Discrete Mathematics Exam

24.01.2025

1 Number theory [10pt]

Let p denote a prime number, k a positive integer and $\mathbb{F}_{p^k}$ the field with p^k elements.

a) What is the characteristic $\texttt{char}(\mathbb{F}_{p^k})$? [1pt]

b) Prove that if $p = 2$ then $a = -a$ for all $a \in \mathbb{F}_{2^k}$. Hint: $a = 1 * a$. [1pt]

c) How many elements does the group of units $\mathbb{F}_{p^k}^{\times}$ have? [1pt]

d) How many elements does the group of units $(\mathbb{Z}/ < p^k >)^{\times}$ have? [3pt]

e) How many elements does the group of units $(\mathbb{Z}/ < 90 >)^{\times}$ have? [4pt]

2 Polynomials in finite fields [10pt]

Let $\mathbb{F}_8$ denote the field with 8 elements. Let $f, g \in \mathbb{F}_8[X]$ with $f = X^4 + X^3 + X + 1$ and $g = X^3 + X + 1$.

a) Determine $\texttt{gcd}(f, g)$. [4pt]

b) Find $\lambda, \mu \in \mathbb{F}_8[X]$ with $\lambda f + \mu g = \texttt{gcd}(f, g)$. [4pt]

c) Is g invertible in $\mathbb{F}_8[X]/ < f >$? If yes, determine its inverse. [2pt]

3 Combinatorics [10pt]

a) Let $A = \{1, 2, \ldots, 6\}$ How many ways are there to partition A into 2-element subsets? (For example, $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$) is one such partition.) [5pt]

b) Show that for any positive integer n the number of positive divisors of n^2 is always odd. (Hint: use prime factor decompositions) [5pt]

4 Graph theory [10pt]

Calculate the number of spanning trees of the following graph: