(\mathfrak{a}) = g(\mathfrak{a}))$$

$$\text{(VI)} \vdash a = \backslash\acute{\varepsilon}(a = \varepsilon)$$

- Some of old axioms and theorems become rules of inference.
- Collection of *twelve rules* for the use of the script.

According to Frege all truths of arithmetic can be derived.

➤ Notions, in particular that of (natural) *number*, need to be defined.
➤ Gives seven principles for introducing new expressions by definition.
➥ Numbers by reference to standard concepts from within logic itself:
- 0 is the number belonging to the concept ‘not identical with itself’,
- 1 is the number belonging to the concept ‘identical with 0’,
- 2 is the number belonging to the concept ‘identical with 0 or with 1’, etc.

➤ Resembles Cantor’s theory of cardinal numbers.
➤ Proof: for every number in the series, one follows directly after it.
➥ Possibility to reduce mathematical induction to laws of logic.

5 Proof of Inconsistency

In a letter by *Bertrand Russell* in 1902 communication of inconsistency.

➤ For simplicity, we sketch it in simplified (modern) notation.

- Restriction to *concepts*, notation Fx rather than $\_\, f(x)$.
- Modern form of first-order and second-order quatification, and
- ‘$\equiv$’ for material equivalence of concepts instead of equality.

➥ E.g., $\forall x(Fx \equiv Gx)$ for $\overset{\mathfrak{a}}{\smile}\, (\_\, f(\mathfrak{a})) = (\_\, g(\mathfrak{a}))$, and $\forall F(Fa \equiv Fb)$ for $\overset{\mathfrak{f}}{\smile}\, (\_\, \mathfrak{f}(a)) = (\_\, \mathfrak{f}(b))$.

- Notation ϵF for the extension of a concept $\acute{\varepsilon}(\_\, f(\varepsilon))$.

From the following simple theorem of Frege’s logic

$$\forall x(Fx \equiv Fx),$$

we derive by existential generalisation

$$\exists G \forall x(Gx \equiv Fx).$$

By his rule of substitution, since F is a free variable, we get

$$\exists G \forall x(Gx \equiv \phi(x)),$$

where $\phi(x)$ denotes any open formula with free variable x.

➤ This principle has later been called *Comprehension Principle*.
➤ Implies existance of a concept for any open formula with free x.
➥ E.g., $\exists G \forall x(Gx \equiv odd(x) \wedge (x > 5))$.

Frege also gave a definition for *membership in an extension*, in modern notation

$$x \in y =_{def} \exists G(y = \epsilon G \;\wedge\; Gx).$$

➤ Hence, x is a member of y iff x falls under a concept, of which y is the extension.

The special case of Grundgesetz V for concepts in our notation is

$$\epsilon F = \epsilon G \equiv \forall x(Fx \equiv Gx).$$

➤ The extension of the concept F is identical to the extension of the concept G iff all and only the objects that fall under F fall under G.

It implies the following corollary, called *Law of Extensions*:

$$\forall F \forall x (x \in \epsilon F \equiv Fx)$$

➤ An object is a member of the extension of a concept iff it falls under that concept.

Frege gives *two derivations* of contradiction:

➤ One using the Comprehension Principle and Grundgesetz V;
➤ the other, which resembles Russell's Paradox, building on the Law of Extensions.

Russell's Paradox: Instantiating the universal quantifier in

	$\forall F \forall x (x \in \epsilon F \equiv Fx)$
with a particular concept G yields	$\forall x (x \in \epsilon G \equiv Gx)$.
Generalizing the extension ϵG gives	$\exists y \forall x (x \in y \equiv Gx)$.
Applying the rule of substitution:	$\exists y \forall x (x \in y \equiv \phi(x))$.

➥ For any formula $\phi(x)$ (defining a condition on objects), there is an extension which has as members all and only the objects that meet the condition.

Let $\phi(x)$ be given by $\neg(x \in x)$, then

$$\exists y \forall x (x \in y \equiv \neg(x \in x))$$

asserts the existence of an object that can be substituted for y.

Suppose that b is such an object, then we conclude

$$\forall x (x \in b \equiv \neg(x \in x)).$$

However, since the claim is universal, we can also substitute b for x:

$$b \in b \equiv \neg(b \in b).$$

6 Frege's Achievements

➤ The *Begriffsschrift* is the *first comprehensive system of formal logic*.
- Showing clearly the relation between primary (propositional) and general (first-order) logic.
- Enlarging the traditional conception of logic with a theory of relations.

➤ Most important: Introduction of *quantifiers to bind variables*.

➤ The *Grundgesetze* contain all essential steps to prove the fundamental propositions of arithmetic (in second-order logic).

- ➥ Replace Grundgesetz V by Hume's Principle (consistent):
  For any two concepts Fx and Gx, the number of objects falling under Fx is equal to that falling under Gx iff there is a one-to-one correspondence between those objects.
- "*Frege's Theorem*": Proof of Dedekind/Peano axioms for number theory from Hume's principle.

Part VI

Logicism and Bertrand Russell

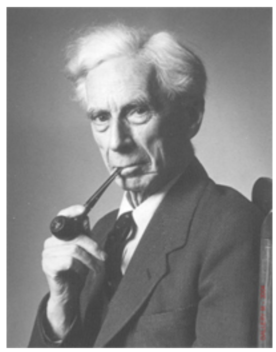
1 Introduction

2^{nd} *Int. Congress of Mathematicians* , Paris, 1900

David Hilbert invited to give the opening talk:

- famous speech on *Mathematical Problems*;
- because of tardiness delivered on third day;
- "We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no *ignorabimus*."
- listed *23 open problems* (10 in the talk) for the new century to come;
- 2^{nd} problem: The *compatibility of the arithmetical axioms*.

2 Bertrand Russell



Portrait of Bertrand Arthur William Russell

- 18 May 1872 - 2 Feb. 1970, born at Trellech, Monmouthshire (Wales),
  - son of Kate (sister of the Countess of Carlisle) and Viscount Amberly Russell.
  - His parents died early: in 1875 and 1876, respectively.
  - Bertrand and and elder brother Frank raised by their grandparents:
  - Frances (Countess Russell) and John Russell (the 1^{st} Earl Russell; Prime Minister in the 1840s and 1860s, died 1878).
- Educated by a series of tutors.
- 1890 won a scholarship to study mathematics at Trinity College, Cambridge University.
  - ➥ Came under the influence of Alfred North Whitehead.

- 1893 B.A. in mathematics, followed by a fellowship in 1895.
- 1896 First publication on German Social Democracy.
  - ➥ Taught German Social Democracy at the London School of Economics.
- 1908 Fellow of the Royal Society.
- 1911 Acquaintance with Ludwig Wittgenstein; encouraged his academic development though various phobias and bouts of despair.
- 1916 Dismissed from Trinity College (opposed British participation in WW I; later imprisoned for 6 month).
- 1920 Travel to Russia and lectures in Peking on philosophy for one year.
- 1931 Frank died, Bertrand became the 3^{rd} Earl Russell.

- 1939 He moved to Santa Barbara to lecture at the University of California, Los Angeles.

- 1940 Appointed professor at the City College of New York.
  - After public outcries ("morally unfit") appointment annulled.
- 1944 Russell rejoined the faculty of Trinity College.
- 1949 Awarded the Order of Merit.
- 1950 Won the Nobel Price in Literature
- 1950s and 1960s political engagement (nuclear disarmament, opposing Vietnam War).
- Published three-volume autobiography in the late 1960s.

2.1 Significant Works

Bertrand Russell made significant contributions to a broad range of subjects, his major publications include:

- Logic and mathematics
  - *Principles of Mathematics*, 1903;
  - *Mathematical Logic as Based on the Theory of Types*, 1908;
  - *Principia Mathematica*, 1910, 1912, 1913;
  - *Introduction to Mathematical Philosophy*, 1919.
- Analytic philosophy
  - Eliminate meaningless and incoherent assertions in philosophy for the sake of *clarity and precision in argument* by the use of exact language, *breaking down propositions into their simplest components*.
  - *On Denoting*, 1905;

- Analytic philosophy (ctd.)
  - *Knowledge by Acquaintance and Knowledge by Description*, 1910;
  - *The Philosophy of Logical Atomism*, 1918, 1919;
  - *Logical Atomism*, 1924;
  - *The Analysis of Mind*, 1921;
  - *The Analysis of Matter*, 1927.
- Social and political philosophy
  - *The Problems of Philosophy*, 1912;
  - *A History of Western Philosophy*, 1945.
  - and many more.

3 Logicism

Logicism is the view that *mathematics can be reduced to logic*.

- According to Russell: "*[...] show that all pure mathematics follows from purely logical premises and uses only concepts definable in logical terms.*"

This is often considered to entail the following claims:

1. *Language Logicism*: The language of mathematics consists of purely logical expressions.
2. *Truth Logicism*: Mathematical truths are true as a matter of pure logic.

These can be further precised:

- Assume a mathematical language $\mathcal{L}$ with an intended model $\mathcal{M}$.
- A *paraphrase-function* $\star$, is a function such that for any sentence ϕ in the domain of $\star$, $\phi^\star$ is a paraphrase of ϕ.

- Language Logicism: There is a paraphrase-function $\star$ such that, for any sentence ϕ of $\mathcal{L}$, $\phi^\star$ contains no nonlogical vocabulary.
- Truth Logicism: There is a paraphrase-function $\star$ such that, for any sentence ϕ of $\mathcal{L}$, which is true (false) according to $\mathcal{M}$,
  - $\phi^\star$ (its negation) is a logical truth (semantic version),
  - $\phi^\star$ (its negation) is derivable from the empty set on the basis of purely logical axioms and rules of inference (syntactic version).
- Note that this leaves the question of *what should be counted as logic* still open.

The main idea of Logicism was put forward by Leibniz already in the late 17^{th} century.
However, it was not until Frege had developed his general logic that it became technically plausible.

4 Russell's Theory of Logical Types

Russell published his paradox in 1903 in an appendix to his *Principles of Mathematics*.

- but remained a logicist (i.e., hoped for a satisfactory demonstration of the identity of arithmetic and logic) and tried to overcome the problem.
- Problem: *vicious circles*, e.g., a definition of a concept referring to the totality of all possible objects under the concept.
- Some propositional functions are *non-predicative*.

Attempt to overcome contradictions: *Russell's Theory of Logical Types* published in:

- an article of 1908 *Mathematical Logic as based on the Theory of Types*, and in
- the introduction of *Principia Mathematica* (*PM*).

As for notation, we use

- $(x)\phi(x)$ as an expression for *whatever you please,* $\phi(it)$;
- $(\exists x)\phi(x)$ as an expression for *there is something* $\phi(which)$;
- $\hat{x}\phi(x)$ as a designation for *the class of all things* $\phi(which)$;
- $\phi(\hat{x})$ as a designation for *the function satisfied by anything* $\phi(which)$.

When we speak of "ϕx", where x is not specified,

- "ϕx" *ambiguously denotes* ϕa, ϕb, ϕc, etc. (i.e., the various values of "ϕx").
- "ϕx" only has a well-defined meaning if the objects ϕa, ϕb, ϕc, etc. are well-defined.
- No function can have among its values anything which *presupposes* the function.

This is a particular case of the *vicious-circle principle*:

- No totality can contain any members involving the totality.
- "*Whatever involves all of a collection must not be one of the collection*".

Hence, the symbol "$\phi(\phi\hat{x})$" *must not* express a proposition.

- "$\phi(\phi\hat{x})$" does not express anything (is not significant);
- there are arguments for which a function has no value, as well as for which it has.

4.1 Basic Types

A *type* may be defined as a *range of significance*,

- ➤ i.e., as the collection of arguments for which a given propositional functions has values, or
- ➤ according to the entities which it admits as proper arguments.

Example 1. Consider men as individuals or entities of type 0. $\Rightarrow$ wisdom is of type 1 (asserted significantly of Socrates) $\Rightarrow$ cardinal virtue is of type 2 (asserted significantly of wisdom).

For *attributive functions*, we may think of types arranged *hierarchically*, but

- ➤ *relational functions* must be distinguished in type (although they may be said to be of the same *level*);
- ➥ cannot be represented adequately by ordinal numbers;
- ➥ functions taking only individuals as proper arguments are of types (0), $(0,0)$, $(0,0,0)$, etc.;
- ➥ an attribute of an attribute of an individual is of type $((0))$, etc.

Example 2. The transitivity of the relation being an ancestor is of type $((0,0))$.

The *general rule* is:

- ➤ *Propositional functions take arguments from appropriate lower type(s) only!*
- ➥ This excludes the property of being a property which does not exemplify itself.

Additionally, Russell uses his *'no classes' theory*:

- ➤ Designations of classes are incomplete symbols, i.e., symbols which cannot be defined alone.
- ➤ "*A proposition about a class is always to be reduced to a statement about a function which defines the class.*"
- ➥ *Class signs can be eliminated without loss*, e.g., $a \in \hat{x}(\phi x)$ by ϕa.

4.2 Ramified Theory of types

In order to avoid other (semantic resp. linguistic) paradoxes, e.g., the Liar paradox, a further development is necessary.

- ➥ This is the *ramified theory of types*, i.e.,
- ➤ the distinction of *orders* within types.

For simplicity consider only functions that take individuals as arguments.

- ➤ First order: It can be defined without the application of quantifiers to any variables other than individual variables.
- ➤ Second order: It can be defined quantifying only over individual variables and/or first order functional signs.
- ➤ ...

A function is called *predicative*, $\phi!\hat{x}$, iff it is of the lowest order compatible with its arguments.

Example 3. The phrase x *has all the first order qualities that make a great general* expresses a function

- ➤ of type (0), since it takes individuals as its arguments, but
- ➤ of the second order: $(\phi)[(y)(y \text{ is a great general} \supset \phi!y) \supset \phi!x]$.

$\Rightarrow$ Drastic *limitations* and enormous *complication* of logic and numbers:

- e.g., theorem that any non-empty set of real numbers with an upper bound has a least upper bound cannot be formulated.
  - Introduction of the *Axiom of Reducibility*:

$$(\phi)(\exists\psi)(x)[\phi x \equiv \psi!x],$$

    i.e., the assumption that for every propositional function there is a predicative function that is satisfied by exactly the same arguments.

- The definition of the infinity of natural numbers (in the style of Frege) is impossible.
  - Introduction of a special *Axiom of Infinity*.
- Russell admits that these *axioms can scarcely be said to be self-evident truths of logic*.
- Nevertheless, Russell's type theory is a powerful tool that led to a monumental achievement in re-establishing rigor in mathematics, in asking and answering the questions:
  - "*What basic principles of a general kind are needed to develop the whole of classical mathematics?*"
  - "*Can these principles be incorporated into a general theory of concepts and objects which is free of the logical paradoxes?*"

5 Principia Mathematica

Bertrand Russell and Alfred Whitehead (1861-1947) started their joint work on foundations in 1900:

- Aim: the seamless development of mathematics from the view of *clearly stated axioms* and *rules of inference* in pure logic.
- "*All pure mathematics follows from purely logical premises and uses only concepts definable in logical terms*" (Russell).
- Opted for the more modern *notation of Peano* instead of Frege's Begriffsschrift.
- Approach was essentially that of Frege, but avoided paradoxes by Russell's Theory of Types.
- *Three volumes*, almost 2000 pages, appeared in 1910, 1912, and 1913.
- A fourth volume on geometry planned, but never appeared.

- *Vol. I:*
  - *Introduction:* explanations of ideas, notation, theory of types;
  - *Part I:* "Mathematical Logic" incl. axioms and rules of inference; elementary results on classes and binary relations;
  - *Part II:* "Prolegomena to Cardinal Arithmetic", def. of numbers;
- *Vol. II:*
  - *Part III:* "Cardinal Arithmetic", definitions and arithmetic properties of cardinal numbers;
  - *Part IV:* "Relation Arithmetic", arithmetic of binary relations;
  - *First half of Part V:* "Series" incl. linear orderings, limit points, continuous functions;
- *Vol. III:*
  - *Second half of Part V:* incl. well-orderings, finite and infinite series and ordinals;
  - *Part VI:* "Quantity" incl. measurement (modulo a quantity).

Part VII

Hilbert's Formalism vs. Brouwer's Intuitionism

1 David Hilbert

Portrait of David Hilbert; 1912

- 23 Jan. 1862 - 14 Feb. 1943, born in Wehlau near Königsberg,
  - son of Maria and Otto Hilbert (a county judge).
  - He attended the Gymnasium at Königsberg and
  - entered the University of Königsberg to study mathematics afterwards.
- 1885 PhD for his thesis *Über invariante Eigenschaften spezieller Formen insbesondere der Kugelfunktionen.*
  - Friendship with Minkowski, also a doctoral student; strongly influenced each other.
- 1886-1895 Faculty member at Königsberg: Privatdozent, 1892 extraordinary professor, 1893 full professor; friendship with Hurwitz.
- Klein wanted to have Hilbert as chair of mathematics in Göttingen:
  - not appointed in 1892 but in 1895.
- After 1900 other institutions tempted him to leave Göttingen.
  - ➥ Used offers to bargain (set up a new chair for Minkowski).
- Hilbert first worked on *invariant theory*.
  - Gordon had proved the finite basis theorem for binary forms.
  - Attempts to generalize it to more than two variables had failed.
  - Visits Kronecker in Berlin to discuss ideas; long talk on what constitutes mathematical existence.
  - Proved the *finite basis theorem* for any number of variables in 1888,
  - in an entirely abstract (non-constructive) way.
- Next he turned to algebraic *number theory*:
  - *Zahlbericht*, 1897; a brilliant synthesis introducing new concepts that shaped the course of further research.
- A systematic study of the axioms of *Euclidean geometry* lead to
  - *Grundlagen der Geometrie*, 1899; proposed 21 axioms and analysed their significance, thus putting geometry in a *formal axiomatic setting*.
- In 1904 Hilbert heard from Zermelo about the *paradoxes in set theory*
  - ➥ "*A downright catastrophic effect*"

  In the same year 3rd Int. Congress of Mathematicians:
  - Kronecker posed that construction by a finite number of integers is the only possible criterion of mathematical existence.
  - Hilbert now insisted that the integer itself "*can and must*" have a foundation.
  - Proposed that *proof* itself should be made *object of mathematical investigation*.
  - Presented a sketch of proof for his second problem.
  - ➥ Formalism, Hilbert program.
- Lacked the logical basis to peruse this further and subsequently turned to integral equations, functional analysis, and mathematical physics. ➥The concept of *Hilbert space*.

2 Formalism and Hilbert's Program

The central doctrine of *symbolic formalism* is that

- the instrumentalist conception of language allows for purely *symbolic* uses of signs in our reasoning,
- uses that do not depend in any essential way on the semantic content of the signs involved or on their even having such content.

This doctrine is itself composed of two key elements:

1. *Creative element*: the idea that the mathematician is free to "create" methods out of considerations of convenience or efficiency as distinct from evaluation of content.
2. *Symbolic element*: that nonsemantical uses of signs may, at least on occasion, constitute such conveniences.

2.1 Hilbert's Program

Principa Mathematica provided the required *logical basis* for an attack on these issues.

In the early 1920s Hilbert put forward a new proposal for the *foundation of classical mathematics*:

- Formalization of all mathematics in *axiomatic form*, together with
- a proof that this axiomatization is consistent.
- The *consistency proof* is to be carried out using only *finitary methods*.

Hilbert's finitary standpoint restricts mathematical thought to:

- Objects which are "intuitively present as immediate experience prior to all thought", and
- operations and methods of reasoning which do not require abstract concepts, in particular completed infinite totalities.

Example 4 (Contentual Number Theory). In contentual number theory,

- Objects are numerals such as 1, 11, 111, etc.
- Basic propositions are equality and inequality.
- Most basic operation is concatenation.

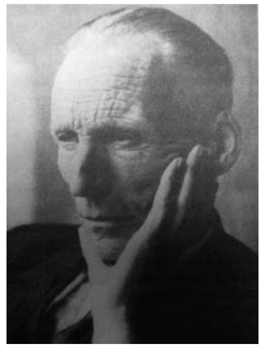
Gnerealized by operations defined by recursion and the usual logical operations including bounded quantification.

Metamathematics operates with *sequences of symbols* (formulas, proofs) that can be *syntactically manipulated* just as the sequences of strokes in contentual number theory.

- ➥ Eventual aim: Justify the abstract concepts and inference principles used in mathematics itself.

- 1914 Hilbert's student Behmann and others begin to study PM.
- 1917 Hilbert himself returned on foundational issus.
- Poses proof of consistency of arithmetic and set theory again as main open problems.
  - ➥ In both cases, nothing more fundamental available to which the consistency could be reduced than to logic itself.
- 1926 *On the Infinite*, most detailed account of finitary standpoint.
- Significant contributions to formal logic together with Behmann and Bernays; *Principles of Theoretical Logic*, 1928 (with Ackermann).
- Strategy for a consistency proof elaborated with Bernays.
- Further contributions by Ackermann, von Neumann, and Bernays.
- 1928 claimed optimistically the consistency proof of number theory had been established.

3 L.E.J. Brouwer



Portrait of Luitzen Egbertus Jan Brouwer

- 27 Feb. 1881 - 2 Dec. 1966, born in Overschie (Rotterdam),
  - son of Henderika and Egbertus Luitzens Brouwer (a teacher).
  - He attended high school in Hoorn and Amsterdam, before he
  - entered the University of Amsterdam to study mathematics and physics.
- 1904 Doctorandus (MA) in mathematics; first publication on rotations in four dimensional space.
- 1907 Doctor title for his thesis *Over de Grondslagen der Wiskunde*
  - under the supervision of Korteweg; marked the beginning of his intuitionistic reconstruction of mathematics.
- 1909 Becomes privaat-docent at the University of Amsterdam.
  - He meets Hilbert, who he admires, in Scheveningen (a Dutch seaside resort):
  - "*[...]a beautiful new ray of light through my life.*"

- 1912 Elected member of the Royal Academy of Sciences and appointed professor extraordinarius.
- 1913 Full professor ordinarius, succeeding Korteweg.
- 1918 Brouwer begins *systematic intuitionistic reconstruction of mathematics*:
  - *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Erster Teil, Allgemeine Mengenlehre.*
- 1919 Professorships offered in Göttingen and Berlin; both declined.
- 1920 Start of the "*Grundlagenstreit*" with his lecture at the "Naturforscherversammlung" in Bad Nauheim.
  - Published one year later as *Besitzt jede reelle Zahl eine Dezimalbruch-Entwicklung?*.
  - Answered by Hilbert in 1922 by his *Neubegründung der Mathematik*

- 1926-1928 Lectures in Göttingen (on good terms with Hilbert for a brief period), Berlin (later assistant Freudenthal is in the audience), and Vienna (Gödel and Wittgenstein attend).
- 1928 Bologna conference conflict.
- 1928-1929 "*Mathematische Annalenstreit*": Hilbert expels Brouwer, who is supported by Einstein, form the board.
- 1929-1934 Foundation of a new mathematical journal: *Compositio Mathematica*; shift in interest to philosophy.
- 1935-1941 Member of the municipal council of Blaricum (Neutral Party).
- During WW II Brouwer assisted resistance but was met with skepticism (declaration of loyalty) and suspended for a few month after liberation.
- 1951 Retirement; Arend Heyting successor as director of the Mathematical Institute; disagreement over future role of Brouwer.

4 Intuitionism

If we refrain from talking of infinity in the way of Cantor, then we do not have to face its paradoxes.

- Gauss: "*There is no place in mathematics for talk of a completed infinity.*"
- More recently, Kronecker had objected the Cantorian programme.
- However, Brouwer was the first to *draw the consequences* of
  - *not allowing any infinity*
  - *except the potential infinity of a sequence that can be continued* *ad libitum*.

Maintains Kant's view that artihmetic must be derived from the *intuition of time*.
From this intuition we get

- the notion of sequence of ordinal numbers,
- the notion of the linear continuum, i.e., "*of the betweenness which is not exhaustible by the inter-position of new units and therefore can never be thought of as a mere collection of units*".

- There are *no sets except the denumerable*, and so
  - there are no transfinite cardinal numbers other than $\aleph_0$;
  - there is no meaning to be given to any phrase such as 'the set of all real numbers between 0 and 1'.

- Even the sequence of natural numbers is to be conceived as an open manifold:
  - always in growth and never brought to finish;
  - we can only comprehend it by understanding *the law of its construction*.

4.1 Constructive Proof

- All *satisfactory proofs* in mathematics *are constructive*!
- Intuition is the mind's clear apprehension of what it has itself constructed.
- Denial of the dependency on any special kind of language and the technique of formailzation.
  - Mathematics does not presuppose a system of logic.
  - It is rather a source of logical principles,
  - that may be generalized only after their validity has been established by an appropriate intuition.

In practice, the most important requirement for a constructive proof is

- that *no existential statement* is admitted
- *unless* it can be *demonstrated by an instance*.

Brouwer: Anyone who fails to obey to this principle has to blame himself if he falls into paradox.
Consider an exitential statement

$$(\exists x)[\phi x. \sim\psi x], \tag{2}$$

and the universal statement

$$(x) \sim[\phi x. \sim\psi x]. \tag{3}$$

If the set denoted by $\hat{x}\phi x$ is finite:

- We can in principle examine all the members
- and decide in favour of one or the other.

If the set denoted by $\hat{x}\phi x$ is denumerable infinite, however:

- Systematic enumeration is still satisfactory for proving (2),

- ➥ since it leads to an instance in a finite number of steps,
- ➤ but is no longer effective for (3),
- ➥ since it is sensless to talk of going through all the members.

How to prove an universal statement?

- ➤ Show that the existential statement involves a contradiction:

$$\sim(\exists x)[\phi x. \sim\psi x] \supset (x) \sim[\phi x. \sim\psi x].$$

- ➤ However the following is not admitted:

$$\sim(x) \sim[\phi x. \sim\psi x] \supset (\exists x)[\phi x. \sim\psi x],$$

- ➥ does not instantiate a thing a of which we can assert $\phi a. \sim\psi a$.

This amounts to the *abandonment of the principle of excluded middle* and the associated principle for the elimination of double negation!

4.2 Intuitionistic Mathematics

For Brouwer the principle of excluded middle was equivalent with the *a priori* assumption that every mathematical problem has a solution, which he rejected.

Serious consequneces in mathematics:

- ➥ Non-constructive methods very common, in particular in the theory of real numbers.

Example 5 (Dedekind's theory).

- ➤ A real number is considered to be an inifinite set of rational numbers.
- ➤ To prove that either $x > y$ or $x = y$ or $x < y$ we need to assert that
- ➤ there either is or is not a rational in x which is not in y.
- ➥ This is an application of excluded middle to infinite sets.

- ➤ How can (a substantial part of) classical analysis be developed without them?

Brouwer builds the continuum from infinite convergent series:

- ➤ A legitimate infinite object must be given by a pricnciple or law.
- ➤ However, generating law need not be fully deterministic.
- ➥ Continuum built from "*choice sequences*".

A *choice sequence*, σ is given by a finite initial segment $\langle\sigma(1), \ldots, \sigma(n)\rangle$ together with a principle which given $\langle\sigma(1), \ldots, \sigma(k)\rangle$ determines the range of possible choices for $\sigma(k+1)$.

Example 6 (Generation of a real number).

- ➤ $\alpha(1) = 1/2$.
- ➤ $\alpha(k+1)$ must be a rational number q such that $|\alpha(k) - q| \leq (1/2)^{k+1}$.
- ➥ This sequence converges to $r_\alpha \in [0, 1]$.

Note that we cannot tell, however, whether $r_\alpha > 1/2$ or $r_\alpha = 1/2$ or $r_\alpha < 1/2$.

Brouwer developed an *intuitionistic set theory* to handle sets that include choice sequences by means of *spreads* and *fans*.

- ➤ A *spread* is a set of rules for admissible finite sequences of natural numbers such that each such admissible finite sequence has at least one admissible successor. An infinite sequence of natural numbers belongs to a spread if each finite subsequence does.
- ➤ A spread which admits only finitely many successors to each admissible finite sequence is called a *fan*.

Brouwer showed that the real numbers within a bounded closed interval can all be generated by the sequences of a single fan.

➥ It shall by now already be clear that the structures that have to be built are *inevitably more complicted* than in classical mathematics.

5 Intuitionistic Logic

One would not expect intuitionists to produce formal systems for branches of intuitionisitc mathematics.

- ➤ In the 1930s Arend Heyting developed *intuitionistic logic* and *formal intuitionistic number theory*.
- ➤ But he does not suggest that mathematical reasoning must be confined to this patterns.
- ➤ Brouwer accepted his system as a correct digest of the logical principles used in intuitionistic mathematics.

The system consists of *modus ponens* as a rule of deduction but

1. all logical signs are taken as *undefined primitives*;
2. the various logical axioms are abandoned in favour of *eleven axioms*.

1. $p \supset (p.p)$	7. $p \supset (p \vee q)$
2. $(p.q) \supset (q.p)$	8. $(p \vee q) \supset (q \vee p)$
3. $(p \supset q) \supset [(p.r) \supset (q.r)]$	9. $[(p \supset r).(q \supset r)] \supset [(p \vee q) \supset r]$
4. $[(p \supset q).(q \supset r)] \supset (p \supset r)$	10. $\neg p \supset (p \supset q)$
5. $q \supset (p \supset q)$	11. $[(p \supset q).(p \supset \neg q)] \supset \neg p$
6. $[p.(p \supset q)] \supset q$	

- ➤ Similar in appearance to theses in classical logic,
- ➥ but in 10. and 11. the *new sign* $\neg$ appears, where we might have expected $\sim$.
- ➤ Allows for many of the ways of reasoning in classical logic,
- ➥ but it is not possible to derive $p \vee \neg p$ and $\neg\neg p \supset p$.

Heyting's calculus should be interpreted as an *axiomatic theory of the notion of provability* by constructive methods.

- ➤ In mathematics to be true is to be provable.
- ➤ Intuitionistic logic is the result of applying this principle to the semantics of the connectives and quantifiers.

The *main idea* is as follows:

- ➤ $p.q$ is proved in a situation iff the situation proves p and q;
- ➤ $p \vee q$ is proved in a situation iff the situation contains evidence indicating that either p or q will eventually be proved;

- $p \supset q$ is proved in a situation iff the situation contains a method for converting any proof of p into a proof of q.
- $\neg p$ is proved in a situation iff the situation contains evidence that a proof of p can be turned into a proof of contradiction.

According to this interpretation, it is not a system of logic in the strict sense,

- because it does not fix the sense of the logical signs.

Later *model-theoretic approaches* provided precise semantic notions of truth and logical validity:

- Evert Beth in 1947, and
- Saul Kripke in 1965.

Their underlying idea is the same:

- Instead of a single model one has a partially ordered *collection of nodes*,
- each of which might be sought of as a model in its own right.
- The partial order among nodes is called *accessibility relation*.
- Truth of compound statements may well depend upon truth values of components at accessible nodes.

Part VIII

Kurt Gödel and Incompleteness

1 Kurt Gödel

Portrait of Kurt Friedrich Gödel

- 28 Apr. 1906 - 14 Jan. 1978, born in Brünn (Austria-Hungary),
  - son of Marianne Handschuh and Rudolf Gödel (managing director in a textile firm).
  - He attended the Gymnasium in Brünn.
  - Best grades though missing very often.
  - Entered the University of Vienna in 1923, decided late to take mathematics main subject.
- Taught by Furtwängler, Hahn, Menger and others.
  - took part on a seminar by Schlick on Russell's *Introduction to mathematical philosophy*.
  - it became slowly obvious that he would stick with logic and was to be Hahn's student.

- 1926 First visit of the Wiener Kreis.
  - Every Thursday evening in the seminar room of the mathematics institute;
  - by invitation only (presumably by Hahn or Schlick).
  - Aimed at developing a theory of scientific "truth".
  - Main impact on Gödel: introducing him to new literature and bringing him into contact with its famed members.
- 1929 PhD for his thesis establishing the completeness of first order calculus.
- 1930 Faculty member (doctorate) of the University of Vienna.
- 1931 *Über formal unentscheidbare Sätze der Principa Mathematica und verwandter Systeme.*

2 Incompleteness Theorems

Gödel considers a formal system $\mathcal{P}$ taking the *logical calculus of PM*, writing $x\Pi(x)$ instead of $(x)\phi(x)$:

- retaining the distinction of types by subscripts attached to his variables, i.e., x_1, x_2, x_3, etc;
- the logical axioms

$$\begin{array}{ll} 1.\ \ p \vee p \supset p & 3.\ \ p \vee q \supset q \vee p \\ 2.\ \ p \supset p \vee q & 4.\ \ (p \supset q) \supset (r \vee p \supset r \vee q)\ ; \end{array}$$

- as well as $v\Pi(a) \supset \mathit{Subst}\ a\binom{v}{c}$ and $v\Pi(b \vee a) \supset b \vee v\Pi(a)$; where
  - a is an arbitrary formula and v an arbitrary variable,
  - b is a formula where v does not occur free,
  - c is a sign of the same type as v, in which does not occur
  - a variable that is bound in a at a position at which v is free.

- the *axiom of reducibility* $(\exists u)(v\Pi(u(v) \equiv a))$;
- the *principle of extensionality* as an infinite sequence of formulae:

$$x_1\Pi(x_2(x_1) \equiv y_2(x_1)) \supset x_2 = y_2$$
$$x_2\Pi(x_3(x_2) \equiv y_3(x_2)) \supset x_3 = y_3 \quad \text{etc.}$$

- and *modus ponens* as the only rule $P \supset Q, P/Q$.

Furthermore Gödel adds *Peano's axiomatization* of number theory to $\mathcal{P}$ by

- introducing two new primitive signs f and 0, and (taking f as meaning 'the successor of')
- versions of Peano axioms three, four, and five:
  (i) $\sim (fx_1 = 0)$
  (ii) $fx_1 = fy_1 \supset x_1 = y_1$
  (iii) $x_2(0) \,.\, x_1\Pi(x_2(x_1) \supset x_2(fx_1)) \supset x_1\Pi(x_2(x_1))$

It is interesting to note that

1. It is unnecessary to introduce Peano axims one and two ('0 *is a number*' and '*if* n *is a number so is* fn')
   ➥ since the sequence of natural numbers is taken as the domain of the system.
2. The principle of mathematical induction cannot be stated in a single axiom without use of a variable of higher type.

As 1904 Hilbert remarked, symbolic logic could be treated as though it were a branch of elementary number theory itself.

➥ Gödel was the first who worked out such an arithmetization by means of his *Gödelnummerierung*.

2.1 Gödelnumbering

Gödelnummerierung:

- Correlate each primitive sign in $\mathcal{P}$ with a natural number:
  '0' with 1, 'f' with 3, '$\sim$' with 5, '$\vee$' with 7,
  'Π' with 9, '(' with 11, ')' with 13, variables of type n with p^n,
  where p is a prime number greater then 13, e.g., x_1 with 17, x_2 with 17^2, y_3 with 19^3, etc.
- If $s_1, \ldots, s_k$ are the numbers assigned to the k symbols of a formula, the whole formula is correlated with $2^{s_1} \cdot 3^{s_2} \cdots p_k^{s_k}$, where p_k is the k-th prime.
- A sequence of m formulas, having assigned numbers $f_1, \ldots, f_m$, respectively, correlates with $2^{f_1} \cdot 3^{f_2} \cdots p_m^{f_m}$.

Example 7. The formula $x_1\Pi(x_2(x_1) \vee \sim x_2(x_1))$ is correlated with $2^{17} \cdot 3^9 \cdot 5^{11} \cdot 7^{17^2} \cdot 11^{11} \cdot 13^{17} \cdot 17^{13} \cdot 19^7 \cdot 23^5 \cdot 29^{17^2} \cdot 31^{11} \cdot 37^{17} \cdot 41^{13} \cdot 43^{13}$.

Given this association between signs and numbers:

- Statements about the structure of a (sequence of) *formula(s)* correspond to statements about (the composition of) *numbers*,
- but this can be very complex in practice.

➥ Gödel constructs 46 abbreviating definitions, e.g., $Z(n)$ for the number associated with the numeral where f occurs n times ('$fff\ldots0$').
  - All except the last are explicit equivalences or recursively defined, as e.g., $0! \equiv 1$ and $(n+1)! \equiv (n+1)n!$.
  - All are functions or predicates of *numbers* definable without reference to signs or formulas.
  - The last is $Bew(x)$ applicable to a number iff it is the number of a propositional formula provable in the system.

Gödel has thus directed attention to a *fragment of number theory* which corresponds in structure to the syntax of $\mathcal{P}$ **and** can be represented in $\mathcal{P}$.

2.2 Theorems

For the sake of presentation, let us introduce another abbreviation:

$$Diag(x) \equiv Subst\ x\binom{17}{Z(x)}.$$

➡ read as '*the number of the formula we obtain by substituting the numeral denoting x for the free variable whose number is 17 in the formula whose number is x*'.

Now, consider the formula $\sim Bew(Diag(x))$:

- Let its number be $\mathcal{N}$.
- Let us further consider the formula $\mathcal{G}$ as $\sim Bew(Diag(\mathcal{N}))$.
- What is the number of $\mathcal{G}$?
  - ➡ The number of $Diag(\mathcal{N})$.

Thus, $\mathcal{G}$ says in effect that it is unprovable.

For illustration, let us introduce another sign Gödel did not use:

- Let $\mathcal{E}_N$ stand for '*the expression whose associated number is denoted by N*', and
- let us consider only propositional expressions which contain just one free variable (i.e., express properties of numbers).

➡ Then we can ask of each number (e.g., say M) whether it has the property expressed by the given expression ($\mathcal{E}_N$), in symbols $[\mathcal{E}_N; M]$.

We can arrange the propositions in a doubly infinite array:

$$\begin{array}{llll} [\mathcal{E}_A;A] & [\mathcal{E}_A;B] & [\mathcal{E}_A;C] & \ldots \\ [\mathcal{E}_B;A] & [\mathcal{E}_B;B] & [\mathcal{E}_B;C] & \ldots \\ [\mathcal{E}_C;A] & [\mathcal{E}_C;B] & [\mathcal{E}_C;C] & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}$$

➡ $Diag(x)$ assumes as values the numbers of formulas that express propositions on the diagonal of the array.

By means of $\mathcal{G}$, Gödel was able to show two remarkable theorems.

Before we can state them we need the following:

Definition 1. *A system is ω-consistent iff there is no formula F with one free variable such that*

(1) Fn is a theorem for every natural number n, and

(2) $\sim(x)Fx$ is also a theorem.

Gödels first incompleteness theorem reads as follows:

Theorem 1 (1). *If $\mathcal{P}$ is ω-consistent, it must be incomplete, with $\mathcal{G}$ as an undecidable formula.*

Proof (Idea).

($\Leftarrow$) Suppose $\mathcal{G}$ can be proved within $\mathcal{P}$. Then there must be some number of the proof of $\mathcal{G}$. But $\mathcal{G}$ asserts the non-existance of such a number. This leads to self-contradiction.

($\Rightarrow$) Suppose $\mathcal{G}$ can be disproved in $\mathcal{P}$. Then $\sim \mathcal{G}$ is provable, which is equivalent to $(\exists x)P(x)$, where $P(x)$ is '*x is the number of a proof of $\mathcal{G}$*'. But for every natural number n, we can prove within $\mathcal{P}$ (in a finite number of steps) that $\sim P(n)$ holds (otherwise it would be possible to construct a proof of $\mathcal{G}$ in $\mathcal{P}$). We conclude that, if $\mathcal{P}$ is ω-consistent, then $\mathcal{G}$ cannot be refuted in $\mathcal{P}$. Hence, if $\mathcal{P}$ is ω-consistent, then $\mathcal{G}$ is undecidable, i.e., $\mathcal{P}$ is incomplete.

➡ Arithmetic can never be completely formalized.

- What happens if we add $\mathcal{G}$ as an axiom to $\mathcal{P}$?
- Same for any system, where Peano's axioms can be obtained (PM, Zermelo's axiomatic theory, ...).

Theorem 2 (2). *If $\mathcal{P}$ is consistent, the consistency of $\mathcal{P}$ cannot be proved within $\mathcal{P}$.*

Proof (Idea). Show the lemma: If $\mathcal{C}$ is a formula of $\mathcal{P}$ which asserts in syntactical interpretation the consistency of $\mathcal{P}$, then $\mathcal{C} \supset \mathcal{G}$ follows. It follows that if we can prove the consistency of $\mathcal{P}$ within $\mathcal{P}$, it would be also possible to prove $\mathcal{G}$; and that, as we have seen, is not the case.

➥ Hilbert's program can never be carried out as originally conceived.
- $\mathcal{P}$ seems to include all arguments commonly called finitary,
- yet it does not allow for a proof of its own consistency (same for more ambitious systems).
- Gödel, however, said his result is not inconsistent with Hilbert's programme (but does not explain his use of "finitary").

3 Kurt Gödel on Intuitionism

In 1932 Gödel showed that

➤ classical restricted calculus of functions
  ➥ (*Eine Interpretation des intuitionistischen Aussagenkalküls*)
➤ and classical elementary arithmetic
  ➥ (*Zur intuitionistischen Arithmetik und Zahlentheorie*).

can both be reconstructed within the corresponding intuitionistic systems.

Although Heyting's signs are to be taken as undefined primitives, nothing prevents us from introducing

➤ *a new disjunction sign* $\oplus$, and *a new implication sign* $\rightarrow$, as

$$p \oplus q = \neg(\neg p . \neg q),$$
$$p \rightarrow q = \neg(p . \neg q).$$

From these and the axioms and rules of inference given by Heyting

➤ it is possible to obtain *all theses of classical propositional logic*,
➤ including the law of excluded middle in the form $p \oplus \neg p$ and
➤ the principle for the elimination of double negation as $\neg\neg p \rightarrow p$.
➥ Since these are merely abbreviations for $\neg(\neg p . \neg\neg p)$ and $\neg(\neg\neg p . \neg p)$, which can both be proved within Heyting's system..

Notice however, that

➤ the new *signs are not to be equated* with the intuitionstic signs, and
➤ $\neg\neg p \rightarrow p$ cannot be used in the intuitionistic calculus
  ➥ since *modus ponens* there refers to $\supset$, not to $\rightarrow$.

Nevertheless, reproduction amounts to a *relative proof of consistency*:

➤ If intuitionistic mathematics is free from contradiction, then so too are those parts of classical mathematics that can be translated into parts of intuitionistic mathematics.

4 Kurt Gödel (ctd.)

- 1932 became Privatdozent, submitting his paper for his habilitation.
- 1933 Hitler first had no effect on Gödel (little interest in politics).
- 1934 First visit and lectures in Princeton (Kleene took notes).
  - An offer for further lecturing followed,
  - but he suffered a nervous breakdown and returned.
- Proved important results on the consistency of the axiom of choice and other axioms in set theory in 1935.
- In the same year Schlick was murdered by a National Socialist student.
  - Gödel had another breakdown in 1936.
  - After his recovery, first call to a guest professorship in USA.
- In summer 1938 he visited Göttingen, lecturing on his set theory research.
- 1938 Gödel married Adele Porkert.
- 1938-39 Second visit in Princeton.

- Has not been extended as a paid lecturer after the Anschluss.
  - Attacked by a gang of youths on Strudlhofstiege.
  - It seems that he was thought to be Jewish.
- In 1940 Gödel and his wife left Europe,
  - becoming ordinary member of the Institute for Advanced Study in Princeton from 1940 to 1946;
  - a permanent member until 1953, U.S. citicen in 1948, and
  - held a chair in Princeton from 1953 to death.
- In 1940 he published his work *The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis with the Axioms of Set Theory*.
- Afterwards he began to study philosophical problems.
- In the early seventies, deeply religious, Gödel circulated among friends what became known as Gödels ontological proof (of God's existence).

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