Complexity Theory and Database Theory

VU 181.142 and VU 181.140, WS 2023

Solutions of Quiz 1

Reinhard Pichler

Institut für Informationssysteme
Arbeitsbereich DBAI
Technische Universität Wien

10 October, 2023

TU WIEN Informatics

Problem Definitions

The Vertex Cover Problem

INSTANCE: An undirected graph $G = (V, E)$ and a positive integer k.

QUESTION: Does there exist a *vertex cover* N of size $\leq k$?
i.e., $N \subseteq V$, s.t. for all $[i, j] \in E$, either $i \in N$ or $j \in N$.

The Dominating Set Problem

INSTANCE: An undirected graph $G = (V, E)$ and a positive integer k.

QUESTION: Does there exist a *dominating set* S of size $\leq k$?
i.e., $S \subseteq V$, s.t. for all $i \in V$, either $i \in S$ or there exists a $j \in S$ with $[i, j] \in E$.

NP-completeness

Theorem

The Dominating Set problem is NP-complete.

NP-hardness proof (by reduction from Vertex Cover)

Let an arbitrary instance of the Vertex Cover problem be given through an undirected graph $G = (V, E)$ and positive integer k with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$.
W.l.o.g., we assume that V contains no isolated nodes (i.e., nodes which are not an endpoint of any edge).

We define an instance $G' = (V', E'), k'$ of Dominating Set as follows:

- $V' = V \cup W$ with $W = \{w_1, \ldots, w_m\}$,
- $E' = E \cup F$ with $F = \{[v_i, w_j] \mid v_i$ is an endpoint of edge e_j in $G\}$,
- $k' = k$.

Question 1

Question 1

Prove the first direction of the correctness of the above problem reduction, namely: If (G, k) is a positive instance of Vertex Cover then (G', k') is a positive instance of Dominating Set, i.e.:

If the graph $G = (V, E)$ has a vertex cover of size k,
then the graph $G' = (V', E')$ has a dominating set of size k'
(with $k' = k$ by the problem reduction).

Solution to Question 1

Suppose that there exists a vertex cover N of G with $|N| \leq k$, i.e., for every $j \in \{1, \dots, m\}$, at least one of the endpoints of $e_j \in E$ is in N.

We define $S = N$. Clearly, we have $|S| = |N| \leq k = k'$.

It remains to show that S is a dominating set of G', i.e., let x be an arbitrary node in $V' = V \cup W$; we show that either $x \in S$ or there exists a vertex y in S with $[x, y] \in E'$.

Solution to Question 1 (continued)

Consider an arbitrary node $x \in V'$. We distinguish two cases:

Case 1. Suppose that $x \in V$. Recall that, in our problem reduction, we assume that V contains no isolated nodes. Hence, x is the endpoint of at least one edge, say e_j. By assumption, N is a vertex cover of G. Hence, at least one of the endpoints of e_j is in N. Thus, either x is in N or the other endpoint, say y, is in N. Since $E \subseteq E'$ (by the problem reduction), either x is in N or there exists y in N with $[x, y] \in E'$.

Case 2. Suppose that $x \in W$. Then x is of the form $x = w_j$ for some $j \in \{1, \ldots, m\}$. By assumption, N is a vertex cover of G. Hence, there exists $y \in N$, s.t. y is an endpoint of edge $e_j \in E$. By the problem reduction, E' contains the edge $[y, w_j] = [y, x]$ (by $w_j = x$).

Since x was arbitarily chosen and since in both cases we either have $x \in N$ or there exists $y \in N$ with $[x, y] \in E'$, we have shown that N (and therefore S) is indeed a dominating set of G'. $\square$

Question 2

Prove the second direction of the correctness of the above problem reduction, namely: If (G', k') is a positive instance of Dominating Set then (G, k) is a positive instance of Vertex Cover, i.e.:

If the graph $G' = (V', E')$ has a dominating set of size k', then the graph $G = (V, E)$ has a vertex cover of size k (with $k' = k$).

Claim

The following property holds:
Let S be an arbitrary dominating set of G' of size k.
Then there exists a dominating set S' of G' of size $\leq k$, s.t. $S' \subseteq V$.

Proof argument for this property: Suppose that S contains a vertex $w_j \in W$ and that edge e_j in G has the form $e_j = [v, v']$. Then the vertex w_j in S only "covers" (i.e., is identical or adjacent to) w_j itself plus the two vertices v and v'. Clearly, if we replace w_j by v (or by v') in S, then we still have a dominating set of G' and its cardinality does not increase. We can carry out this replacement for every vertex in $S \setminus V$ to arrive at the desired dominating set $S' \subseteq V$ with $|S'| \leq |S| \leq k$.

Solution to Question 2

Suppose that there exists a dominating set S of G' with $|S| \leq k'$, i.e., for every $x \in V'$, either $x \in S$ or there exists a $y \in S$ with $[x, y] \in E'$.
By the above claim, we may assume w.l.o.g., that $S \subseteq V$, since otherwise we could transform S into a dominating set S' of G' with $S' \subseteq V$ and $|S'| \leq |S|$.

We define $N = S$. Clearly, we have $|N| = |S| \leq k' = k$.

It remains to show that N is a vertex cover of G, i.e., let e_j be an arbitrary edge of G; we show that at least one endpoint of e_j is in N.

Consider the vertex $w_j \in W \subseteq V'$. Since we are assuming that S is a dominating set of G', the vertex w_j must be "covered" by some vertex in S, i.e., either $w_j \in S$ or S contains some vertex x with $[x, w_j] \in E'$.
Recall that we are assuming w.l.o.g. that $S \subseteq V$. Hence, $w_j \notin S$.
Therefore, there exists a vertex $x \in S$ with $[x, w_j] \in E'$. Since $x \in V$ and $w_j \in W$, we may conclude by the problem reduction that x is an endpoint of the edge e_j in G. But then the edge e_j in G is indeed "covered" by the vertex $x \in S = N$. $\square$