Hybrid Systems
Modeling, Analysis and Control

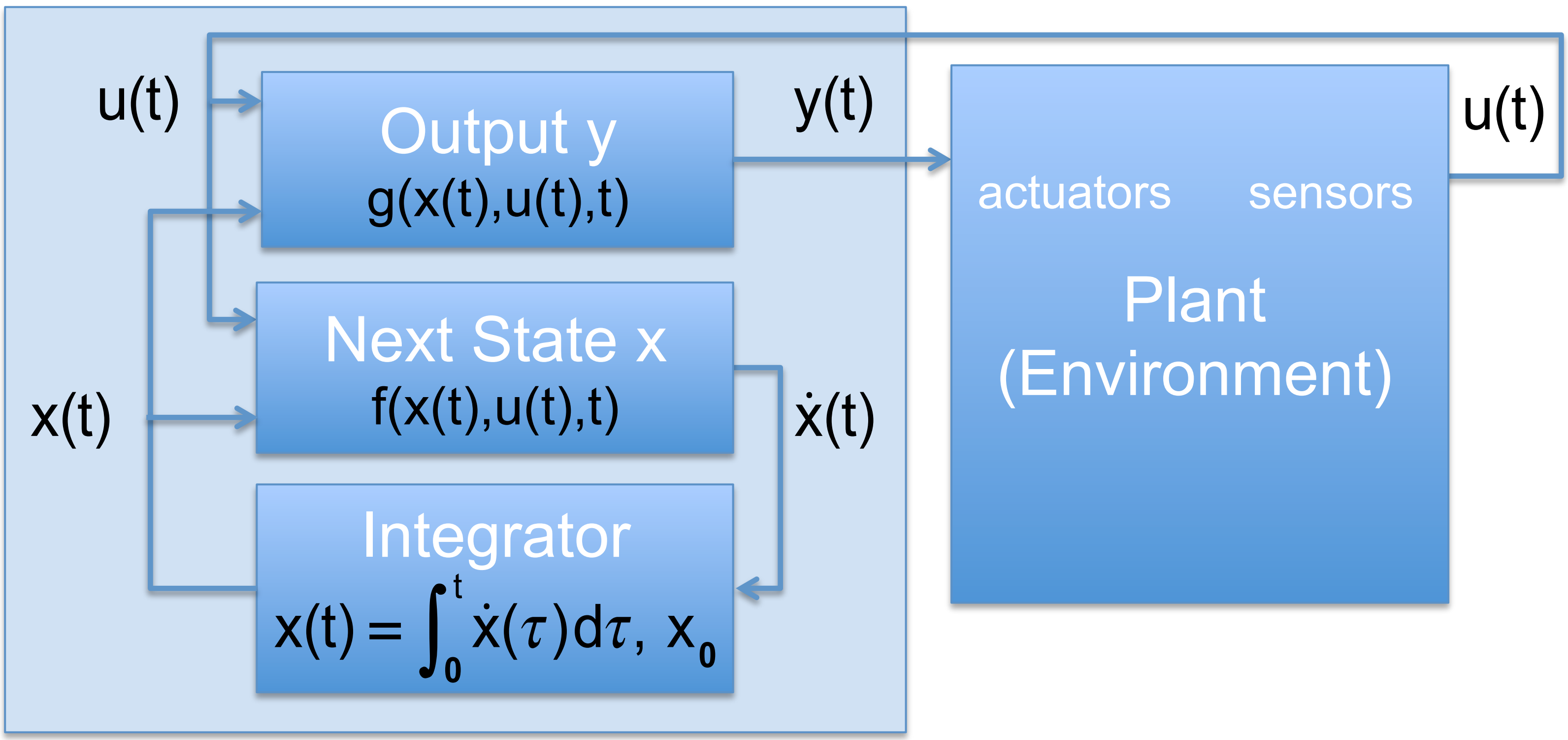
Radu Grosu
Vienna University of Technology

Lecture 2

Physical Model: Description

Differential Equations:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), & \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \end{cases}$$



Physical Model: Description

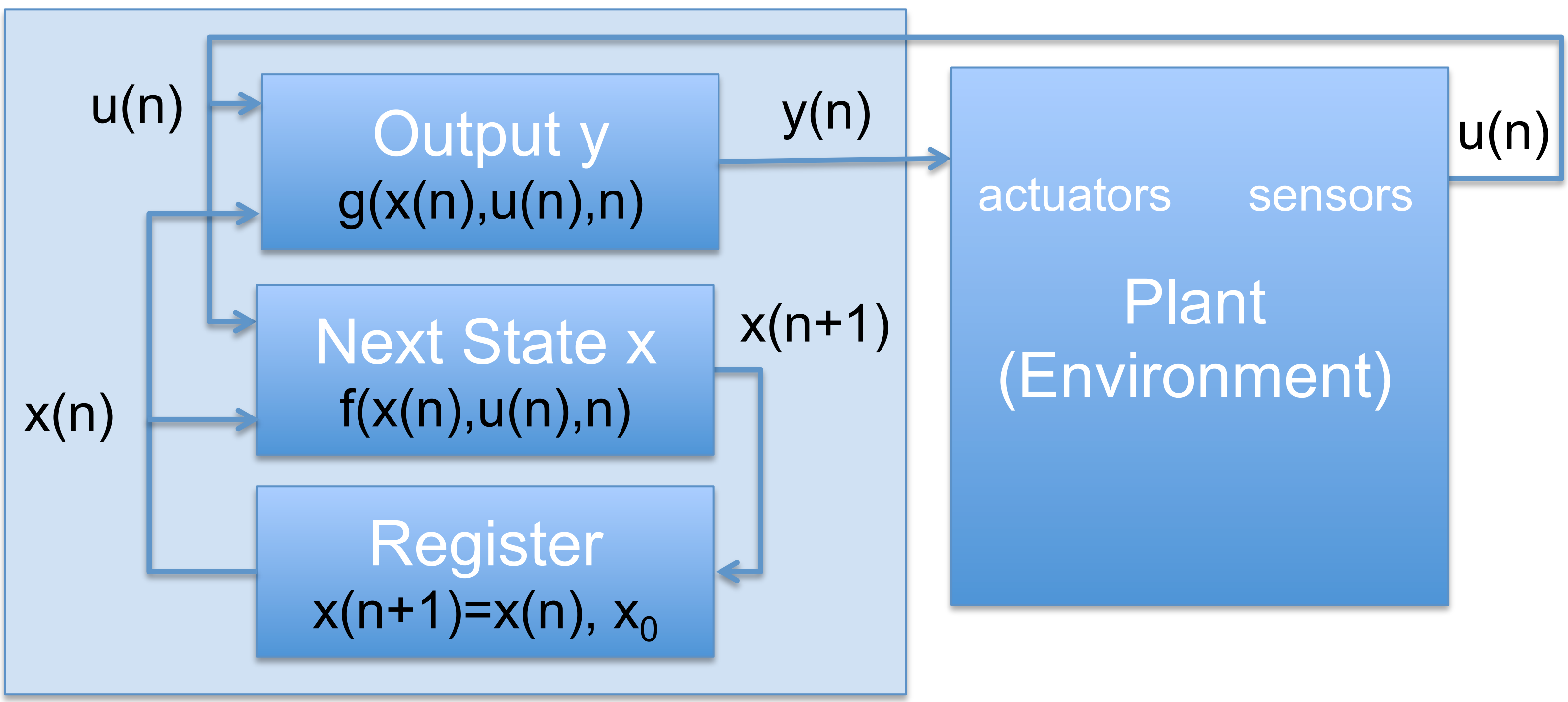
Differential Equations:
$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), & \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \end{cases}$$

- **State vector:** $\mathbf{x}(t) \in \mathbb{R}^n$, **input vector:** $\mathbf{u}(t) \in \mathbb{R}^k$, **output vector:** $\mathbf{y}(t) \in \mathbb{R}^m$
- **Next (infinitesimal) state function:** $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^n$
  - **Time invariant:** $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$, $\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))$, no exp. dep. on t
  - **Linear:** $\mathbf{f}(a_1 x_1 + a_2 x_2, u, t) = a_1 \mathbf{f}(x_1, u, t) + a_2 \mathbf{f}(x_2, u, t)$, similar for u
- **Output (observation) function:** $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^m$
  - **Moore:** if $\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), t)$ depends only on x and t

Physical Model: Description

Difference Equations:

$$\begin{cases} \mathbf{x}(n+1) = \mathbf{f}(\mathbf{x}(n), \mathbf{u}(n), n), & \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(n) = \mathbf{g}(\mathbf{x}(n), \mathbf{u}(n), n) & \end{cases}$$



Physical Model: Description

Difference Equations:
$$\begin{cases} \mathbf{x}(n+1) = \mathbf{f}(\mathbf{x}(n), \mathbf{u}(n), n), & \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(n) = \mathbf{g}(\mathbf{x}(n), \mathbf{u}(n), n) \end{cases}$$

- State vector: $\mathbf{x}(t) \in \mathbb{R}^n$, input vector: $\mathbf{u}(t) \in \mathbb{R}^k$, output vector: $\mathbf{y}(t) \in \mathbb{R}^m$
- Next (tick) state function: $\mathbf{f}: \mathbf{R}^n \times \mathbf{R}^k \times \mathbf{N} \to \mathbf{R}^n$
  - Time invariant: $\mathbf{x}(n+1) = \mathbf{f}(\mathbf{x}(n), \mathbf{u}(n))$, $\mathbf{y}(n) = \mathbf{g}(\mathbf{x}(n), \mathbf{u}(n))$, no exp. dep. on n
  - Linear: $\mathbf{f}(a_1 x_1 + a_2 x_2, u, n) = a_1 \mathbf{f}(x_1, u, n) + a_2 \mathbf{f}(x_2, u, n)$, similar for u
- Output (observation) function: $\mathbf{g}: \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{N} \to \mathbb{R}^m$
  - Moore: if $\mathbf{y}(n) = \mathbf{g}(\mathbf{x}(n), n)$ depends only on x and n

Fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{F}_2$

A structure $\mathcal{F} = (F, +, \cdot, 0, 1, -, ^{-1})$ such that:

$(F, +, 0, -)$ is a commutative group:

(ass)	$\forall x, y, z \in F.$	$(x + y) + z$	$= x + (y + z)$
(com)	$\forall x, y \in F.$	$x + y$	$= y + x$
(ntr)	$\forall x \in F.$	$x + 0$	$= x$
(inv)	$\forall x \in F.$	$x + (-x)$	$= 0$

$(F \setminus \{0\}, \cdot, 1, ^{-1})$ is a commutative group:

(ass)	$\forall x, y, z \in F.$	$(x \cdot y) \cdot z$	$= x \cdot (y \cdot z)$
(com)	$\forall x, y \in F.$	$x \cdot y$	$= y \cdot x$
(ntr)	$\forall x \in F.$	$x \cdot 1$	$= x$
(inv)	$\forall x \in F \setminus \{0\}.$	$x \cdot (x)^{-1}$	$= 1$

Compatibility of addition and multiplication:

(dis) $\forall x, y, z \in F.\ x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

Vector Space: $\mathbb{Q}^n$, $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{F}_2^n$

A structure $\mathcal{V} = (\mathcal{F}, \mathcal{T}, \bullet)$ such that:

- $\mathcal{F} = (F, +, \cdot, 0, 1, -, ^{-1})$ **is a field (of scalars)**
- $\mathcal{T} = (T, +, 0, -)$ **is commutative group (of vectors)**
- **Multiplication $s \bullet \mathbf{x}$ of vector x with scalar s satisfies:**

(dis$_1$) $\forall a \in F,\ \mathbf{x}, \mathbf{y} \in T.\quad a \bullet (\mathbf{x} + \mathbf{y}) = a \bullet \mathbf{x} + a \bullet \mathbf{y}$

(dis$_2$) $\forall a, b \in F,\ \mathbf{x} \in T.\quad (a + b) \bullet \mathbf{x} = a \bullet \mathbf{x} + b \bullet \mathbf{x}$

(cmp) $\forall a, b \in F,\ \mathbf{x} \in T.\quad a \cdot (b \bullet \mathbf{x}) = (a \cdot b) \bullet \mathbf{x}$

(ntr) $\forall \mathbf{x} \in T.\quad 1 \bullet \mathbf{x} = \mathbf{x}$

Typical example: $\mathcal{V} = (\mathbb{R}, \mathbb{R}^n, \cdot)$

$$[x_1, x_2] + [y_1, y_2] = [x_1 + y_1, x_2 + y_2], \qquad a \cdot [x_1, x_2] = [a \cdot x_1, a \cdot x_2]$$

Vector Space Basis

Linear independence (LI) in ordered set $\mathcal{B} = [\mathbf{b_1},\ldots,\mathbf{b_n}]$:

$$a_1\mathbf{b_1} + \ldots + a_n\mathbf{b_n} = \mathbf{0} \text{ implies } a_1 = \ldots = a_n = 0$$

Otherwise $\mathcal{B}$ is called linearly dependent.

Linear combination

Span: $\text{span}(\mathcal{B}) = \{\mathbf{v} \mid \exists a_1 \ldots a_n .\ \mathbf{v} = a_1\mathbf{b_1} + \ldots + a_n\mathbf{b_n}\}$

Basis of $\mathcal{V}$: LI ordered subset $\mathcal{B}$ of $\mathcal{V}$ st. $\text{span}(\mathcal{B}) = \mathcal{V}$

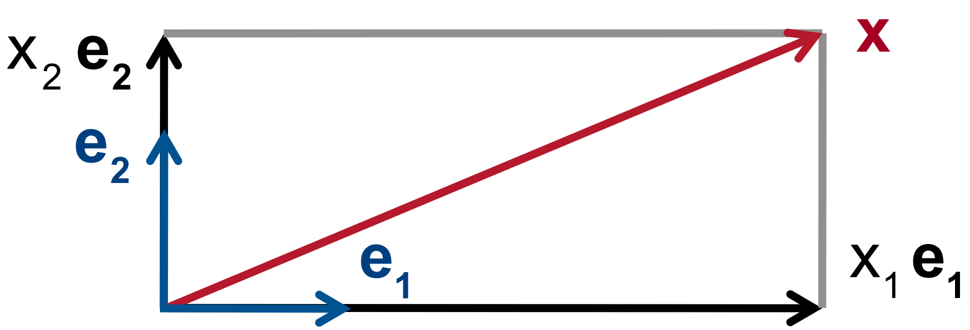
Thm: All bases of $\mathcal{V}$ have same cardinality.

Dimension of $\mathcal{V}$: The cardinality of its bases.

Matrix Representation of a Linear System

Define the canonical basis $\mathcal{E} = [\mathbf{e}_1, \ldots, \mathbf{e}_n]$ where:

$$\mathbf{e}_1 = [1\,0\,\ldots\,0]^T,\ \mathbf{e}_2 = [0\,1\,\ldots\,0]^T, \ldots,\ \mathbf{e}_n = [0\,0\,\ldots\,1]^T$$



With respect to $\mathcal{E}$ every vector $\mathbf{x} \in \mathbb{R}^n$ becomes:

$\mathbf{x} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$ **Hence,** $\mathbf{x}_{\mathcal{E}} = [x_1\,x_2\,\ldots\,x_n]^T$

where scalar x_i is the i^{th} element of vector $\mathbf{x}$

Matrix Representation of a Linear System

If f is linear, we obtain by superposition:

$$\mathbf{f(x)} = \mathbf{f}(x_1\,\mathbf{e_1} + \ldots + x_n\,\mathbf{e_n}) = x_1\,\mathbf{f(e_1)} + \ldots + x_n\,\mathbf{f(e_n)}$$

$$\mathbf{f(e_j)} = a_{1j}\,\mathbf{e_1} + \ldots + a_{nj}\,\mathbf{e_n} \quad \textbf{Hence,} \quad \mathbf{f(e_j)}_{\mathcal{E}} = [a_{1j} \ldots a_{nj}]^T$$

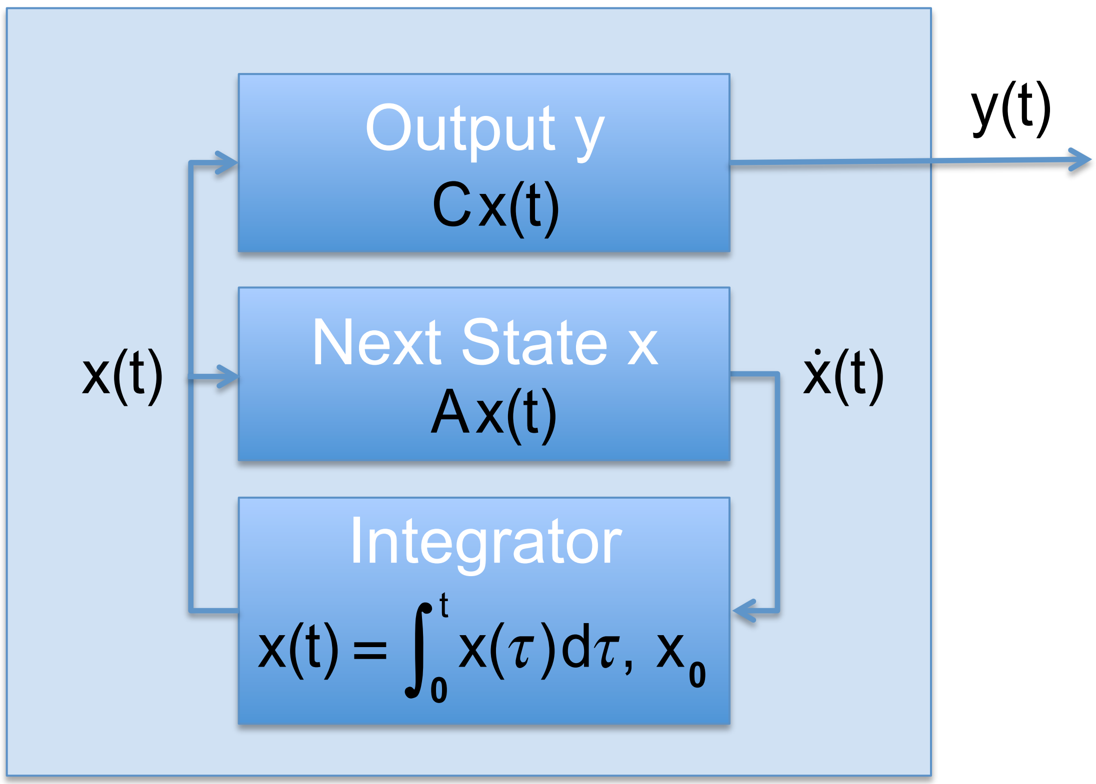
Consequently, $f(x)_{\mathcal{E}}$ can be written as:

$$\mathbf{f(x)}_{\mathcal{E}} = x_1 \begin{bmatrix} a_{11} \\ \ldots \\ a_{n1} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ \ldots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}\ a_{12} \ldots a_{1n} \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{n1}\ a_{n2} \ldots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix} = \mathbf{Ax}$$

$$\mathbf{A} = [\mathbf{f(e_1)}_{\mathcal{E}} \ \ldots \ \mathbf{f(e_n)}_{\mathcal{E}}] = [\mathbf{f(e_1)} \ \ldots \ \mathbf{f(e_n)}]_{\mathcal{E}} = \mathbf{f([e_1 \ \ldots \ e_n])}_{\mathcal{E}} = \mathbf{f}(\mathcal{E})_{\mathcal{E}}$$

Considering input u: $\mathbf{f(x,u)} = \mathbf{[A\ B]\ [x\ u]}^T = \mathbf{Ax} + \mathbf{Bu}$

Solution of: $\dot{x} = Ax$, $y = Cx$, $x(0) = x_0$
(Autonomous System)



Solution (fixpoint) of $\dot{x} = Ax,\ x(0) = x_0$

The solution is: $\mathbf{x}(t) = e^{At}x_0$

Proof:

1) $\dot{x} = Ae^{At}x_0 = Ax$
2) $x(0) = e^{A0}x_0 = e^0 x_0 = Ix_0 = x_0$

Some matrix properties (by Taylor expansion):

$$e^{at} = \sum_{n=0}^{\infty} (e^{at})^{(n)}(0)t^n / n! = \sum_{n=0}^{\infty} a^n t^n / n! \quad \Rightarrow \quad e^{At} = \sum_{n=0}^{\infty} A^n t^n / n!$$

1) $e^{A0} = I$
2) $d(e^{At})/dt = Ae^{At}$
3) $e^{\mathrm{diag}(\lambda_1,\ldots,\lambda_n)t} = \mathrm{diag}(e^{\lambda_1 t},\ldots,e^{\lambda_n t})$

To do: Find basis $\mathcal{B}$ such that $f(\mathcal{B})_{\mathcal{B}} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$

Finding $\mathcal{B}$ such that $f(\mathcal{B})_{\mathcal{B}} = \text{diag}(\lambda_1,\ldots,\lambda_n)$

Consider the basis: $\mathcal{B} = [b_1 \;\ldots\; b_n]$. Obviously, $\mathcal{B}_{\mathcal{B}} = I$

We want that:

$$f(\mathcal{B})_{\mathcal{B}} = [f(b_1) \;\ldots\; f(b_n)]_{\mathcal{B}} = \text{diag}(\lambda_1,\ldots,\lambda_n) = [\lambda_1 b_1 \;\ldots\; \lambda_n b_n]_{\mathcal{B}}$$

eigenvalue eigenvector

Consequently, we need that: $f(b_i) = \lambda_i b_i$ for $i = 1{:}n$

If we represent b_i in $\mathcal{E}$ we have that:

$$f(b_i)_{\mathcal{E}} = Ab_i = \lambda_i b_i \quad \text{Hence,} \quad (\lambda_i I - A)b_i = 0$$

Finding $\mathcal{B}$ such that $f(\mathcal{B})_{\mathcal{B}} = \text{diag}(\lambda_1, \ldots, \lambda_n)$

Let $(\lambda_i I - A) = C = [c_1 \ \ldots \ c_n]$. Then:

Characteristic polynomial (roots in $\mathbb{C}$)

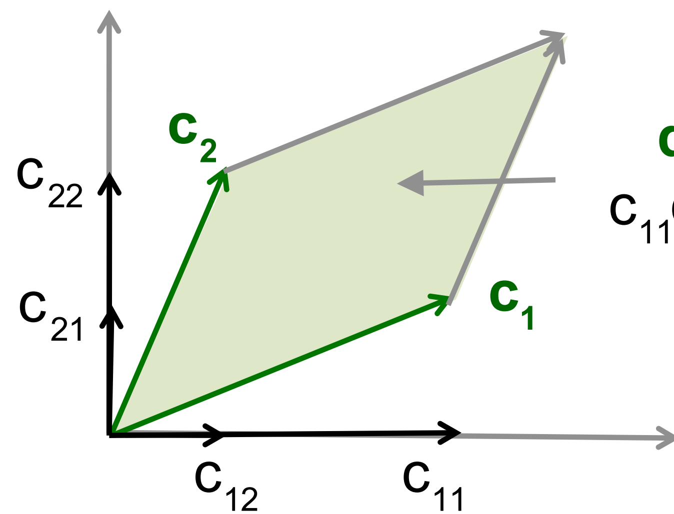
$$(\lambda_i I - A)b_i = Cb_i = c_1 b_{1i} + \ldots + c_n b_{ni} = 0$$

Hence C is linearly dependent $\quad \det(C) = \det(\lambda_i I - A) = 0$

Determinant: det(C) associates to C its signed volume.

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

(columns labelled c_1, c_2)

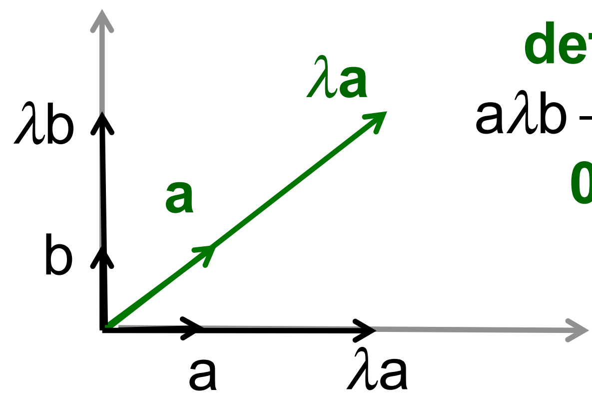


$\det(C) = c_{11}c_{22} - c_{12}c_{21}$

Computing the Final Solution

$\mathbf{x} \quad \lambda\mathbf{x}$

$$\mathbf{C} = \begin{bmatrix} a & \lambda a \\ b & \lambda b \end{bmatrix}$$



$\mathbf{det(C)} =$
$a\lambda b - \lambda ab =$
$\mathbf{0}$

Solve by Gaussian elimination: $\mathbf{A b_i = \lambda_i b_i}$ to get $\mathcal{B}$

Change bases: $\bar{\mathbf{x}} = \mathcal{B}^{-1}\mathbf{x} = \mathbf{x}_{\mathcal{B}} = \alpha_1 \mathbf{b}_1 + \cdots + \alpha_n \mathbf{b}_n$

$$\dot{\bar{\mathbf{x}}} = \mathcal{B}^{-1}\dot{\mathbf{x}} = \mathcal{B}^{-1}\mathbf{A}\mathbf{x} = \mathcal{B}^{-1}\mathbf{A}\mathcal{B}\,\bar{\mathbf{x}} = \mathcal{B}^{-1}\mathcal{B}\,\Lambda\,\bar{\mathbf{x}} = \Lambda\,\bar{\mathbf{x}} = (\mathbf{A}\mathcal{B})_{\mathcal{B}}\,\bar{\mathbf{x}}$$

Consequently: $\bar{\mathbf{x}} = \mathbf{e}^{\Lambda t}\bar{\mathbf{x}}_0 = \mathbf{diag}(\mathbf{e}^{\lambda_1 t}, \ldots, \mathbf{e}^{\lambda_n t})\bar{\mathbf{x}}_0$

Finally: $\mathbf{x} = \mathcal{B}\bar{\mathbf{x}} = \mathcal{B}\,\mathbf{diag}(\mathbf{e}^{\lambda_1 t}, \ldots, \mathbf{e}^{\lambda_n t})\mathcal{B}^{-1}\mathbf{x}_0$

Example: Fixpoint Computation

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. **Compute the solution of** $\dot{x} = Ax$.

1. Compute the eigenvalues of A

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)(\lambda - 2) - 1 = \underset{a}{\lambda^2} \underset{b}{- 4\lambda} + \underset{c}{3}$$

Hence: $\lambda_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{4 \pm \sqrt{16 - 12}}{2} = \dfrac{4 \pm 2}{2}$

We obtain: $\lambda_1 = 1$ **and** $\lambda_2 = 3$

Example: Fixpoint Computation

2. Compute the eigenvectors of A

$$(\lambda_1 = 1) \quad \overset{\mathbf{A}}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \overset{\mathbf{b_1}}{\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}} = \overset{\mathbf{A\,b_1}}{\begin{bmatrix} 2b_{11} + b_{21} \\ b_{11} + 2b_{21} \end{bmatrix}} = \overset{\lambda_1\mathbf{b_1}}{\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}}$$

$b_{21} = -b_{11}$ **Take** $\mathbf{b_1} = [1 \;\; -1]^T$

$$\mathcal{B} = \overset{\mathbf{b_1} \;\; \mathbf{b_2}}{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}$$

$$(\lambda_2 = 3) \quad \overset{\mathbf{A}}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \overset{\mathbf{b_2}}{\begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix}} = \overset{\mathbf{A\,b_2}}{\begin{bmatrix} 2b_{12} + b_{22} \\ b_{12} + 2b_{22} \end{bmatrix}} = \overset{\lambda_2\mathbf{b_2}}{\begin{bmatrix} 3b_{12} \\ 3b_{22} \end{bmatrix}}$$

$b_{12} = b_{22}$ **Take** $\mathbf{b_2} = [1 \;\; 1]^T$

Example: Fixpoint Computation

3. Check diagonalization:

$$\mathcal{B}^{-1}\mathbf{A}\mathcal{B} = (\mathbf{A}\mathcal{B})_{\mathcal{B}} = \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right)_{\mathcal{B}} = \left(\begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix}\right)_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

4. Compute the solution $\mathbf{x} = \mathcal{B}\,\mathbf{diag}(\mathbf{e}^{\lambda_1 \mathbf{t}}, \ldots, \mathbf{e}^{\lambda_n \mathbf{t}})(\mathbf{x}_0)_{\mathcal{B}}$:

$\mathcal{B}$ $\quad \mathbf{diag}(\mathbf{e}^{\lambda_1 \mathbf{t}}, \mathbf{e}^{\lambda_2 \mathbf{t}})$ $\quad (\mathbf{x}_0)_{\mathcal{B}}$ $\qquad$ $\mathcal{B}$ $\mathbf{diag}(\mathbf{e}^{\lambda_1 \mathbf{t}}, \mathbf{e}^{\lambda_2 \mathbf{t}})$ $\quad (\mathbf{x}_0)_{\mathcal{B}}$

$$\mathbf{x}(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{t} + e^{3t} \\ -e^{t} + e^{3t} \end{bmatrix}$$

Example: Fixpoint Computation

5. Check initial condition:

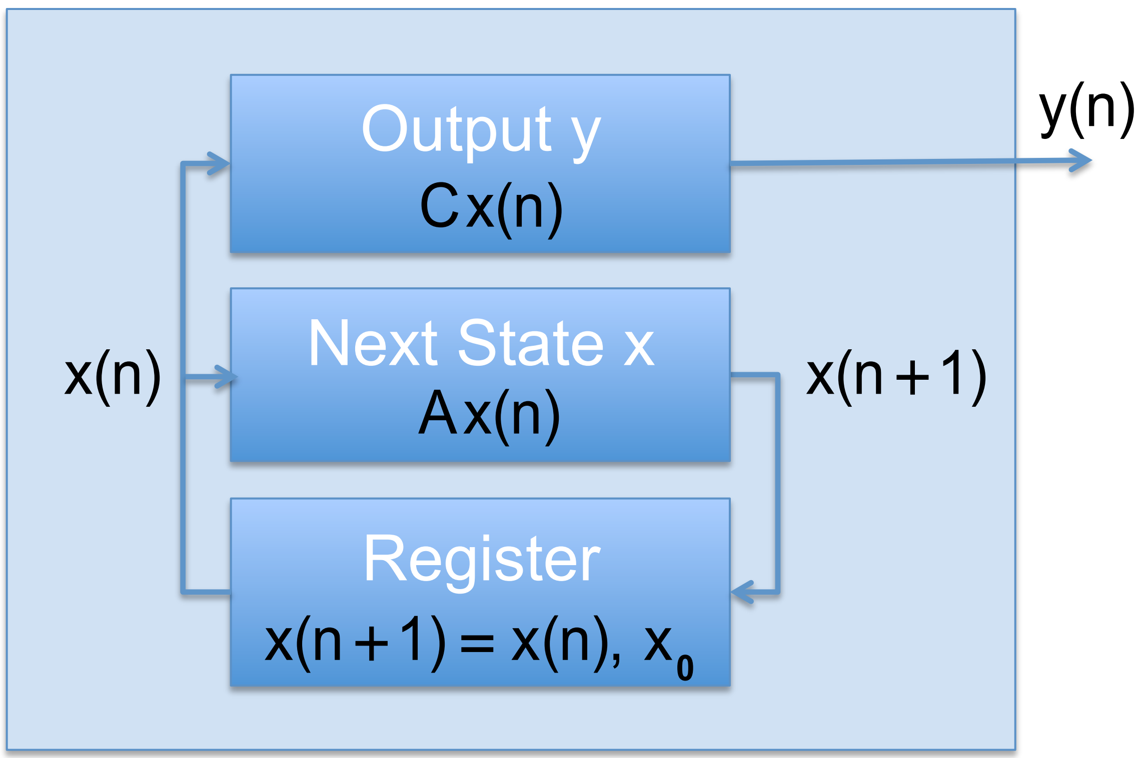
$$\mathbf{x}(0) = \frac{1}{2}\begin{bmatrix} e^{0} + e^{3\times 0} \\ -e^{0} + e^{3\times 0} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

6. Check next infinitesimal state equation:

$$\dot{\mathbf{x}} = \frac{1}{2}d\begin{bmatrix} e^{t} + e^{3t} \\ -e^{t} + e^{3t} \end{bmatrix} / dt = \frac{1}{2}\begin{bmatrix} e^{t} + 3e^{3t} \\ -e^{t} + 3e^{3t} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} e^{t} + e^{3t} \\ -e^{t} + e^{3t} \end{bmatrix} = \mathbf{A}\mathbf{x}$$

Solution of: $x(n+1) = Ax(n),\ y(n) = Cx(n),\ x(0) = x_0$

(Autonomous System)



Solution (fixpoint) of $\mathbf{x}(n+1) = \mathbf{A}\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0$

The solution is: $\mathbf{x}(n) = \mathbf{A}^n \mathbf{x}_0$

Proof:

1) $\mathbf{x}(n+1) = \mathbf{A}^{n+1}\mathbf{x}_0 = \mathbf{A}(\mathbf{A}^n \mathbf{x}_0) = \mathbf{A}\mathbf{x}(\mathbf{n})$
2) $\mathbf{x}(0) = \mathbf{A}^0 \mathbf{x}_0 = \mathbf{I}\mathbf{x}_0 = \mathbf{x}_0$

Change bases: $\bar{\mathbf{x}} = \mathcal{B}^{-1}\mathbf{x} = \mathbf{x}_{\mathcal{B}}$

$$\bar{\mathbf{x}}(n+1) = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)\bar{\mathbf{x}}(n)$$

Consequently: $\bar{\mathbf{x}}(n) = \mathbf{diag}(\lambda_1^n, \ldots, \lambda_n^n)\bar{\mathbf{x}}_0$

Finally: $\mathbf{x}(n) = \mathcal{B}\bar{\mathbf{x}}(n) = \mathcal{B}\,\mathbf{diag}(\lambda_1^n, \ldots, \lambda_n^n)\mathcal{B}^{-1}\mathbf{x}_0$

Example: Fixpoint Computation

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. **Compute solution of** $\mathbf{x}(n+1) = \mathbf{A}\mathbf{x}(n)$.

1. The eigenvectors basis is:

$$\mathcal{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad (\text{columns } \mathbf{b}_1, \mathbf{b}_2)$$

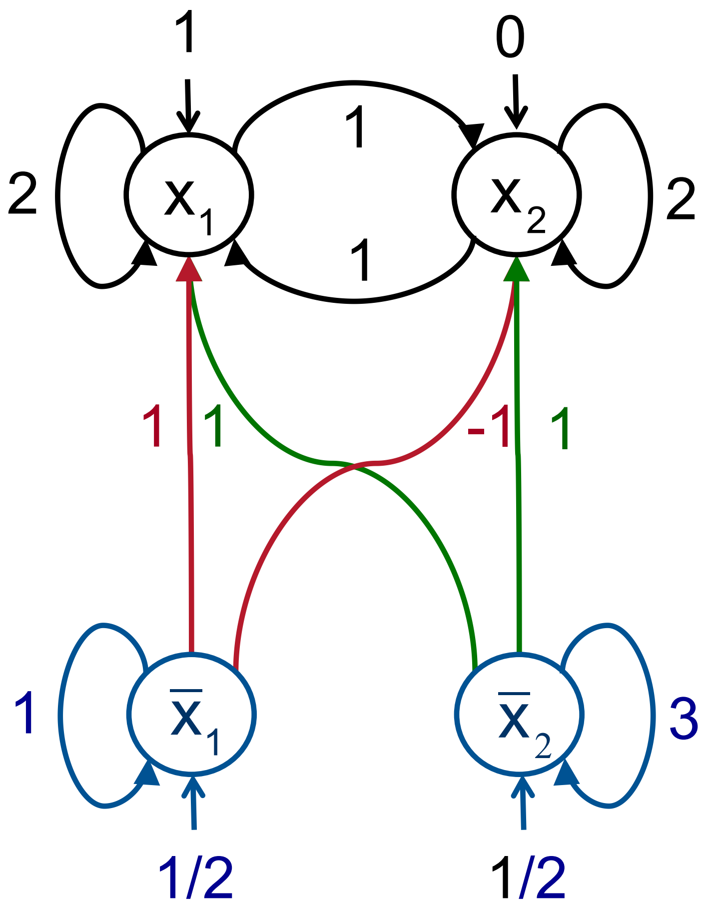
2. The diagonal matrix $\mathcal{B}^{-1}\mathbf{A}\mathcal{B} = (\mathbf{A}\mathcal{B})_{\mathcal{B}}$ **is:**

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

3. The solution is:

$$\mathbf{x}(n) = \mathcal{B}\Lambda^n (x_0)_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + 3^n \\ -1 + 3^n \end{bmatrix}$$

Example: Change of Basis



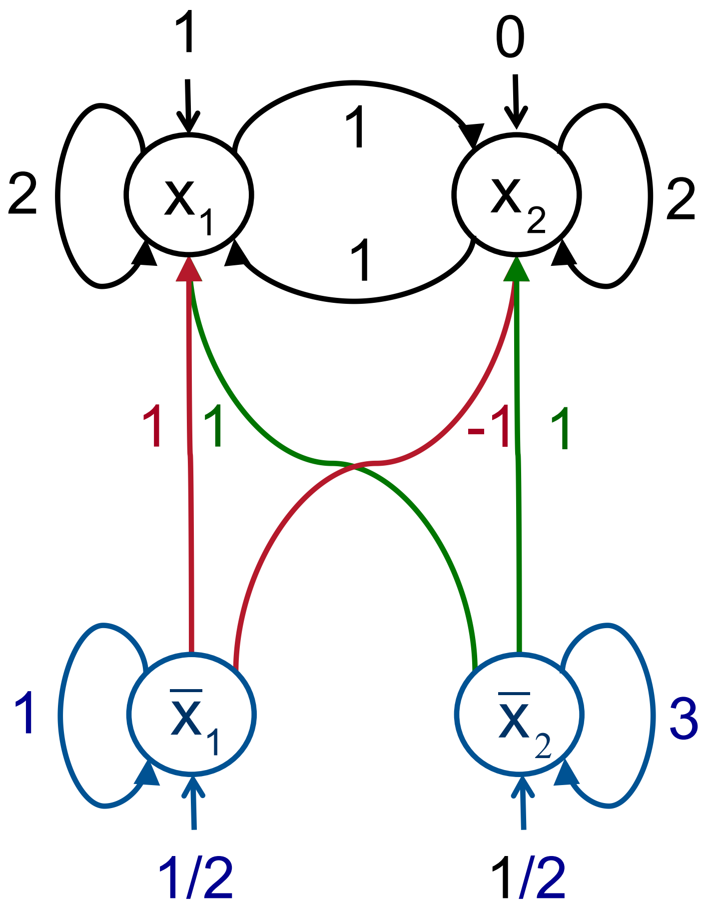
$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad x = \mathcal{B}\bar{x}$$

(columns $\mathbf{b}_1$, $\mathbf{b}_2$)

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Example: Change of Basis



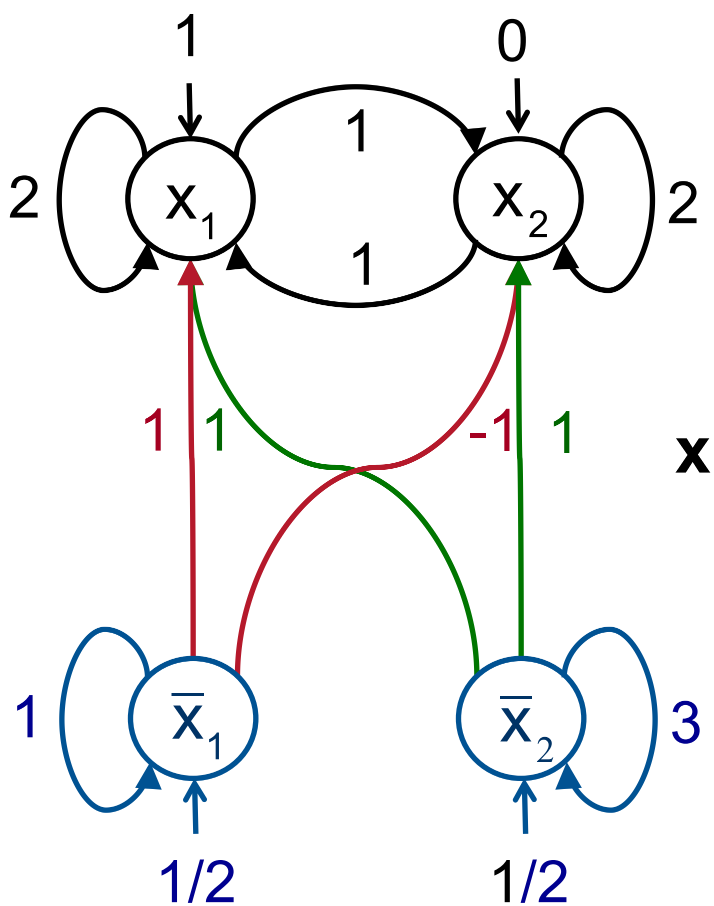
$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0 = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}2\\ 1\end{bmatrix}$$

$\mathbf{b}_1$ $\mathbf{b}_2$

$$\mathbf{x}_1 = \mathcal{B}\bar{\mathbf{x}}_1 = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1/2\\ 3/2\end{bmatrix} = \begin{bmatrix}2\\ 1\end{bmatrix}$$

$$\bar{\mathbf{x}}_1 = \Lambda\bar{\mathbf{x}}_0 = \begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}1/2\\ 1/2\end{bmatrix} = \begin{bmatrix}1/2\\ 3/2\end{bmatrix}$$

Example: Change of Basis



$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = ?$$

$$\mathbf{x}_1 = \mathcal{B}\bar{\mathbf{x}}_1 = \overset{\mathbf{b}_1 \;\; \mathbf{b}_2}{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}} \begin{bmatrix} 1/2 \\ 3^n/2 \end{bmatrix} = \begin{bmatrix} (3^n + 1)/2 \\ (3^n - 1)/2 \end{bmatrix}$$

$$\bar{\mathbf{x}}_n = \Lambda^n \bar{\mathbf{x}}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3^n/2 \end{bmatrix}$$