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Montag, 17. Mai 2010
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Sampling and Reconstruction

Overview

- Introduction
- Sampling Theory
  - Fourier Transform
  - Convolution & Convolution Theorem
- Reconstruction
  - Sampling Theorem
  - Reconstruction in theory and practice
- Interpolation - Zero Insertion

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Image Data

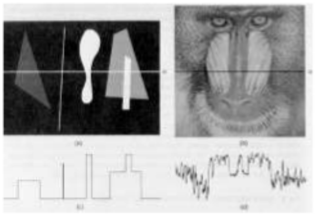

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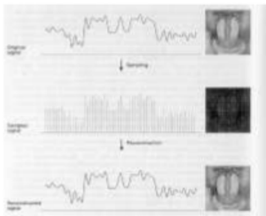
Image Storage and Retrieval



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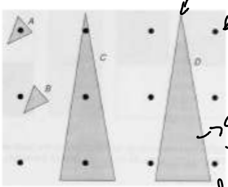
Spitze wird nicht erwischt werden

Sampling Problems

Pixel o. D



→ Dreiecke zw. Samples nicht rekonstruierbar

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Sampling Theory

- Relationship between Signal and Samples
- View Image Data as Signals
- Signals can be plotted as intensity vs. time - spatial domain
- Signals can be represented as sum of sine waves - frequency domain

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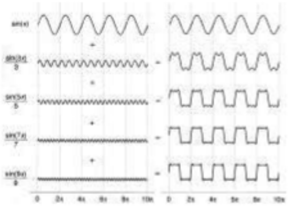
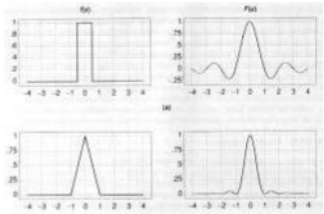
Alle Signale d. Summe v. Sinuswellen darstellbar

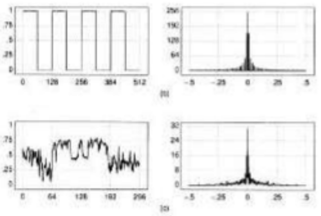
Square Wave Approximation



Box & Tent



Square Wave & Scanline



Fourier Transform

Link between spatial and frequency domain

$$f(x) = \int_{-\infty}^{\infty} F(?)e^{2\pi j ? x} d\omega$$

$$F(?) = \int_{-\infty}^{\infty} f(x)e^{-2\pi j ? x} dx$$

Fourier Transform

- Yields complex functions for frequency domain
- Extends to higher dimensions
- Complex part is phase information - usually ignored

Alternative: Hartley transform

Discrete Fourier Transform

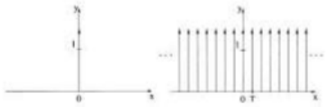
For discrete signals (i.e. sets of samples)

$$f(x) = \sum_{?=0}^{N-1} F(?) \cdot e^{2\pi j ? x / N}$$

$$F(?) = \frac{1}{N} \sum_{?=0}^{N-1} f(x) \cdot e^{-2\pi j ? x / N}$$

N samples: O(N^2) complexity
Fast FT (FFT): O(N log N)

Base Functions



Dirac Pulse Comb Function

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FT of Base Functions

- Impulse function: constant 1, i.e. equal energy at all frequencies
- Comb function: comb with reciprocal spacing

$$\mathrm{comb}_T(x) \Leftrightarrow \mathrm{comb}_{1/T}(?)$$

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Convolution

- Operation on two functions
- Produces a new function which is a sliding weigthed average of a function. The second function provides the weights.

$$(f_1 * f_2)(x) = \int_{-\infty}^{\infty} f_1(x') f_2(x - x') dx'$$

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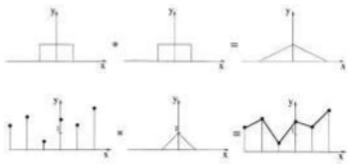
Convolution - Examples



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Convolution Theorem

The spectrum of the convolution of two functions is equivalent to the product of the transforms of both input signals, and vice versa.

$$f_1 * f_2 \equiv F_1 F_2$$
$$F_1 * F_2 \equiv f_1 f_2$$

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Example - Low-Pass

Low-pass filtering performed on Mandrill scanline

Spatial domain: convolution with sinc function

Frequency domain: cutoff of high frequencies - multiplication with box filter

Sinc function corresponds to box function and vice versa!

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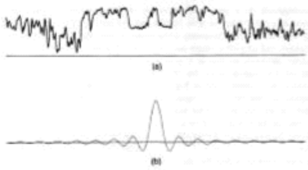
Low-Pass in Spatial Domain 1



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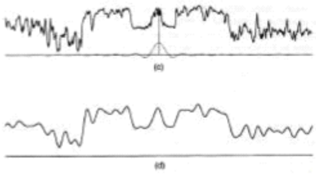
Low-Pass in Spatial Domain 2



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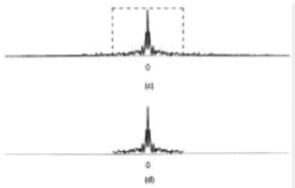
Low-Pass in Frequency Domain



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Sampling

The process of sampling is a multiplication of the signal with a comb function.

$$f_s(x) = f(x) \cdot \mathrm{comb}_T(x)$$

The frequency response is convolved with a transformed comb function.

$$F_s(\omega) = F(\omega) * \mathrm{comb}_{1/T}(\omega)$$

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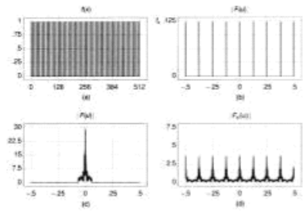
Spectrum of a Sampled Signal



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Reconstruction

Recovering the original function from a set of samples

- Sampling theorem
- Ideal reconstruction
  - Sinc function
- Reconstruction in practice

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Definitions

- A function is called *band-limited* if it contains no frequencies outside the interval [-u,u]. u is called the *bandwidth* of the function
- The *Nyquist frequency* of a function is twice its bandwidth, i.e. w = 2u

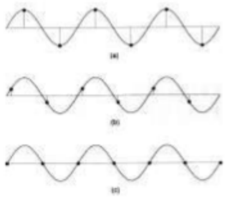
Sampling Theorem

A function f(x) that is

- band-limited and
- sampled above the Nyquist frequency

is completely determined by its samples.

Sampling at Nyquist Frequency



Sampling Below Nyquist *f*

Ideal Reconstruction

- Replicas in frequency domain must not overlap
- Multiplying the frequency response with a box filter of the width of the original bandwidth restores original
- Amounts to convolution with Sinc function

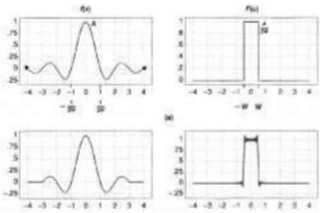
Sinc function

- Infinite in extent
- Ideal reconstruction filter
- FT of box function

$$\mathrm{sinc}(x) = \begin{cases} \frac{\sin \pi x}{\pi x} & \mathit{if}\ x \neq 0 \\ 1 & \mathit{if}\ x = 0 \end{cases}$$

Prob: sinc ist unendlich
Faltung nicht möglich
1. sinc Filter abschneiden?
im Ortsraum

Sinc & Truncated Sinc



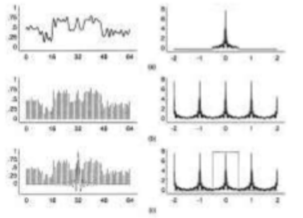
Reconstruction: Examples

Sampling and reconstruction of the Mandrill image scanline signal

- with adequate sampling rate
- with inadequate sampling rate
- demonstration of band-limiting

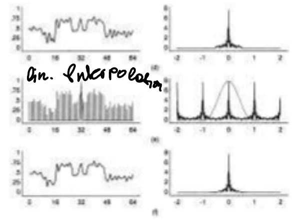
With Sinc and tent reconstruction kernels

Adequate Sampling Rate

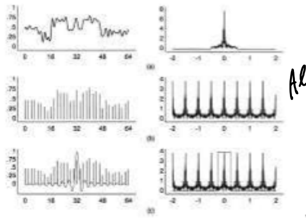


Adequate Sampling Rate

lin. Interpolation

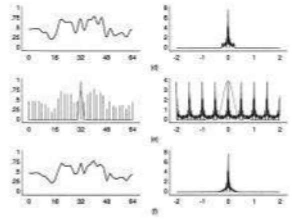


Inadequate Sampling Rate

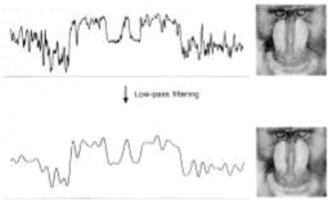


Aliasing

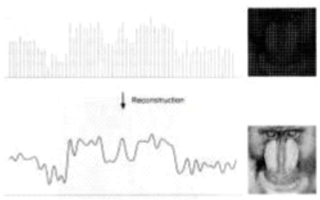
Inadequate Sampling Rate



Band-Limiting a Signal



Band-Limiting a Signal



Reconstruction in Practice

Problem: which reconstruction kernel should be used?

- Genuine Sinc function unusable in practice
- Truncated Sinc often sub-optimal
- Various approximations exist; none is optimal for all purposes

Tasks of Reconstruction Filters

- Remove the extraneous replicas of the frequency response
- Retain the original undistorted frequency response

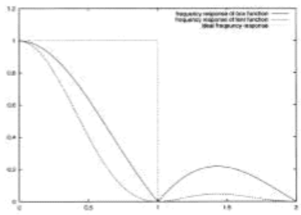
Nearest Neighbour mit rechtecksfkt gefaltet

Used Reconstruction Filters

- Nearest neighbour
- Linear interpolation
- Symmetric cubic filters
- Windowed Sinc

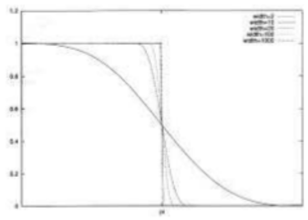
  More sophisticated ways of truncating the Sinc function

Box & Tent Responses



Konstruktionskern bei linearen Interpolation: Faltung mit Dreiecksfunktion
FFT der Dreiecksfunktion:sinc
fkt*sincfkt=sinc^2
Dreiecksfunktion=zweimal gefaltete Rechtecksfunktion
FFt einer Faltung=Multiplikation
Hohe Frequenzen stärker geglättet

Windowed Sinc Responses



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Sampling & Reconstruction Errors

- Aliasing: due to overlap of original frequency response with replicas - information loss
- Truncation Error: due to use of a finite reconstruction filter instead of the infinite Sinc filter
- Non-Sinc error: due to use of a reconstruction filter that has a shape different from the Sinc filter

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Nich interessiert eine Ableitung, nicht die Funktion.
Faltung mit der Ableitung der sinc Fkt (anderer faltungskern: cosc)

Interpolation - Zero Insertion

Operates on series of *n* samples
Takes advantage of DFT properties

- Perform DFT on series
- Append zeros to the sequence
- Perform the inverse DFT

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Zero Insertion - Properties

- Preserves frequency spectrum
- Original signal has to be sampled above Nyquist frequency
- Values can only be interpolated at evenly spaced locations
- The whole series must be accessible, and it is always completey processed

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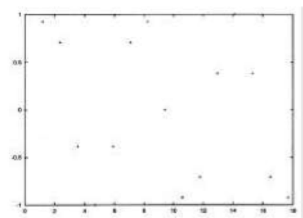
Interpolation über den Frequenzraum

Zero Insertion - Original Series



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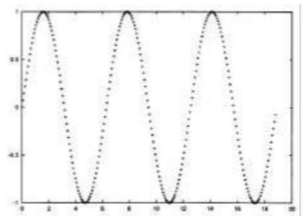
Zero Insertion - Interpolation



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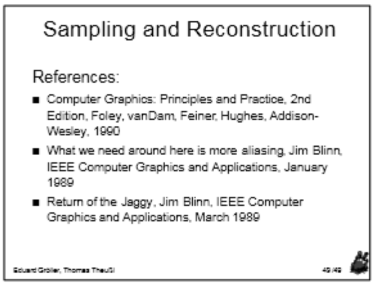

Conclusion
Sampling
Going from continous to discrete signal
Multiplication with comb funktion
Sampling theorem: how many samples are needed

Reconstruction
Sinc is ideal filter but not practicable
Reconstruction in practise
aliasing

Was ist die grenzfrequenz eines signals?
Sinusschwingung in der höchsten frequenz die ich brauche
Im Frequenzraum: spektrum. Wo sinus die gerade schneidet n_

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