

Problem 1

(1) Car licence plates

A certain state's car licence plates have four letters of the alphabet followed by a four-digit number.

- (a) How many different licence plates are possible if all four-letter sequences are permitted and any number from 0000 to 9999 is allowed?
- (b) Mary witnessed a hit-and-run accident. She knows that the first letter on the licence plate of the offender's car was a T, that the second letter was an A or H or M, and that the last number was a 9. How many state's licence plates fit this description?

- (a) for this problem we have to choose 4 letters out of 26, in this sequence the letters can occur more than once so the sequence could be "aaaa" or "abcd" etc.

each position has 26 possibilities to build the sequence we have $26^4 = 456976$ possibilities. The number sequence can be built in the same way, we have four numbers and choose out of 10 possible (0 - 10), that means there are $10^4 = 10000$ possibilities.

In conclusion there are $456976 \cdot 10000 = 4569760000$ possible license plates.

- (b) In this case the first letter is fixed and the second letter is chosen out of three letters the last to can again be chosen out of 26 letters. The number sequence consists now out of three random numbers and the last one is fixed.

$1 \cdot 3 \cdot 26^2 \cdot 10^3 \cdot 1 = 2028000$ license plates fit that description.

Problem 2

(2) Symphony orchestra program

A symphony orchestra has in its repertoire 30 Haydn symphonies, 15 modern works, and 9 Beethoven symphonies. Its program always consists of a Haydn symphony followed by a modern work, and then a Beethoven symphony.

- (a) Assume that each piece can be played more than once. How many different programs can it play? How many different programs are there if the three pieces can be played in any order?
 - (c) Assume that each piece cannot be played more than once. How many different three-piece programs are there if more than one piece from the same category can be played and they can be played in any order?
- (a) The program always consists out of the set $\{H, M, B\}$ in that order which can be played **more** than once: $30 \cdot 15 \cdot 9 = 4050$ different programs.
If the pieces can be played in any order we can permute the set which gives us: $3! \cdot 4050 = 24300$ different programs.
- (b) In this case we have to combine and permute the set of playable pieces. We can use the binomial coefficient to compute the number of different programs. There are $n = 30 + 15 + 9 = 54$ different pieces to choose from, we choose $k = 3$ pieces from that list.

$$\binom{n}{k} = \frac{54!}{(54-3)! \cdot 3!}$$

which gives us 24804 possibilities when the order is relevant, but we want to play the pieces in **any** order so we have to permute the set: $3! \cdot 24804 = 148824$

Problem 3

(3) **Poker game**

A deck of 52 cards has 13 ranks (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A) and 4 suits (\heartsuit , \spadesuit , \diamondsuit , \clubsuit). A poker hand is a set of 5 cards randomly chosen from a deck of 52 cards.

- (a) A full house in poker is a hand where three cards share one rank and two cards share another rank. How many ways are there to get a full-house? What is the probability of getting a full-house?
- (b) A royal flush in poker is a hand with ten, jack, queen, king, ace in a single suit. What is the probability of getting a royal flush?

- (a) A full house consists of a triple and a pair of cards.

The possible ways to get a triple can be computed by choosing one of the possible ranks and choose three out of 4 suits.

$$\binom{13}{1} \cdot \binom{4}{3} = 52$$

The same can be done for the pair. But we only have 12 ranks left and only choose 2 suits.

$$\binom{12}{1} \cdot \binom{4}{2} = 72$$

So there are $52 \cdot 72 = 3744$ possibilities for a full house.

The compute the possibility to get a full house we have to compute the total number of hands.

$$\binom{52}{5} = 2598960$$

$$P(\text{FullHouse}) = \frac{3744}{2598960} = 1.44... \cdot 10^{-3} = 0.144..%$$

- (b) A royal flush can only be received in **four** possible ways (there are only four suits), so there are four possible hands.

$$\frac{4}{2598960} = 1.539... \cdot 10^{-6}$$

Problem 4

(4) Coin game

Two players, A and B, alternately and independently flip a coin and the first player to obtain a head wins. Assume player A flips first. Suppose that $P(\text{head}) = p$, not necessarily $\frac{1}{2}$. What is the probability that the player B wins?

First we consider the possibility that B wins in the first round: $(1-p) \cdot p$. If B loses in the first round they need to flip the coin again: $(1-p) \cdot p + (1-p) \cdot (1-p) \cdot (1-p) \cdot p$ and so on, therefore we can build a sum.

$$\sum_{n=0}^{\infty} (1-p)^{2n+1} \cdot p = p \cdot \sum_{n=0}^{\infty} (1-p)^{2n+1}$$

$$= p \cdot -\frac{1-p}{(-2+p) \cdot p} = \frac{1-p}{2-p}$$

$$s_n = (1-p) + (1-p)^3 + \dots + (1-p)^{2n+1}$$

$$(1-p)^2 \cdot s_n = (1-p)^3 + \dots + (1-p)^{2n+3}$$

$$s_n - (1-p)^2 \cdot s_n = (1-p) - (1-p)^{2n+3}$$

$$s_n \cdot (1 - (1-p)^2) = (1-p) - (1-p)^{2n+3} \rightarrow s_n \cdot (-p^2 + 2p) = (1-p) - (1-p)^{2n+3}$$

$$s_n = \frac{(1-p) - (1-p)^{2n+3}}{(-p^2 + 2p)} \rightarrow \lim_{n \rightarrow \infty} \frac{(1-p) - (1-p)^{2n+3}}{(-p^2 + 2p)}$$

Problem 5

(5) Student athletes

A random sample of 400 college students was asked if college athletes should be paid. The following table gives a two-way classification of the responses.

	Should be paid	Should not be paid
Student athlete	80	20
Student nonathlete	220	80

- (a) If one student is randomly selected from these 400 students, find the probability that this student
- is in favor of paying college athletes
 - is an athlete and favors paying student athletes
 - is a nonathlete or is against paying students athletes
- (b) Are the events *student athlete* and *should be paid* mutually exclusive? Justify your answer.

- (a) i. out of 400 student 300 are in favor for paying college athletes.
 $\frac{300}{400} = 0.75 = 75\%$
- ii. out of 400 students 80 are athletes **and** in favor for paying athletes.
 $\frac{80}{400} = 0.2 = 20\%$
- iii. out of 400 students 100 are not in favor for paying college athletes and 300 are nonathlete. To compute that probability we have to find out the number that are both and subtract that number from the overall number (inclusion-exclusion).

A... nonathletes $\rightarrow 300$

B... not in favor $\rightarrow 100$

$$|A \cup B| = |A| + |B| - |A \cap B| = 300 + 100 - 80 = 320$$

$$\frac{320}{400} = 0.8 = 80\%$$

- (b) mutually exclusive would mean that the intersection of A & B is the empty set ($A \cap B = \emptyset$). In the table above we can see that 80 students are athlete **and** in favor for paying college students. So these events are **not** mutually exclusive!

Problem 6

(6) Binomial coefficients

- (a) Prove that $\binom{n}{j} = \binom{n}{n-j}$ holds for $n \in \mathbb{N}$ and $0 \leq j \leq n$.
 (b) Prove that $\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$ holds for $n \geq 2$ and $0 < j < n$.
 (c) Find integers n and r such that the equation $\binom{13}{5} + 2\binom{13}{6} + \binom{13}{7} = \binom{n}{r}$ is true.

(a)

$$\binom{n}{j} = \frac{n!}{j! \cdot (n-j)!}$$

$$\binom{n}{n-j} = \frac{n!}{(n-j)! \cdot (n-(n-j))!} = \frac{n!}{(n-j)! \cdot j!}$$

(b)

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$$

$$= \frac{(n-1)!}{(n-1-j)! \cdot j!} + \frac{(n-1)!}{(n-1-(j-1))! \cdot (j-1)!}$$

$$= \frac{(n-1)!}{(n-j-1)! \cdot j!} + \frac{(n-1)!}{(n-j)! \cdot (j-1)!}$$

$$= \frac{(n-1)! \cdot (n-j)}{(n-j-1)! \cdot j! \cdot (n-j)} + \frac{(n-1)! \cdot j}{(n-j)! \cdot (j-1)! \cdot j}$$

$$= \frac{(n-1)! \cdot (n-j)}{(n-j)! \cdot j!} + \frac{(n-1)! \cdot j}{(n-j)! \cdot j!}$$

$$= \frac{(n-1)! \cdot (n-j) + (n-1)! \cdot j}{(n-j)! \cdot j!}$$

$$= \frac{(n-1)! \cdot (n-j+j)}{(n-j)! \cdot j!} = \frac{(n-1)! \cdot n}{(n-j)! \cdot j!}$$

$$= \frac{n!}{j! \cdot (n-j)!} = \binom{n}{j}$$

(c)

$$\text{Notice : } \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

$$\binom{n}{r} = \binom{13}{5} + 2 \cdot \binom{13}{6} + \binom{13}{7}$$

$$= \binom{13}{5} + \binom{13}{6} + \binom{13}{6} + \binom{13}{7}$$

$$= \binom{14}{6} + \binom{14}{7} = \binom{15}{7}$$