Mid-term Exam
Signal and Image Processing
SS 2021

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University of Vienna

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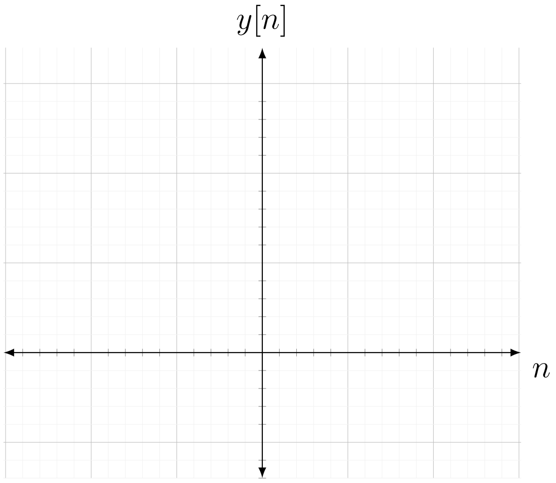
Part	A	B	C	Total
Score	/ 30	/ 30	/40	/ 100

Part A: Signal and System [30 P]

a.) Find the output $y[n]$ of the LTI system with impulse response $h[n] = 2\delta[n+1]+1\delta[n-1]$ given the input $x[n] = 2\delta[n+1]+3\delta[n]+(-1)\delta[n-2]$. **Write down the mathematical expression for $y[n]$ AND plot $y[n]$ in the below coordinate. (Carefully label both axes!)**
Hint: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$



[/ 12]

b.) Describe which kind of convolution has been used for continuous Fourier transform (FT), discrete time Fourier transform (DTFT), and discrete Fourier transform (DFT) respectively?

[/ 3]

c.) Now, consider a new system $h_1[n]$ which take $x[n]$ as input and output $y[n]$. The system equation is expressed as $y[n] = \alpha x[n] + \beta y[n-1]$, where $\alpha = 2, \beta = 0.5$. Is $h_1[n]$ a linear system and is it stable? If another system $h_2[n]$ is with the same system equation but different coefficients $\alpha = 0.5, \beta = 2$, is it also linear and stable? **Provide and justify your answer!**

[/ 8]

d.) Explain why is the amplitude spectrum $|X(\omega)|$ of a real-valued signal $x(t)$ symmetric, i.e., $|X(\omega)| = |X(-\omega)|$? (**Hint**: $X(\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau$; $e^{j\theta} = \cos\theta + j\sin\theta$)

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Part B: Filter Design [30 P]

a.) For an ideal low-pass filter, its phase spectrum should be either zero or linear. Zero-phase can be easily understood as it means there is no time delay for all frequency components, i.e., all of them start at the same time. However, a low-pass filter filter with a linear phase spectrum can also be considered as ideal. **Explain why the linear-phase filter is desirable!**

[/ 10]

b). There are two dominant ways to design an FIR filter, i.e., either applying a windowing function or leveraging the least square method. **Describe how to design an FIR filter with zero-phase via using the least square method.**

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c.) Now, if you are asked to design an FIR filter with **linear phase** with the least square method, which modification(s) should be done in your answer for question b)? **Provide and justify your answer!**

[/ 8]

Part C: Hilbert Transform and Analytic Signal [40 P]

As a complex-valued signal, the analytic signal plays an essential role in telecommunication. To compute such a complex-valued signal from a normal real-valued signal, we usually leverage the power of the Hilbert transform to compute the imaginary part of the analytic signal.

$$h(t) = \frac{1}{\pi t}, t \in [-\infty, \infty] \tag{1}$$

$$H(f) = \begin{cases} j = e^{+\frac{j\pi}{2}}, & \text{for } f < 0, \\ 0, & \text{for } f = 0, , \\ -j = e^{-\frac{j\pi}{2}}, & \text{for } f > 0. \end{cases} \tag{2}$$

where $h(t)$ and $H(f)$ represent the Hilbert transform function in time- and frequency domain respectively. Therefore, for a real-valued signal $g(t)$, the corresponding analytic signal can be expressed as:

$$\hat{g}(t) = g(t) + j\mathcal{H}\left(g(t)\right) = g(t) + j\left[g(t) * h(t)\right], \tag{3}$$

where $\hat{g}(t)$ is the analytic signal, $\mathcal{H}\left(g(t)\right)$ indicates the Hilbert transformation of $g(t)$ and $g(t) * h(t)$ represents the convolution between $g(t)$ and $h(t)$. (Hint: $g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau$)

1. Given a real-valued signal $x(t) = \cos(2\pi f_0 t)$, compute its analytic signal $\hat{x}(t)$. (**Hint**: $e^{j2\pi f_0 t} = \cos(2\pi f_0 t) + j\sin(2\pi f_0 t) \overset{FT}{\longleftrightarrow} \delta(f - f_0)$ and represent the sine function as the supercomposition of exponential functions before applying Fourier transform)

[/ 10]

2. Now, assuming that a direct current signal $c(t) = C$ is added into as $x(t)$ an offset (C is a constant), will $\mathcal{H}(x(t))$ be equivalent to $\mathcal{H}(x(t) + c(t))$? **Provide and justify your answer!** (Hint: Eq. (2)) [/ 10]

3. If we treat the Hilbert transform as a system which outputs the Hilbert transformed signal. Is this system causal? **Provide and justify your answer!** [/ 3]

4. In previous questions, we focus on the Hilbert transform for the continuous signal. On the other hand, the Hilbert transform is more practically applied for discrete signal. **Describe how to implement the discrete Hilbert transform with the help of DTFT.** [/ 10]

5. One important application of Hilbert transform is the single-sideband (SSB) modulation which can make the best use of the scarce spectrum resources. For better understanding the idea of "wasting spectrum resources", the below figures show the amplitude spectrum of base-band and modulated signal.

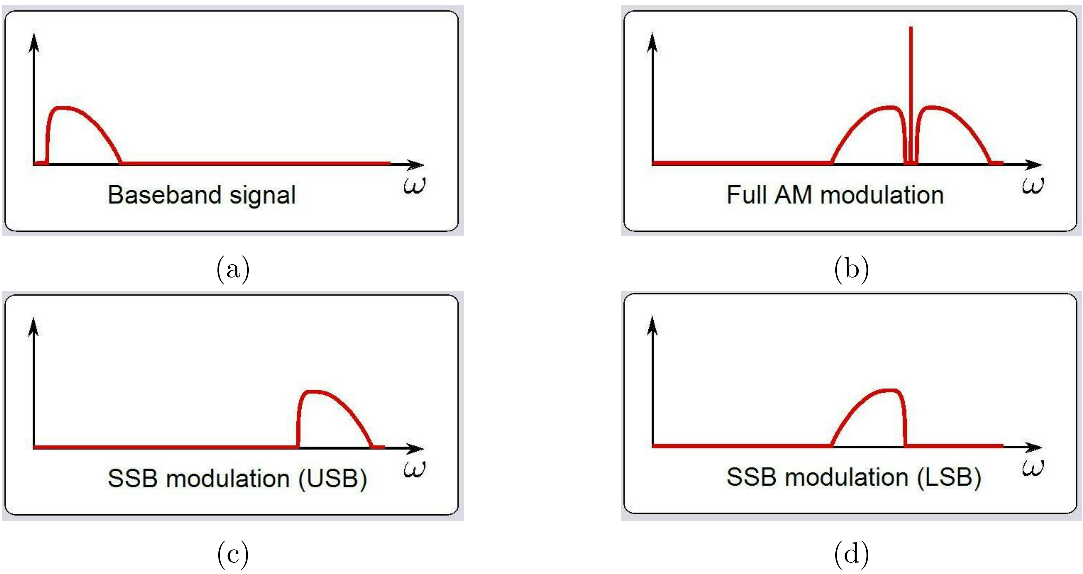

Figure 1: Amplitude spectrum of baseband and modulated signal

As Figure 1. (b) shows, the amplitude modulation (AM) mirrors and shifts the base-band signal spectrum around the carrier frequency, and the information contained in the AM modulation is apparently duplicated as the right part of its spectrum carries exactly the same information as the left part. Hence, the SSB modulation is preferable as it can avoid unnecessary spectrum occupancy, as shown in Figure 1. (c) and Figure 1. (d). To realize SSB modulation, one typical pipeline which adopts Hilbert transformer is illustrated in Figure 2,

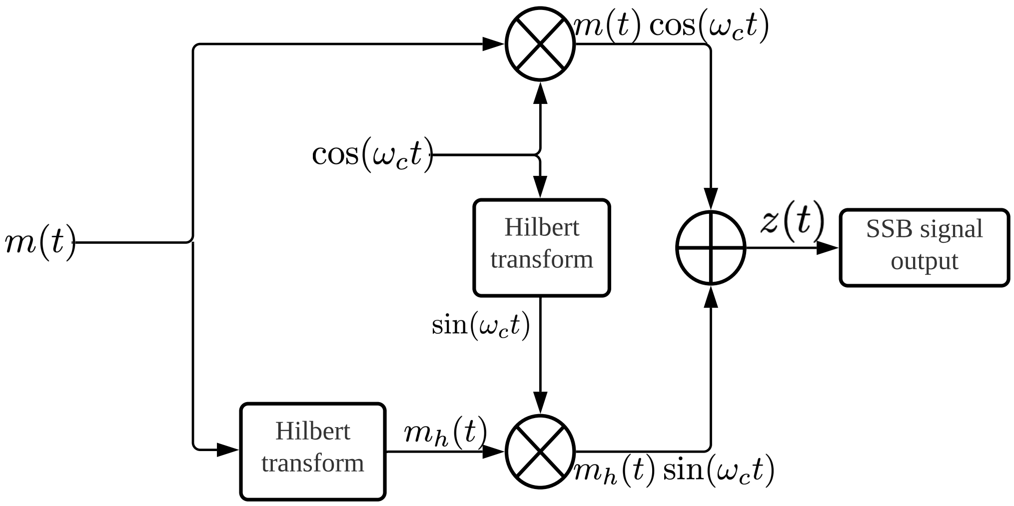

Figure 2: SSB modulation pipeline leveraging Hilbert transform

where $m(t)$ is the baseband signal and $m_h(t)$ is the Hilbert transformed signal of $m(t)$. The SSB modulated output signal is denoted as $z(t) = m(t)\cos(\omega_c t) + m_h(t)\sin(\omega_c t)$. **Prove why the spectrum of output signal $z(t)$ should be single-sideband ?**

Hints:

(a) $\cos\omega_c t \overset{\text{FT}}{\longleftrightarrow} \frac{1}{2}[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]$

(b) $FT(f(t)g(t)) = F(\omega) * G(\omega)$

(c) $X(\omega) * \delta(\omega-\omega_c) = X(\omega-\omega_c)$

(d) Equation (2) in Part C

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