a) Energy used by cells per day:

$$3000 \text{ kcal} \times 40\% = 1200 \text{ kcal/day}$$

Moles of ATP per day:

$$1200 \text{ kcal/day} \times \frac{1}{12.5 \text{ kcal/mole}}$$

$$= 96 \text{ moles/day}$$

Weight of ATP:

$$96 \text{ moles/day} \times 507 \text{ g/mole}$$

$$= 48672 \text{ g of ATP/day}$$

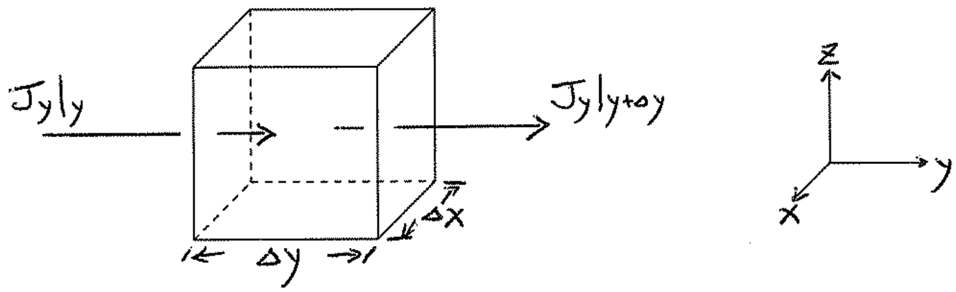
b) Recycle time:

$$5\text{ g} \times \frac{1}{48672 \text{ g/day}}$$

$$= 5\text{ g} \times \frac{1}{48672 \text{ g}/24(3600)\text{ s}}$$

$$= 8.88 \text{ s}$$

a)



where J is the solute flux ($[J]$ = mass/time/area)

By mass balance:

$$-\Delta x \Delta y \Delta z \frac{\partial c}{\partial t} = (J_x|_{x+\Delta x} - J_x|_x)\Delta y \Delta z + (J_y|_{y+\Delta y} - J_y|_y)\Delta x \Delta z + (J_z|_{z+\Delta z} - J_z|_z)\Delta x \Delta y$$

$$\Rightarrow -\frac{\partial c}{\partial t} = \frac{J_x|_{x+\Delta x} - J_x|_x}{\Delta x} + \frac{J_y|_{y+\Delta y} - J_y|_y}{\Delta y} + \frac{J_z|_{z+\Delta z} - J_z|_z}{\Delta z}$$

In the limit of $\Delta x, \Delta y, \Delta z \rightarrow 0$

$$-\frac{\partial c}{\partial t} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$

Also, by Fick's law, solute fluxes can be written as:

$$J_x = -D\frac{\partial c}{\partial x} \quad ; \quad J_y = -D\frac{\partial c}{\partial y} \quad ; \quad J_z = -D\frac{\partial c}{\partial z}$$

Therefore

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial c}{\partial z}\right)$$

$$\Rightarrow \frac{\partial c}{\partial t} = \nabla \cdot (D \nabla c)$$

q.e.d.

b)

Initial condition (I.C.):

at $t=0$ $\quad c(x,t) = C_{init}$

Boundary Condition (B.C.):

at $t>0$ $\quad c(x=0) = C_{init}$, $c(x=L)=0$

For equation 2.71 to satisfy I.C. and B.C., it requires conditions on the coefficients a_n and b_n:

$$\text{I.C.:} \quad \frac{x}{L} = \sum_{n=1}^{\infty}\left[a_n \sin\frac{2\pi n x}{L} + b_n \cos\frac{2\pi n x}{L}\right]$$

$$\text{B.C.:} \quad \text{at } x=0 \Rightarrow \sum_{n=1}^{\infty} b_n e^{-(2\pi n)^2 Dt/L^2} = 0$$

$$\text{at } x=L \Rightarrow \sum_{n=1}^{\infty} b_n e^{-(2\pi n)^2 Dt/L^2} = 0$$

From the above, we deduce $b_n=0$ for all n,

and $\frac{x}{L} = \sum_{n=1}^{\infty} a_n \sin\frac{2\pi n x}{L}$

Aside: to get the a_n, we use Fourier theory:

$$\int_0^L \sin\frac{2\pi m x}{L}\cdot\frac{x}{L}\,dx = \sum_{n=1}^{\infty} a_n \int_0^L \sin\frac{2\pi n x}{L}\sin\frac{2\pi m x}{L}\,dx$$

$$\frac{L}{(2\pi m)^2}\int_0^{2\pi m} \theta\sin\theta\,d\theta = \sum_{n=1}^{\infty} a_n \delta_{mn}\cdot\frac{L}{2}$$

$$\frac{-L}{(2\pi m)^2}\left(\theta\cos\theta - \sin\theta\right)\Big]_0^{2\pi m} = a_m\cdot\frac{L}{2}$$

$$\Rightarrow \frac{2}{L}\frac{-L\,2\pi m}{(2\pi m)^2} = a_m \quad \Rightarrow \quad a_m = \frac{-1}{\pi m}$$

b) continued...

equation 2.70 : $\frac{\partial c}{\partial t} = \nabla \cdot (D \nabla c) = \frac{\partial}{\partial x}\left(D \frac{\partial c}{\partial x}\right)$

for 1-D system.

Show 2.71 satisfies 2.70 :

L.H.S :

$$\frac{\partial c}{\partial t} = C_{init}\left[\sum_{n=1}^{\infty}\left[a_n \sin\frac{2\pi n x}{L} + b_n \cos\frac{2\pi n x}{L}\right]\left(-\frac{(2\pi n)^2 D}{L^2}\right) e^{-(2\pi n)^2 Dt/L^2}\right]$$

R.H.S :

$$D\frac{\partial c}{\partial x} = D\cdot C_{init}\left[-\frac{1}{L} + \sum_{n=1}^{\infty}\left(a_n\left(\frac{2\pi n}{L}\right)\cos\left(\frac{2\pi n x}{L}\right) - b_n\left(\frac{2\pi n}{L}\right)\sin\left(\frac{2\pi n x}{L}\right)\right) e^{-(2\pi n)^2 Dt/L^2}\right]$$

$$= D\cdot C_{init}\left[-\frac{1}{L} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2\pi n x}{L} - b_n \sin\frac{2\pi n x}{L}\right]\left(\frac{2\pi n}{L}\right) e^{-(2\pi n)^2 Dt/L^2}\right]$$

$$\frac{\partial}{\partial x}D\frac{\partial c}{\partial x} = D\cdot C_{init}\left[\sum_{n=1}^{\infty}\left[-a_n\left(\frac{2\pi n}{L}\right)\sin\frac{2\pi n x}{L} - b_n\left(\frac{2\pi n}{L}\right)\cos\frac{2\pi n x}{L}\right]\left(\frac{2\pi n}{L}\right) e^{-(2\pi n)^2 Dt/L^2}\right]$$

$$= C_{init}\left[\sum_{n=1}^{\infty}\left[a_n \sin\frac{2\pi n x}{L} + b_n \cos\frac{2\pi n x}{L}\right]\left(-\frac{(2\pi n)^2 D}{L^2}\right) e^{-(2\pi n)^2 Dt/L^2}\right]$$

$\therefore$ L.H.S = R.H.S

$\therefore$ 2.71 satisfies 2.70

c)

To minimize the decay of the exponential term, n must be minimum $\longrightarrow n=1$

$$\therefore \exp[-(2\pi n)^2 Dt/L^2] = \exp[-(2\pi)^2 Dt/L^2]$$

$$= \exp(-t/\tau]$$

$$\therefore \tau = \frac{L^2}{(2\pi)^2 D}$$

If $D = 4.5\times10^{-10}\ m^2/s$ and $L = 0.5\,\mu m$,

$$\tau = \frac{(0.5\times10^{-6}\ m)^2}{(2\pi)^2 \cdot 4.5\times10^{-10}\ m^2/s} = 1.407\times10^{-5}\ s$$

$$= 14.07\ \mu s$$

d)

Muscle produces 40 W/kg of muscle

To be on the safe side, use a time-lag of $3\times\tau = 42.21\ \mu s$

During that 3τ time lag, muscle needs:

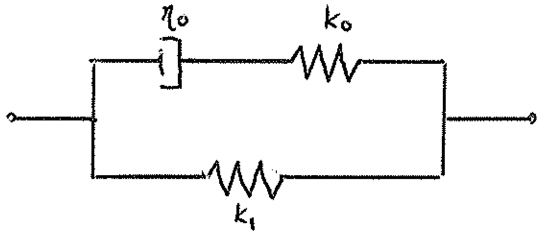
$$40\ W/\text{kg of muscle} \times 42.21\times10^{-6}\ s = 1.69\times10^{-3}\ J/\text{kg of muscle}$$

Since 1 mole of ATP (507 g) yields 104 J, we need:

$$\frac{1.69\times10^{-3}\ J/\text{kg of muscle}}{104\ J/\text{mole of ATP}} \times 507\ g/\text{mole of ATP}$$

$$= 8.2\times10^{-3}\ g \text{ of ATP}/\text{kg of muscle}$$

Therefore, 8.2 mg of ATP is needed per kg of muscle to safely overcome the time lag until diffusion begins transporting the ATP.



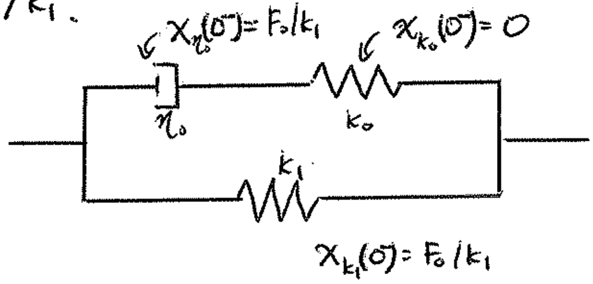
Kelvin body

From equation 2.17, $\frac{x(t)}{F_o} = \frac{1}{k_1}\left(1 - \frac{k_o}{k_o + k_1} e^{-t/\tau}\right)$,

we notice that at steady state the length of the Kelvin body is $x_{final} = \frac{F_o}{k_1}$.

This implies that at steady state, all of the force is being carried in the lower spring with constant k_1. This makes sense because if there were any force in the upper leg of the Kelvin body, the dashpot would be moving and the system would not be at steady state.

Therefore, before the force is released, the length of the upper spring must be zero (since there is no force), and the length of the dashpot and the lower spring are both F_o/k_1.



Note: All lengths are measured with respect to the resting condition (i.e. no external force on the Kelvin body).

continued . . .

When the load is released, the length of dashpot stays at F_o/k_1 at $t=0^+$ (i.e. $x_{\eta_o}(0^+) = F_o/k_1$)

By a geometrical constraint, the lengths of the 2 legs of the kelvin body must be the same.
Therefore,

$$x_{k_1}(0^+) = x_{k_o}(0^+) + x_{\eta_o}(0^+)$$
$$= x_{k_o}(0^+) + F_o/k_1 \quad -①$$

Since the dashpot doesn't change length at $t=0^+$, the only way this can happen is if the upper spring is in compression ($x_{k_o}(0^+) < 0$) and the lower spring is in tension ($x_{k_1}(0^+) > 0$).
We also observe that the force experienced by the dashpot and the spring with constant k_o are the same, since they are in the same 'leg'. Since there is no net external force on the Kelvin body, a force balance gives:
$k_o x_{k_o}(0^+) + k_1 x_{k_1}(0^+) = 0 \Rightarrow x_{k_1}(0^+) = -\frac{k_o}{k_1} x_{k_o}(0^+) \quad -②$

Also, from ①, $x_{k_o}(0^+) = x_{k_1}(0^+) - F_o/k_1$

$\therefore$ ② $\Rightarrow x_{k_1}(0^+) = -\frac{k_o}{k_1}\left(x_{k_1}(0^+) - F_o/k_1 \right)$

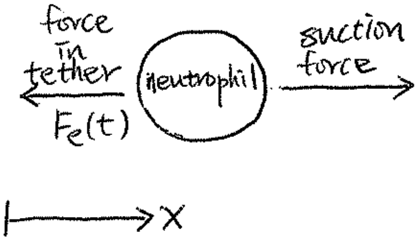
$$\Rightarrow x_{k_1}(0^+) = \frac{k_o F_o}{k_1(k_1+k_o)}$$

Since the length of the lower spring is the length of the kelvin body, $x_{k_1}(0^+)$ gives the initial condition $\Rightarrow x(0^+) = \frac{k_o F_o}{k_1(k_1+k_o)}$

The displacement for all time is then just $\underline{x(t) = \frac{k_o F_o}{k_1(k_1+k_o)} e^{-t/\tau}}$

a) Given $x(t) = x_{final}(1 - e^{-t/\tau})$

Let $u \equiv$ velocity of neutrophil



$$u = \frac{dx}{dt} = \frac{x_{final}}{\tau} e^{-t/\tau}$$

$$\text{Initial velocity} = u(0) = \frac{x_{final}}{\tau}$$

b) Force balance on neutrophil : $\xrightarrow{+} \sum F_x = m a_x = m \frac{du}{dt}$

$$\frac{du}{dt} = -\frac{x_{final}}{\tau^2} e^{-t/\tau}$$

$$F_{suction} - F_e(t) = m \frac{du}{dt} = -m \cdot \frac{x_{final}}{\tau^2} \cdot e^{-t/\tau}$$

$$\Rightarrow F_e(t) = F_{suction} + m \cdot \frac{x_{final}}{\tau^2} \cdot e^{-t/\tau}$$

From equation 2.6, $F_{suction} = \Delta p \pi R_p^2 \left(1 - \frac{u}{u_\infty}\right)$

and $u_\infty = u(0) = x_{final}/\tau \quad \Rightarrow F_{suction} = \Delta p \pi R_p^2 (1 - e^{-t/\tau})$

$$\therefore F_e(t) = \Delta p \pi R_p^2 (1 - e^{-t/\tau}) + m \cdot \frac{x_{final}}{\tau^2} e^{-t/\tau}$$

c) The second term of the above equation represents inertia. For it to be ignored for all time, we need:

$$m \cdot \frac{x_{final}}{\tau^2} \cdot e^{-t/\tau} << \Delta p \pi R_p^2 e^{-t/\tau} \quad \text{or} \quad \frac{\pi R_p^2 \Delta p \tau^2}{m x_{final}} >> 1$$

d) If we neglect inertia $\Rightarrow F_e = \Delta p \pi R_p^2 (1 - e^{-t/\tau})$

$$\text{Spring constant } k = \frac{F_e}{x} = \frac{\Delta p \pi R_p^2 (1 - e^{-t/\tau})}{x_{final}(1 - e^{-t/\tau})} = \frac{\Delta p \pi R_p^2}{x_{final}}$$

a)

For a unit cell model shown in Fig. 2-42, beam deflection $\delta \propto \frac{F l^3}{E_s I}$,

where l is the side length.

Since we are interested in unit volume, set $l=1$

$$\therefore \delta \propto \frac{F}{E_s I} \qquad -①$$

Energy stored in the network, W, is the same as the work done to the network.

$$\text{Work done} = F \cdot \delta = W \qquad -②$$

Combine ① and ②, we get

$$W \propto \frac{F^2}{E_s I}$$

b)

Given :- Energy stored : $W = E^* \varepsilon^2/2$

-Equation 2.44 : $\frac{\rho^*}{\rho_s} \propto \left(\frac{t}{l}\right)^2$

-From a), $W \propto \frac{F^2}{E_s I}$

$$\Rightarrow E^* \varepsilon^2 \propto \frac{F^2}{E_s I}$$

$$\Rightarrow \frac{E^*}{E_s} \propto \frac{F^2}{E_s^2 I \varepsilon^2}$$

, but $F \propto \sigma l^2$
where σ is the externally imposed stress
and $I \propto t^4$
where t is the fibre thickness

$$\Rightarrow \frac{E^*}{E_s} \propto \frac{\sigma^2 l^4}{E_s^2 t^4 \varepsilon^2} = \left(\frac{\sigma}{\varepsilon}\right)^2 \frac{1}{E_s^2} \cdot \left(\frac{l}{t}\right)^4$$

$$= E^{*2} \cdot \frac{1}{E_s^2} \cdot \left(\frac{l}{t}\right)^4$$

$$\Rightarrow \frac{E^*}{E_s} \propto \left(\frac{E^*}{E_s}\right)^2 \left(\frac{l}{t}\right)^4$$

$$\Rightarrow \frac{E^*}{E_s} \propto \left(\frac{t}{l}\right)^4 \propto \left(\frac{\rho^*}{\rho_s}\right)^2$$

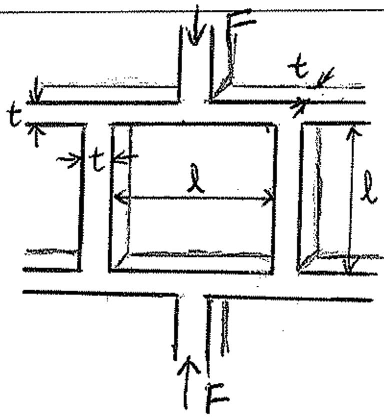
, because $\frac{\rho^*}{\rho_s} \propto \left(\frac{t}{l}\right)^2$

$$\therefore \frac{E^*}{E_s} = C_1 \left(\frac{\rho^*}{\rho_s}\right)^2$$

, which is equation 2.47

q.e.d.

a)



l is the side length
t is the fibre thickness
F is the externally applied force

b)

$F = \sigma l t$ ①, where σ is the externally imposed stress

$$\frac{\rho^*}{\rho_s} \propto \frac{lt}{l^2} \propto \frac{t}{l}$$

and $$\delta \propto \frac{Fl^3}{E_s I} \propto \frac{Fl^3}{E_s t^4}$$ from equation 2.45, and $I \propto t^4$

also, $$\varepsilon = \frac{\delta}{l} \propto \frac{Fl^2}{E_s t^4} \propto \frac{\sigma l^3}{E_s t^3}$$, by ①

but $$\varepsilon = \frac{\sigma}{E^*}$$

$$\therefore \frac{\sigma}{E^*} \propto \frac{\sigma l^3}{E_s t^3} \Rightarrow \frac{1}{E^*} \propto \frac{1}{E_s}\left(\frac{l}{t}\right)^3$$

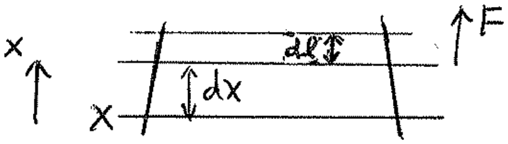
$$\Rightarrow \frac{E^*}{E_s} \propto \left(\frac{t}{l}\right)^3 \propto \left(\frac{\rho^*}{\rho_s}\right)^3$$ q.e.d.

From the text,

$$\frac{E^*}{E_s} = C_1\left(\frac{\rho^*}{\rho_s}\right)^2 \quad \text{, and } C_1 = 1$$

$$\Rightarrow \quad \frac{E^*}{E_s} = \left(\frac{\rho^*}{\rho_s}\right)^2$$

Given, $\rho^*(x) = \rho_0 e^{-kx} \Rightarrow \dfrac{E^*}{E_s} = \left(\dfrac{\rho_0}{\rho_s}\right)^2 e^{-2kx}$



l is the total change in vertical length

By definition, $\dfrac{\sigma(x)}{E^*(x)} = \dfrac{dl}{dx}$

$$\Rightarrow \quad dl = \frac{\sigma(x)}{E^*(x)}\, dx.$$

Integrate both sides,

$$l = \int_0^{4\mu m} \frac{\sigma(x)}{E^*(x)}\, dx$$

but $\sigma(x) = F/A(x)$
$A(x) = A_0 e^{-cx}$
$E^*(x) = E_s\left(\frac{\rho_0}{\rho_s}\right)^2 e^{-2kx}$

$$\Rightarrow \quad l = \frac{F}{A_0}\cdot\frac{1}{E_s}\cdot\left(\frac{\rho_s}{\rho_0}\right)^2 \int_0^{4\mu m} e^{(2k+c)x}\, dx$$

$$l = \frac{F}{A_0 E_s}\left(\frac{\rho_s}{\rho_0}\right)^2 \cdot \frac{1}{2k+c}\left[e^{4(2k+c)} - 1\right]$$

continued...

$F = 700 \times 10^{-9}$ N

$A_o = 350 \times 10^{-12}$ m^2

$\rho_o = 25$ mg/ml

$E_s = 2 \times 10^9$ Pa

$\rho_s = 730$ mg/mL

$c = 0.05$ μm^{-1}

$k = 0.1$ μm^{-1}

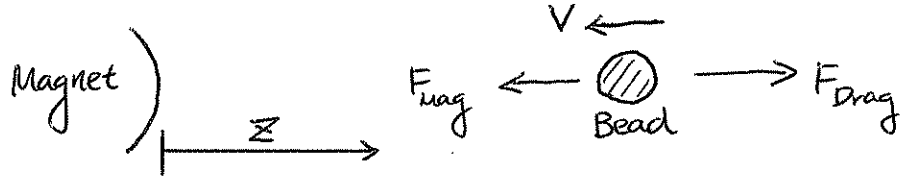
$\Rightarrow 2k + c = 0.25$ μm^{-1}

$$\therefore \ell = \frac{700 \times 10^{-9}\,\text{N}}{(350 \times 10^{-12}\,\text{m}^2)(2 \times 10^9\,\text{Pa})} \left(\frac{730\,\text{mg/mL}}{25\,\text{mg/mL}}\right)^2 \frac{1}{0.25\,\mu\text{m}^{-1}} \left[e^{4(0.25)} - 1\right]$$

$$= 5.86 \times 10^{-3}\ \mu\text{m}$$

a)

$$F_{mag} = \frac{\beta}{z} \quad ; \quad F_{Drag} = 3\pi\mu D V = -3\pi\mu D \frac{dz}{dt}$$



$$F_{mag} - F_{Drag} = m \cdot a = 0 \quad , \quad \because \text{neglecting the mass of the bead}$$

$$\Rightarrow \frac{\beta}{z} = -3\pi\mu D \frac{dz}{dt} \quad \Rightarrow \int_0^t \beta\, dt = \int_{z_0}^{z} -3\pi\mu D\, z\, dz$$

$$\Rightarrow \beta = \frac{3\pi\mu D(z_0^2 - z^2)}{2t} \quad - \text{①}$$

b) Given: $D = 4.5\ \mu m$, $\mu = 1.2\ g/cm \cdot s = 1.2 \times 10^{-4}\ g/\mu m \cdot s$

designed $\beta = 200\ nN\ \mu m = 200\ g\ \mu m^2/s^2$

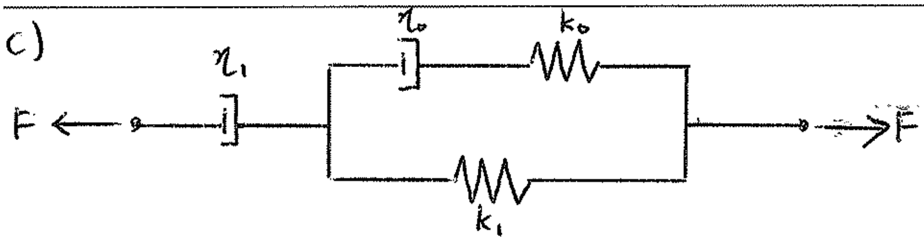
Use ① from a), the corresponding β's can be calculated in the given table:

time (sec)	z (μm)	β ($nN\ \mu m$)
0.000	100	—
0.095	80	96.43
0.158	60	103.08
0.214	40	99.89
0.246	20	99.30

Average $\beta = 99.68\ nN\ \mu m$

$\therefore$ These data are not compatible with the design specification.

c)



$$\frac{X(t)}{F_0} = \frac{1}{k_1}\left(1 - \frac{k_0}{k_0+k_1} e^{-t/\tau}\right) + \frac{t}{\eta_1} \quad — ②$$

We are interested in the response at time $t=0^+$, where $X = 0.85\ \mu m$ (given) and $F_0 = \frac{\beta}{Z} = \frac{200\ nN\ \mu m}{50\ \mu m} = 4\ nN$

Plug X and F_0 at $t=0^+$ in ②,

$$\frac{0.85\ \mu m}{4\ nN} = \frac{1}{k_1}\left[1 - \frac{k_0}{k_0+k_1}\right] = \frac{1}{k_0+k_1}$$

$$\therefore k_1 + k_0 = \frac{4\ nN}{0.85\ \mu m} = 4.71\ nN/\mu m.$$

d) This measurement is very sensitive to noise, since we are essentially trying to fit the data to a "jump" in position of the bead at time $t=0^+$

e) From the description, Latrunculin-B will break down the actin component of the cytoskeleton, and therefore decrease k_0+k_1.

a)

$$\text{Energy stored} = \text{work} = \int_0^{\epsilon L} T\,dx$$

$$= \int_0^{\epsilon L} (\sigma A)(d\epsilon L)$$

$$= AL \int_0^{\epsilon} \sigma\, d\epsilon$$

$$= AL \int_0^{\epsilon} (\epsilon E_{tissue})\, d\epsilon$$

$$= AL E_{tissue} \cdot \frac{\epsilon^2}{2}$$

b) For series configuration, $\sigma_{cell} = \sigma_{ECM} = \sigma$

$$\text{Energy stored} = \phi AL \int_0^{\epsilon_{cell}} \sigma_{cell}\, d\epsilon_{cell} + (1-\phi)AL \int_0^{\epsilon_{ECM}} \sigma_{ECM}\, d\epsilon_{ECM}$$

$$= \frac{\phi AL}{E_{cell}} \int_0^{\sigma} \sigma_{cell}\, d\sigma_{cell} + \frac{(1-\phi)AL}{E_{ECM}} \int_0^{\sigma} \sigma_{ECM}\, d\sigma_{ECM}$$

$$= AL \cdot \frac{\sigma^2}{2} \left[\frac{\phi}{E_{cell}} + \frac{1-\phi}{E_{ECM}} \right]$$

but, from a), $\text{Energy stored} = AL E_{tissue} \cdot \frac{\epsilon^2}{2} = AL \cdot \frac{1}{E_{tissue}} \cdot \frac{\sigma^2}{2}$

$$\therefore AL \cdot \frac{\sigma^2}{2} \cdot \left[\frac{\phi}{E_{cell}} + \frac{1-\phi}{E_{ECM}} \right] = AL \cdot \frac{1}{E_{tissue}} \cdot \frac{\sigma^2}{2}$$

$$\Rightarrow \frac{1}{E_{tissue}} = \frac{\phi}{E_{cell}} + \frac{1-\phi}{E_{ECM}}$$

q.e.d.

c)

For parallel configuration, $\varepsilon_{cell} = \varepsilon_{ECM} = \varepsilon$

$$\text{Energy stored} = \phi AL \int_0^{\varepsilon} \sigma_{cell}\, d\varepsilon_{cell} + (1-\phi) AL \int_0^{\varepsilon} \sigma_{ECM}\, d\varepsilon_{ECM}$$

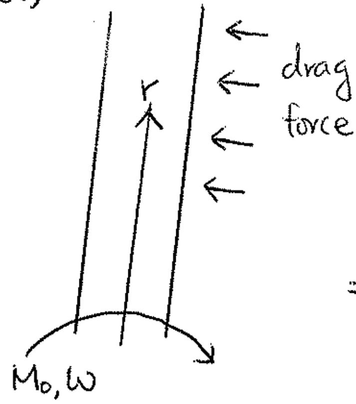
$$= \phi AL E_{cell} \int_0^{\varepsilon} \varepsilon_{cell}\, d\varepsilon_{cell} + (1-\phi) AL \int_0^{\varepsilon} \varepsilon_{ECM}\, d\varepsilon_{ECM}$$

$$= \frac{AL\varepsilon^2}{2} \left[\phi E_{cell} + (1-\phi) E_{ECM} \right]$$

but from a), Energy stored $= \dfrac{AL}{E_{tissue}} \cdot \dfrac{\sigma^2}{2}$

$$\therefore E_{tissue} = \phi E_{cell} + (1-\phi) E_{ECM}.$$

a)



$\circlearrowleft + \sum M_o = I_o \alpha = 0$, $\because \alpha$ = angular acceleration $= 0$

$\therefore M_o$ balances torque due to the fluid drag

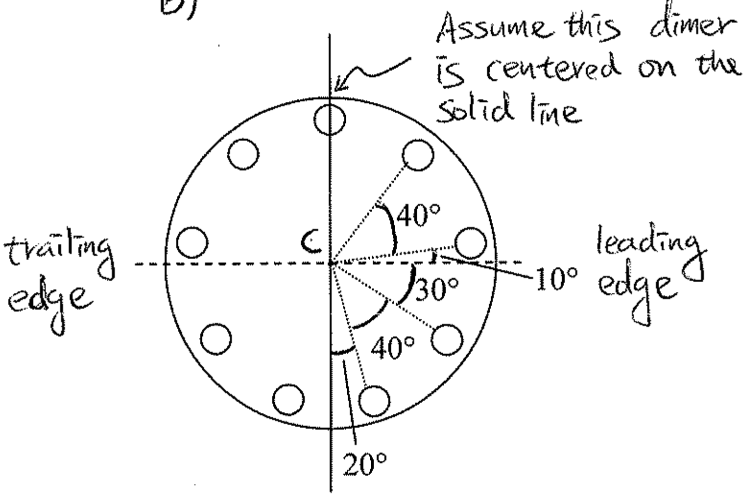
$\Rightarrow M_o = \int_0^L r(f\,dr)$, force on an element of length dr is $f \cdot dr$

$= \int_0^L r k_1 \mu (\omega r) dr$, $\because f = k_1 \mu U$ and $U = \omega r$

$= k_1 \mu \omega \int_0^L r^2 dr$

$\therefore M_o = \frac{1}{3} k_1 \mu \omega L^3$

b)



Dimers are $360°/9 = 40°$ apart

Let the cilium diameter be d

The torque about center point, C, is:

$$(F_{max})\left(\frac{d}{2}\right)\left[2\cos 10° + 2\cos 30° + 2\cos(10°+40°) + 2\cos(30°+40°)\right]$$

$$= (F_{max})\left(\frac{d}{2}\right)(5.671)$$

continued...

This must be equal to the result from part (a)

$$\therefore \frac{k_1 \mu \omega L^3}{3} = (F_{max})\left(\frac{d}{2}\right)(5.671)$$

$$\Rightarrow F_{max} = \frac{2}{3}\frac{k_1 \mu \omega L^3}{5.671 \cdot d}$$

Given: $k_1 = 2$; $\omega = 0.2$ rad/s ; $\mu = 1$ cP $= 10^{-2}$ g/cm·s

$L = 2\times10^{-4}$ cm ; $d = 0.4\times10^{-4}$ cm

$$F_{max} = \frac{2}{3} \cdot \frac{(2)(10^{-2}\ \text{g/cm·s})(0.2/\text{s})(2\times10^{-4}\ \text{cm})^3}{(5.671)(0.4\times10^{-4}\ \text{cm})}$$

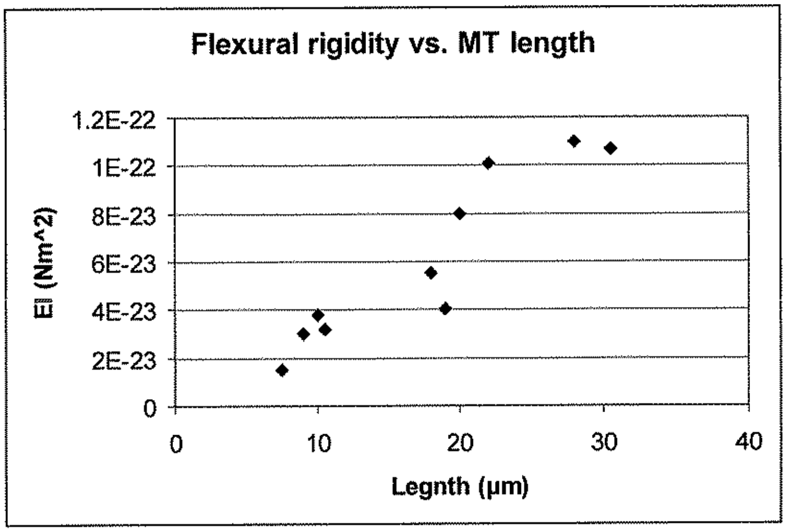
$$= 9.405\times10^{-11}\ \text{dynes}$$

$$= 9.405\times10^{-16}\ \text{N}$$

a) $P_{cr} = \frac{\pi^2 EI}{L^2} \Rightarrow EI = \frac{P_{cr} L^2}{\pi^2}$

MT length (μm)	EI (Nm^2)
7.5	1.5 x 10^{-23}
9	3 x 10^{-23}
10	3.8 x 10^{-23}
10.5	3.2 x 10^{-23}
18	5.5 x 10^{-23}
19	4 x 10^{-23}
20	8 x 10^{-23}
22	10.1 x 10^{-23}
28	11 x 10^{-23}
30.5	10.7 x 10^{-23}

Average $EI = 6.1 \times 10^{-23}$ Nm^2



Clearly EI is not constant with length in these experimental data. Potential sources of error include:
- reliable identification of initial buckling load
- microtubles are not loaded directly in compression (bead twisting may occur)

c) From the text, we estimate the inner and outer diameters of a microtubule to be 16 nm and 24 nm, respectively. Based on these estimates,

$$I = \frac{\pi}{64}(D_o^4 - D_i^4) = \frac{\pi}{64}(24^4 - 16^4)\ \text{nm}^4 = 1.31\times10^{-32}\ \text{m}^4$$

Using this estimate for the moment of inertia and $EI = 6.1\times10^{-23}\ \text{Nm}^2$ from part a), we get $E \approx 4.65$ GPa

d) The ability of a cytoskeletal filament to withstand high bending or buckling forces without excessive deformation depends on its flexural rigidity, EI. Assuming actin filaments have a diameter of ~8 nm (from the text) and can be modeled as a solid circular rod, then

$$I_{actin} = \frac{\pi}{4}r^4 \approx 2\times10^{-34}\ m^4$$

The differences between microtubules and actin filaments are summarized below:

	Microtubule	Actin filament	Ratio MT/Actin
E (GPa)	~4.65	~2	~2.3
I (m^4)	~1.31×10^{-32}	~2×10^{-34}	~65
EI (Nm^2)	~6.1×10^{-23}	~4×10^{-25}	~153

Due to their structure, microtubules are over 150 times more rigid in bending than are actin filaments despite having only about twice the Young's modulus.

a) Penetration depth at the probe tip $\delta = u_z$ when $r=0$.
Solving equation (2.76) for a with $u_z = \delta$ and $r=0$ gives:

$$a = \frac{2}{\pi}\delta \tan\alpha$$

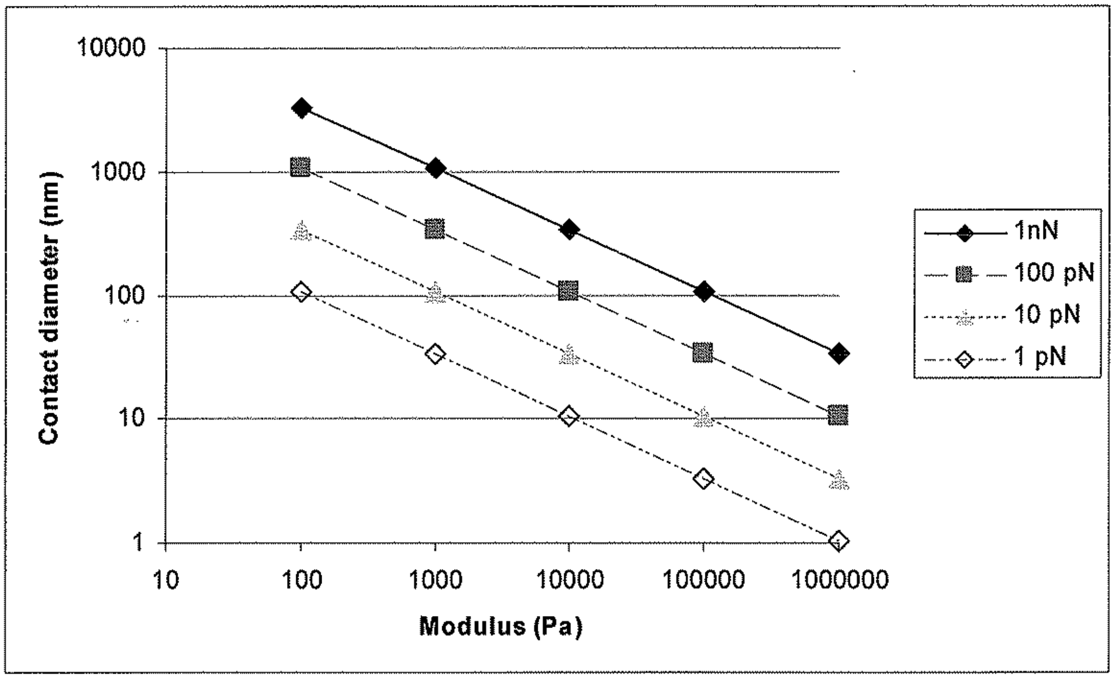
b) From the text (equation 2.4),

$$\delta^2 = \frac{\pi}{2}\cdot\frac{F(1-\nu^2)}{E\tan\alpha}$$

Substituting the expression for δ into the expression from part a) gives

$$a = \frac{2}{\pi}\tan\alpha\left(\frac{\pi}{2}\frac{F(1-\nu^2)}{E\tan\alpha}\right)^{1/2} = \left(\frac{2}{\pi}\tan\alpha\,\frac{F(1-\nu^2)}{E}\right)^{1/2}$$

c)



d) A typical range of cell stiffness is 100 Pa – 10 kPa. Typical forces applied by AFM for probing cells are 10-100 pN. Therefore, from the graph we would expect contact diameters in the range of ~800 nm – 0.8 μm. For a typical cell 20 μm in diameter, and therefore having a projected area of ~300 μm^2, a 'worst case' vertical resolution of < 1 μm^2 represents ~0.3% of the cell area. We can therefore conclude that the AFM offers reasonably high spatial resolution for the purposes of elasticity mapping.

a)

Assuming the actin filament is cylindrical rod,

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{F}{EA} = \frac{F}{E\pi r^2}$$

substituting the values given in the problem, the strain at which the actin filament will break (or yield) is 0.009.

b) Substituting the breaking force and the yield strain of the actin filament into equation 2.77

$$E_0 = \frac{58300}{L_0^2}$$

where E_0 is in dyn/cm^2 and L_0 is in μm.
see plot below.

c)

The question states that $P_0 = \sqrt{6}\, F_0$ or alternatively, $F_0 = \frac{1}{\sqrt{6}} P_0$.
We can therefore write that the incremental modulus in terms of the force in the microtubules is

$$E_0 = \frac{15.6}{\sqrt{6}} \frac{P_0}{L_0^2} \frac{1+4\varepsilon_0}{1+12\varepsilon_0}$$

When the microtubules are on the verge of buckling, P_0 is the critical buckling force (see equation 2.75) and therefore

$$E_0 = \frac{15.6}{\sqrt{6}} \frac{\pi^2 EI}{L_0^4} \frac{1+4\varepsilon_0}{1+12\varepsilon_0}$$

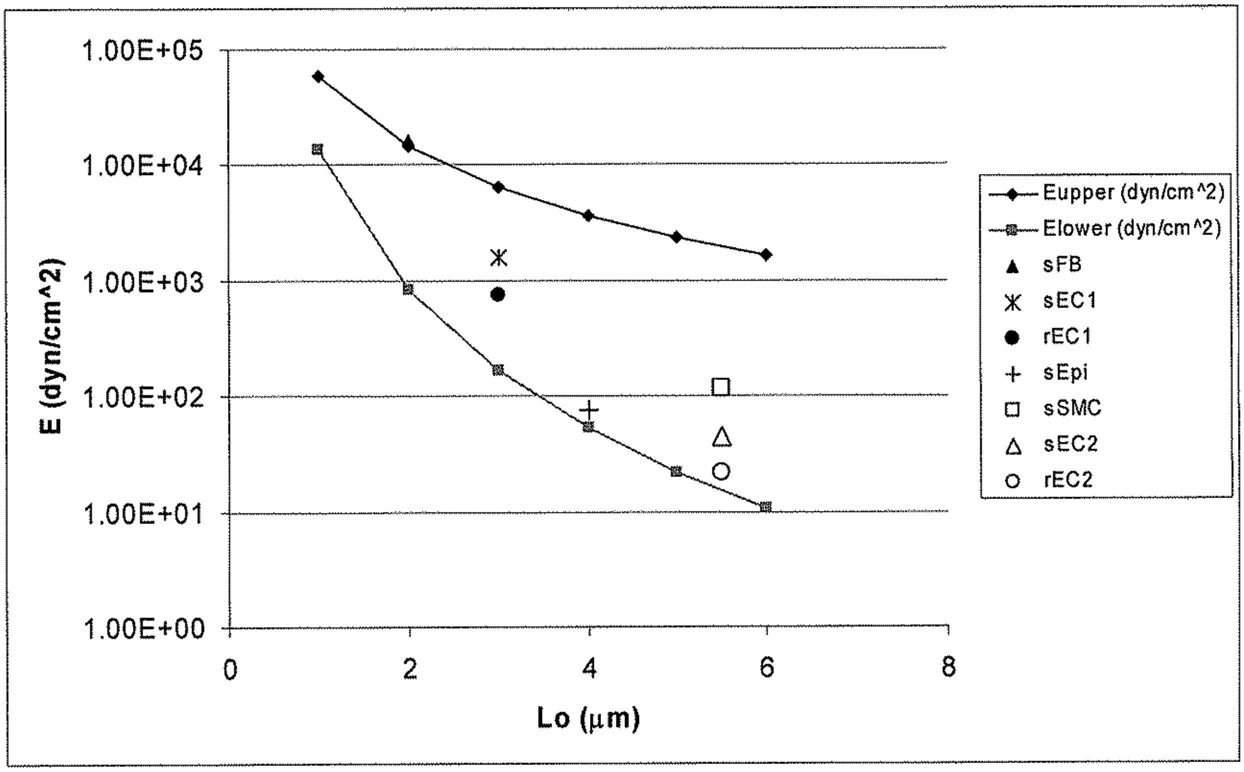
d) For small strain, we can neglect the contribution of ε_o, and using $EI = 21.5\ pN \cdot \mu m^2$, the lower bound for the incremental modulus is therefore

$$E_o = \frac{1369}{L_o^4}$$

where E_o is in dyn/cm^2 and L_o is in μm. See plot below.

e) see plot below

f) Experimental measurements fall within bounds predicted by tensegrity and generally are closer to the lower bound. The lower bound assumes the microtubules are on the verge of buckling, so the model predictions and experimental observations suggest microtubule buckling may be an important determinant of cellular elasticity. However, the bounds are quite far apart (2 orders of magnitude for large L_o), limiting the conclusions that can be made from this analysis alone. Discussion and critique of this model is given in Stamenović and Coughlin (2000) *J Biomech Eng* 122: 39-43 and subsequent publications by Stamenović.


1.00E+05
1.00E+04
1.00E+03
1.00E+02
1.00E+01
1.00E+00
E (dyn/cm^2)
0
2
4
6
8
Lo (μm)
Eupper (dyn/cm^2)
Elower (dyn/cm^2)
sFB
sEC1
rEC1
sEpi
sSMC
sEC2
rEC2

There are actually more than 3 reasons. They are presented roughly in order of decreasing importance:

1. Poiseuille's law assumes that flow is steady, which is evidently not the case on the arterial side. This has a substantial effect on the velocity profile and particularly on the wall shear stress profile.

2. Poiseuille's law assumes fully developed flow in an infinitely long cylindrical tube. Real arteries are tapered, curved, branched ... etc. All of these geometrical effects influence the flow.

3. Poiseuille's law assumes rigid tube walls. In fact, the arteries are not rigid. The pressure of distensible walls, together with unsteadiness, causes reflections at junctions. It also causes vessel walls to move.

4. Poiseuille's law assumes a Newtonian fluid. As we have discussed in the text, blood is in fact non-Newtonian. Such effects are generally important in smaller vessels.

a)

$$V = \frac{4}{3}\pi r^3 = 98\ \mu m^3$$

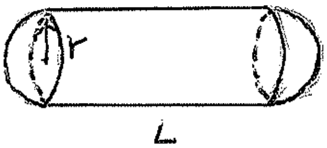
$\Rightarrow r = 2.86\ \mu m$

$\Rightarrow$ diameter of the smallest pore = 5.72 μm

b)

$$V = \frac{4}{3}\pi r^3 + \pi r^2 L = 98\ \mu m^3 \quad -①$$

$$A_s = 4\pi r^2 + 2\pi r L = 130\ \mu m^2 \quad -②$$



$$\begin{matrix} ① \times 2 \\ ② \times r \end{matrix} \Rightarrow \begin{cases} 8/3\,\pi r^3 + 2\pi r^2 L = 196 \\ 4\pi r^3 + 2\pi r^2 L = 130r \end{cases}$$

$$\Rightarrow 4.19 r^3 - 130 r + 196 = 0$$

solve by iteration:

$r_1 = 1.6533\ \mu m$, $r_2 = -6.2104\ \mu m$, $r_3 = 4.5571\ \mu m$

but $0 < r < 2.86\ \mu m$

$\therefore$ the smallest pore size for RBC is

$$d = 2(1.65) = 3.30\ \mu m$$

c)

- Non-spherical shape allows the RBC to deform and fit through smaller pores then if it were spherical
- A biconcave shape has a larger surface area then a spherical shape for the same volume, and can therefore allow O_2 and CO_2 to diffuse more rapidly.

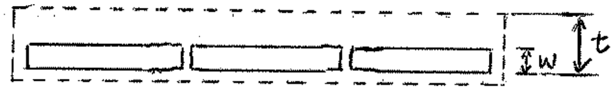
We assume that:

- the plate gap is much greater than the RBC thickness ($h >> w$)
- RBCs are rigid
- the gap between RBC "lines", t, is uniform.

We know from basic Couette flow physics that the shear stress across the gap is uniform and given by:

$$\tau = \mu_{eff} \frac{U}{h} \qquad (1)$$

(This is actually the definition of μ_{eff})
Also, the velocity profile in the gap is linear.
Now consider one "line" plus a fluid gap



Evidently: $H = \frac{w}{w+t}$

Also, the total velocity difference from the top to the bottom plate is U. The velocity gradient across the boxed region above must therefore be:

$$U\left(\frac{w+t}{h}\right)$$

This occurs in the fluid portion (since RBCs are rigid), so that in the fluid gap:

$$\tau = \mu_p \frac{U\left(\frac{w+t}{h}\right)}{t} \qquad (2)$$

where μ_p = plasma viscosity

continued...

Equating ① and ② gives

$$\mu_{eff} = \mu_p \frac{w+t}{t} = \frac{\mu_p}{1-H}$$

(Note: $H = \frac{w}{w+t}$)

Given $\mu_p = 1.2$ cP , $H = 0.75$

$\Rightarrow$ $\mu_{eff} = 4.8$ cP

a)

V = Volume of non-pulmonary capillaries = 2% of 5L
$= 0.1\,L = 10^{-4}\,m^3$

Since D = capillary diameter $= 8\times10^{-6}\,m$,
the total length of non-pulmonary capillaries, L_{TOT}, is

$$L_{TOT} = \frac{V}{\pi\left(\frac{D}{2}\right)^2} = \frac{10^{-4}\,m^3}{\pi\left(\frac{8\times10^{-6}\,m}{2}\right)^2} = 1.989\times10^{6}\,m.$$

b)

If the average capillary length L_{avg} is $10^{-3}\,m$,

then the number of capillaries is

$$N = \frac{L_{TOT}}{L_{avg}} = 1.989\times10^{9}\ \text{capillaries}$$

c)

If cardiac output $= 5L/min = 5\cdot\frac{10^{-3}m^3}{60s} = 8.33\times10^{-5}\,m^3/s$,

then the flow rate through each capillary is

$$Q = \frac{8.33\times10^{-5}\,m^3/s}{1.989\times10^{9}} = 4.189\times10^{-14}\,m^3/s$$

Assume Newtonian laminar flow $\Rightarrow Q = \frac{\pi R^4}{8\mu_{eff}}\frac{\Delta P}{\Delta L}$

$$\Rightarrow \Delta P = \frac{128\,\mu_{eff}\,Q L}{\pi D^4} = \frac{128(3.5\times10^{-3}\,kg/m\cdot s)(4.189\times10^{-14}\,m^3/s)(10^{-3}\,m)}{\pi(8\times10^{-6}\,m)^4}$$

$$= 1.459\,kPa = 1.459\,kPa\times\frac{10^3 Pa}{kPa}\times\frac{1\,mmHg}{133\,Pa} = 10.97\,mmHg.$$

Since the total systemic pressure drop is 85mmHg, this represents $\frac{10.97}{85} = 12.9\%$

(a) Given $L_n/D_n = 16$, we have $L_n = 16 D_n$.

$$\alpha = \frac{\text{area of daughter tubes}}{\text{area of parent tube}}$$

$$= 2\,\frac{\pi/4\, D_n^2}{\pi/4\, D_{n-1}^2} = \frac{2 D_n^2}{D_{n-1}^2}$$

$$\therefore D_n^2 = \left(\frac{\alpha}{2}\right) D_{n-1}^2$$

$$= \left(\frac{\alpha}{2}\right)\left(\frac{\alpha}{2}\right) \cdot D_{n-2}^2$$

$$\vdots$$

$$= \left(\frac{\alpha}{2}\right)^n D_0^2$$

$$\therefore D_n = \left(\frac{\alpha}{2}\right)^{n/2} D_0$$

Note that this also implies that $L_n = \left(\frac{\alpha}{2}\right)^{n/2} L_0$

(b) Assume Poiseuille's law holds

$$Q = \frac{\pi (D/2)^4}{8 \mu_{eff}} \frac{\Delta p}{L}$$

$$\therefore \Delta p_n = \frac{8 \mu_{eff} Q_n L_n}{\pi (D_n/2)^4} = \frac{8 \mu_{eff} Q_n (\alpha/2)^{n/2} L_0}{\pi (D_0/2)^4 (\alpha/2)^{2n}}$$

[Pressure drop in generation n.]

But $Q_n = \frac{1}{2} Q_{n-1} = \frac{1}{2}\cdot\frac{1}{2} Q_{n-2} = \cdots = \left(\frac{1}{2}\right)^n Q_0$

$$\therefore \Delta P_n = \frac{8 \mu_{eff} Q_0 L_0}{\pi (D_0/2)^4} \frac{1}{(\alpha/2)^{3n/2}} \left(\frac{1}{2}\right)^n = \Delta p_0 \left(\frac{2}{\alpha^3}\right)^{n/2}$$

Now to get the total pressure drop from generation 0 to the end of generation N

$$\Delta p_{0-N} = \sum_{n=0}^{N} \Delta p_n = \sum_{n=0}^{N} \left(\frac{2}{\alpha^3}\right)^{n/2} \Delta p_0$$

Using the hint for a geometric series sum:

$$\Delta p_{0-N} = \Delta p_0 \left\{ \frac{\left(2/\alpha^3\right)^{\frac{N+1}{2}} - 1}{\left(2/\alpha^3\right)^{1/2} - 1} \right\} \qquad \underline{QED}$$

(c) Since there are 2^n vessels in generation n, we can write the volume of generation n as:

$$V_n = 2^n \frac{\pi}{4} D_n^2 L_n = 2^n \frac{\pi}{4} \left(\alpha/2\right)^n D_0^2 \left(\alpha/2\right)^{n/2} L_0$$

$$= \left(\frac{\alpha^3}{2}\right)^{n/2} \frac{\pi}{4} D_0^2 L_0$$

$$= \left(\alpha^3/2\right)^{n/2} V_0$$

The volume from generation 0 to N is therefore

$$V_{0-N} = \sum_{n=0}^{N} V_n = V_0 \sum_{n=0}^{N} \left(\alpha^3/2\right)^{n/2} = V_0 \left\{ \frac{1 - \left(\alpha^3/2\right)^{\frac{N+1}{2}}}{1 - \left(\alpha^3/2\right)} \right\}$$

(d) Using the result from (c) in the limit of large N, and requiring $\alpha^3/2$ to be less than 1, we have

$$V = \lim_{N \to \infty} V_{0-N} = V_0 / \left[1 - \left(\alpha^3/2\right)\right]$$

Given $V = 1200$ mL and $V_0 = 100$ mL, plug in numbers and solve for $\alpha = 1.1889$

(e) From part (a), $D_n = \left(\alpha/2\right)^{n/2} D_0$ — (*)

Given $D_n = 20\ \mu m = 2\times10^{-5}$ m

$D_0 = 2$ cm $= 2\times10^{-2}$ m

$\alpha = 1.1889$

Rearrange (*) to get $n = \dfrac{\ln\left(D_n/D_0\right)}{\ln(\alpha/2)} \cdot 2$

$$= \frac{\ln\left(2\times10^{-5}/2\times10^{-2}\right)}{\ln\left(1.1889/2\right)} \cdot 2$$

$$\approx 27$$

∴ The model is suitable up to ~27 generations.

(f) Given data: $Q = 5$ L/min $= 5/60 \times 10^{-3}$ m³/s

$\mu = 3.5$ cP $= 0.0035$ kg/(m s)

$D_0 = 2$ cm $= 2\times10^{-2}$ m

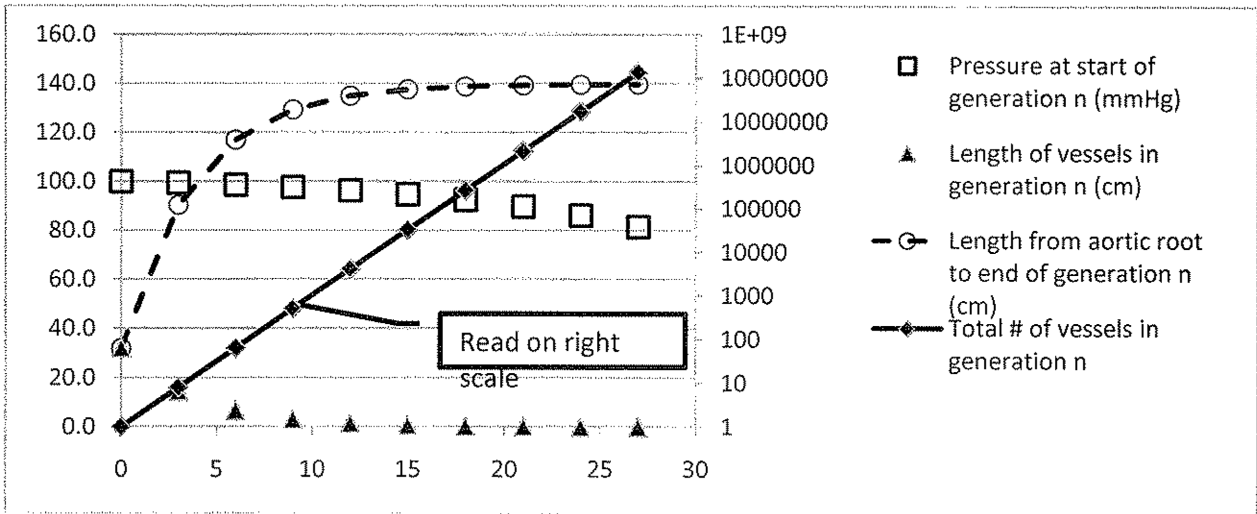
$L_n/D_n = 16$

$\alpha = 1.1889$

$$\Delta p_0 = \frac{8\mu Q_0 L_0}{\pi (D_0/2)^4} = \frac{8\mu Q_0 16 D_0}{\pi D_0^4}\cdot 16 = 23.767\ \text{Pa} = 0.178\ \text{mmHg}$$

Then
$$\Delta p_{0-N} = \Delta p_0 \left\{ \frac{\left(2/\alpha^3\right)^{\frac{N+1}{2}} - 1}{\left(2/\alpha^3\right)^{1/2} - 1} \right\} = \Delta p_0 \left\{ \frac{1.0909^{N+1} - 1}{1.0909 - 1} \right\} = 1.961 \left[1.0909^{N+1} - 1\right] \text{ mmHg}$$

n	Total # of vessels in generation n	Pressure at start of generation n (mmHg)	Length of vessels in generation n (cm)	Length from aortic root to end of generation n (cm)
0	1	100.0	32.000	32.00
3	8	99.4	14.666	90.36
6	64	98.7	6.722	117.11
9	512	97.7	3.081	129.37
12	4096	96.4	1.412	134.99
15	32768	94.7	0.647	137.56
18	262144	92.6	0.297	138.74
21	2097152	89.8	0.136	139.28
24	16777216	86.1	0.062	139.53
27	134217728	81.4	0.029	139.65



Here we use the fact that total vessel length from the aortic root to the end of generation N is given by

$$\sum_{K=0}^{N} L_K = 16 \sum_{K=0}^{N} D_n = 16 \sum_{K=0}^{N} D_0 \left(\frac{\alpha}{2}\right)^{K/2} = 16 D_0 \left\{ \frac{\left(\frac{\alpha}{2}\right)^{\frac{N+1}{2}} - 1}{\left(\frac{\alpha}{2}\right)^{1/2} - 1} \right\}$$

Also, pressure at the start of generation n is given by

$$100 \text{ mmHg} - \Delta p_{0-N} \quad \text{with } N = n-1.$$

$$= 100 - 1.961\left[1.0909^{n} - 1\right] \text{ mmHg}.$$

a) Let V_n = average fluid velocity in the n^{th} generation

$$T_n = \frac{L_n}{V_n} = \frac{L_n A_n}{Q_n} \quad \text{where } A_n = \frac{\pi}{4} D_n^2$$

$$\text{So } T_n = \frac{L_n \pi D_n^2}{4Q_n} \quad \text{but } L_n = 16 D_n = 16\left(\frac{\alpha}{2}\right)^{n/2} D_0$$
$$Q_n = \left(\frac{1}{2}\right)^n Q_0$$

$$\therefore T_n = \frac{\pi}{4} \cdot 16 \cdot \left(\frac{\alpha}{2}\right)^{3n/2} \cdot \frac{D_0^3}{Q_0} \cdot 2^n$$
$$= 4\pi \cdot \left(\frac{\alpha^3}{2}\right)^{n/2} \cdot \frac{D_0^3}{Q_0}$$

Given: $Q_0 = 5\ L/min = 8.33 \times 10^{-5}\ m^3/s$
$D_0 = 2\ cm = 0.02\ m$
$L_0 = 16 \cdot D_0 = 0.32\ m$
$\alpha = 1.19$

$$\therefore T_n = 1.21(0.84)^{n/2} \text{ sec}$$

b) Total transit time

$$T_{0\text{-}N} = \sum_{n=0}^{N} T_n = \sum_{n=0}^{N} 1.21(0.84)^{n/2}$$
$$= 1.21 \cdot \frac{1 - 0.84^{\frac{N+1}{2}}}{1 - 0.84^{1/2}}$$
$$= 14.5\left(1 - 0.84^{(N+1)/2}\right) \text{ sec}$$

c)

For $N = 27$

$$T_{0\text{-}N} = 14.5\left(1 - 0.84^{(27+1)/2}\right) = 13.2 \text{ sec}$$

a) For laminar Newtonian flow in a tube of radius R:

$$\cdot\ \tau = \frac{r}{2}\left|\frac{dP}{dx}\right| \Rightarrow \tau_w = \frac{R}{2}\left|\frac{dP}{dx}\right|$$

$$\cdot\ Q = -\frac{dP}{dx}\frac{\pi R^4}{8\mu} \Rightarrow V = \left|\frac{Q}{\pi R^2}\right| = \left|\frac{dP}{dx}\right|\frac{R^2}{8\mu}$$

$$\therefore \left|\frac{dP}{dx}\right| = \frac{8\mu}{R^2}\cdot V$$

$$\therefore \tau_w = \frac{R}{2}\cdot\frac{8\mu}{R^2}\cdot V = \frac{4\mu V}{R}$$

b) Given: $\mu = 3\ cP = 3\times10^{-2}\ g/cm\cdot s$

Vessel	Mean Velocity (cm/s)	Radius (cm)	τ_w (dyne/cm²)
ascending aorta	V = 20	R = 0.75	3.2
abdominal artery	V = 15	R = 0.45	4.0
femoral artery	V = 10	R = 0.2	6.0
arteriole	V = 0.3	R = 0.0025	14.4
inferior vena cava	V = 12	R = 0.5	2.88

c)

Yes, since all the wall shear stress are much greater than the yield shear stress (i.e. $R_c \ll R$), blood flow can be approximated as Newtonian.

Given: $d = 8\ mm = 8\times10^{-3}\ m$ $\mu = 3.5\ cP = 3.5\times10^{-3}\ kg/m\cdot s$

$\Rightarrow R = 4\times10^{-3}\ m$

$$Q = 1.4\ L/min = 1.4\cdot\frac{10^{-3}\ m^3}{60\ s} = 2.33\times10^{-5}\ m^3/s$$

$$Q = \left|\frac{dP}{dx}\right|\frac{\pi R^4}{8\mu}$$

$$\Rightarrow 2.33\times10^{-5}\ m^3/s = \left|\frac{dP}{dx}\right|\frac{\pi(4\times10^{-3}\ m)^4}{8\times3.5\times10^{-3}\ kg/m\cdot s}$$

$$\Rightarrow \left|\frac{dP}{dx}\right| = 811.2\ kg/m^2\cdot s^2 = 811.2\ N/m^3$$

$\therefore$ Wall shear stress is

$$\tau_w = \frac{R}{2}\left|\frac{dP}{dx}\right|$$

$$= \frac{4\times10^{-3}\ m}{2}\times 811.2\ N/m^3$$

$$\tau_w = 1.62\ N/m^2$$

Also given: surface area, $A_s = 550\ \mu m^2 = 550\times10^{-12}\ m^2$

strength per integrin complex : $100\ pN = 100\times10^{-12}\ N$

$\therefore$ Number of integrin complexes needed is

$$N = \frac{1.62\ N/m^2\cdot 550\times10^{-12}\ m^2}{100\times10^{-12}\ N} \simeq 9$$

a) For a Casson fluid

$$Q = -\frac{\pi R^4}{8\mu}\frac{dP}{dx}F(\xi)$$

where $F(\xi) = 1 - \frac{16}{7}\xi^{1/2} + \frac{4}{3}\xi - \frac{1}{21}\xi^4$

$$\tau_y = \frac{R_c}{2}\frac{dP}{dx} \longrightarrow \left|\frac{dP}{dx}\right| = \frac{2\tau_y}{R_c}$$

$$\therefore Q = \frac{\pi R^4}{8\mu}\frac{2\tau_y}{R_c}F(\xi)$$

$$= \frac{\pi R^3 \tau_y}{4\mu}\left[\frac{R}{R_c}F(\xi)\right]$$

But $\frac{R_c}{R} = \xi \Rightarrow Q = \frac{\pi R^3 \tau_y}{4\mu}\left[\frac{1}{\xi}F(\xi)\right]$

where $\frac{1}{\xi}F(\xi) = \frac{1}{\xi} - \frac{16}{7}\xi^{-1/2} + \cancel{\frac{4}{3}} - \cancel{\frac{1}{21}\xi^3}$ $\longrightarrow$ can be neglected when $\xi \ll 1$

$$\simeq \frac{1}{\xi} - \frac{16}{7}\xi^{-1/2}$$

Therefore $Q = \frac{\pi R^3 \tau_y}{4\mu}G(\xi)$ with $G(\xi) = \frac{1}{\xi} - \frac{16}{7\sqrt{\xi}}$

b) Let V be mean velocity; $Q = VA = V\pi R^2$

Use Q from a):

$$\frac{\pi R^3 \tau_y}{4\mu}G(\xi) = V\pi R^2$$

$$\Rightarrow G(\xi) = \frac{4\mu V}{R\tau_y} = k \rightarrow \text{a constant for each vessel}$$

$$\therefore \frac{1}{\xi} - \frac{16}{7\sqrt{\xi}} = k \Rightarrow 1 - \frac{16}{7}\xi^{1/2} - k\xi = 0.$$

b) continued... We can compute ξ using quadratic formula:

$$\xi = \left[\frac{\frac{16}{7} - \sqrt{(\frac{16}{7})^2 - 4(-k)(1)}}{-2k}\right]^2$$

$$= \left[\frac{-1.143 + \sqrt{1.306 + k}}{k}\right]^2$$

$$Q_{blood} = -\frac{\pi R^4}{8\mu}\frac{dP}{dx_{blood}} F(\xi) \simeq -\frac{\pi R^4}{8\mu}\frac{dP}{dx_{blood}}\xi G(\xi)$$

$$Q_N = -\frac{\pi R^4}{8\mu}\frac{dP}{dx_N}$$

∴ For the same flow rate

$$\frac{dP/dx_{blood}}{dP/dx_N} = \frac{1}{\xi G(\xi)} = \frac{1}{\xi k} = \frac{k}{(1.143 - \sqrt{1.306 + k})^2} \quad (*)$$

For each vessel, compute k from $k = \frac{4\mu V}{R\tau_y}$ and plug into (*) to get the pressure gradient ratio:

Vessel	V (cm/s)	R (cm)	k	$dP/dx_{blood} : dP/dx_N$
ascending aorta	20	0.75	74.67	1.302
abdominal artery	15	0.45	93.33	1.266
femoral artery	10	0.2	140	1.213
Arteriole	0.3	0.0025	336	1.137
Inferior vena cava	12	0.5	67.2	1.320

a) when $\tau_w = \tau_y$ (or $R_c = R$), the Δp is the minimum Δp required, i.e. the blood will be on the verge of flowing.

Given: - $\tau_y = 0.05$ dynes/cm^2
- $\Delta L = 35$ cm
- $R_c = R = 0.5\text{mm} = 0.05\text{cm}$

we know $\tau_y = \frac{R_c}{2} \cdot \frac{\Delta p}{\Delta L}$ from the text

$$\therefore \; 0.05\,\text{dynes/cm}^2 = \frac{0.05\,\text{cm}}{2} \cdot \frac{\Delta p}{35\,\text{cm}}$$

$$\therefore \; \Delta p = 70\,\text{dynes/cm}^2$$

$$= 70 \cdot \frac{10^{-5}\,\text{N}}{10^{-4}\,\text{m}^2} = 7\,\text{N/m}^2 = 7\,\text{Pa}$$

$\therefore$ when $\Delta p = 7$ Pa, the blood begin to flow.

b)

$$Q = -\frac{\pi R^4}{8\mu}\frac{dp}{dx}F(\xi) \quad ; \; R_c = \tau_y \cdot 2 \cdot \frac{\Delta L}{\Delta p}$$

$$= 0.05\,\text{dynes/cm}^2 \cdot 2 \cdot \frac{35\,\text{cm}}{10\,\text{Pa}}$$

$$= 0.035\,\text{cm}$$

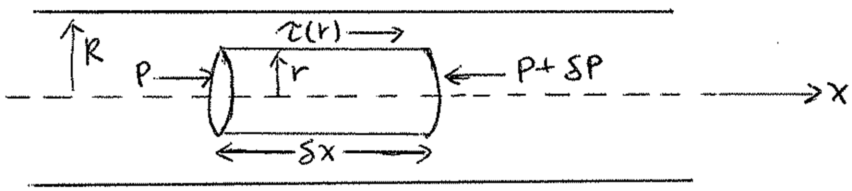
$$\xi = \frac{R_c}{R} = \frac{0.035\,\text{cm}}{0.05\,\text{cm}} = 0.7$$

$$F(\xi) = 1 - \frac{16}{7}\sqrt{0.7} + \frac{4}{3} \cdot 0.7 - \frac{1}{21} \cdot 0.7^4 = 9.534 \times 10^{-3}$$

$$\therefore \; Q = \frac{\pi (0.05\,\text{cm})^4}{8\,(3.5\times10^{-2}\,\text{g/cm·s})} \cdot \frac{100\,\text{dynes/cm}^2}{35\,\text{cm}} \cdot 9.534\times10^{-3}$$

$$= 1.91 \times 10^{-6}\,\text{cm}^3/\text{s}$$

a)



For a tube of radius R:

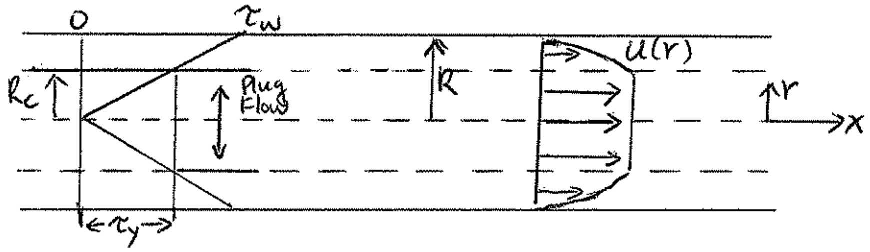
Force balance:

$$p\pi r^2 + \tau(r)2\pi r\,\delta x - (p+\delta p)\pi r^2 = 0$$

In the limit of $\delta x \to 0$

$$\Rightarrow \quad \tau(r) = \frac{r}{2}\frac{dP}{dx}$$

For a Bingham fluid:



For region $R_c < r < R$,

$$\tau = \frac{r}{2}\frac{dP}{dx} = \tau_y + \mu\dot{\gamma}$$

where $\dot{\gamma} = \frac{du}{dr}$ and $\tau_y = \frac{R_c}{2}\frac{dP}{dx}$

$$\Rightarrow \quad \frac{du}{dr} = \frac{1}{2\mu}\frac{dP}{dx}(r - R_c) \quad \Rightarrow u(r) = \frac{1}{2\mu}\frac{dP}{dx}\left(\frac{1}{2}r^2 - R_c r\right) + \underset{\substack{\uparrow \\ \text{constant}}}{C}$$

with boundary condition $u(R) = 0$ (assuming no-slip)

$$\Rightarrow u(R) = \frac{1}{2\mu}\frac{dP}{dx}\left(\frac{1}{2}R^2 - R_c R\right) + C = 0$$

$$\Rightarrow C = -\frac{1}{2\mu}\frac{dP}{dx}\left(\frac{1}{2}R^2 - R_c R\right)$$

a) continued...

$$\therefore u(r) = \frac{1}{2\mu}\frac{dP}{dx}\left(\frac{1}{2}r^2 - R_c r - \frac{1}{2}R^2 + R_c R\right)$$

$$= \frac{R^2}{4\mu}\left(-\frac{dP}{dx}\right)\left[1-\left(\frac{r}{R}\right)^2 - \frac{2R_c}{R}\left(1-\frac{r}{R}\right)\right]$$

$$\text{or } u(r) = \frac{R_c}{2}\frac{dP}{dx}\left(-\frac{R^2}{2\mu}\right)\cdot\frac{1}{R_c}\left[1-\left(\frac{r}{R}\right)^2 - \frac{2R_c}{R}\left(1-\frac{r}{R}\right)\right]$$

$$= -\frac{\tau_y R^2}{2\mu}\cdot\frac{1}{R_c}\left[1-\left(\frac{r}{R}\right)^2 - \frac{2R_c}{R}\left(1-\frac{r}{R}\right)\right]$$

For region $0 < r < R_c$

$\frac{du}{dr} = 0$, so $u(r) = u(r=R_c)$

$$u(r) = \frac{R^2}{4\mu}\left(-\frac{dP}{dx}\right)\left[1-\left(\frac{R_c}{R}\right)^2 - 2\left(\frac{R_c}{R}\right) + 2\left(\frac{R_c}{R}\right)^2\right]$$

$$= \frac{R^2}{4\mu}\left(-\frac{dP}{dx}\right)\left(1-\frac{R_c}{R}\right)^2$$

$$\text{or } u(r) = -\frac{\tau_y R^2}{2\mu}\cdot\frac{1}{R_c}\left(1-\frac{R_c}{R}\right)^2$$

Now, find flow rates using $Q = VA$

For region $R_c < r < R$

$$Q_2 = 2\pi\int_{R_c}^{R} r\,u(r)\,dr$$

$$= 2\pi\frac{R^2}{4\mu}\left(-\frac{dP}{dx}\right)\int_{R_c}^{R}\left[1-\left(\frac{r}{R}\right)^2 - 2\left(\frac{R_c}{R}\right)\left(1-\frac{r}{R}\right)\right]r\,dr$$

To make solving easier, non-dimensionalize equation, set $\hat{r} = \frac{r}{R}$ and $\hat{R}_c = \frac{R_c}{R}$

a) continued...

$$\therefore\ Q_2 = \frac{\pi R^4}{2\mu}\left(-\frac{dP}{dx}\right)\int_{\hat{R}_c}^{1}\left(\hat{r} - \hat{r}^3 - 2\hat{R}_c\hat{r} + 2\hat{R}_c\hat{r}^2\right)d\hat{r}$$

$$\therefore\ Q_2 = \frac{\pi R^4}{2\mu}\left(-\frac{dP}{dx}\right)\left[\frac{1}{2}\hat{r}^2 - \frac{1}{4}\hat{r}^4 - \hat{R}_c\hat{r}^2 + \frac{2}{3}\hat{R}_c\hat{r}^3\right]_{\hat{R}_c}^{1}$$

$$= \frac{\pi R^4}{2\mu}\left(-\frac{dP}{dx}\right)\left(\frac{1}{4} - \frac{1}{3}\hat{R}_c - \frac{1}{2}\hat{R}_c^2 + \hat{R}_c^3 - \frac{5}{12}\hat{R}_c^4\right)$$

For region $0 < r < R_c$,

$$Q_1 = \frac{R^2}{4\mu}\left(-\frac{dP}{dx}\right)\left(1 - \frac{R_c}{R}\right)^2\left(\pi R_c^2\right)$$

$$= \frac{\pi R^2}{4\mu}\left(-\frac{dP}{dx}\right)\left(1 - 2\frac{R_c}{R} + \left(\frac{R_c}{R}\right)^2\right)R_c^2$$

Once again, non-dimensionalize with $\hat{R}_c = \frac{R_c}{R}$

$$Q_1 = \frac{\pi R^2}{4\mu}\left(-\frac{dP}{dx}\right)\left(1 - 2\hat{R}_c + \hat{R}_c^2\right)\hat{R}_c^2 R^2$$

$$= \frac{\pi R^4}{2\mu}\left(-\frac{dP}{dx}\right)\cdot\frac{\hat{R}_c^2}{2}\left(1 - 2\hat{R}_c + \hat{R}_c^2\right)$$

$$= \frac{\pi R^4}{2\mu}\left(-\frac{dP}{dx}\right)\left(\frac{1}{2}\hat{R}_c^2 - \hat{R}_c^3 + \frac{1}{2}\hat{R}_c^4\right)$$

$$Q_{TOT} = Q_1 + Q_2$$

$$= \frac{\pi R^4}{2\mu}\left(-\frac{dP}{dx}\right)\left(\frac{1}{4} - \frac{1}{3}\hat{R}_c + \frac{1}{12}\hat{R}_c^4\right)$$

a) continued...

Now, put Q_{TOT} in the same form as Casson fluid:

$$Q_{TOT} = \frac{\pi R^4}{8\mu}\left(-\frac{dP}{dx}\right)\left(1 - \frac{4}{3}\hat{R}_c + \frac{1}{3}\hat{R}_c^4\right)$$

Since $\hat{R}_c = \frac{R_c}{R} = \xi$, let $F(\xi) = 1 - \frac{4}{3}\xi + \frac{1}{3}\xi^4$

$$\therefore Q_{TOT} = \frac{\pi R^4}{8\mu}\left(-\frac{dP}{dx}\right)F(\xi)$$

b) From the sketch, it can be seen that the yield stress prevents a full parabolic velocity profile from developing. Therefore, this non-Newtonian fluid has a smaller flow rate than a Newtonian fluid of the same viscosity.

Given: $R = 1$ cm

$$\frac{dP}{dx} = 0.4 \text{ dynes/cm}^3$$

$$\tau_y = 0.06 \text{ dynes/cm}^2 = \frac{R_c}{2}\frac{dP}{dx} \Rightarrow R_c = 0.30 \text{ cm}$$

From the text,

$$u_{plug} = -\frac{1}{4\mu}\frac{dP}{dx}\left[R^2 - \frac{8}{3}\sqrt{R^3 R_c} + 2RR_c - \frac{1}{3}R_c^2\right]$$

$$\therefore Q_{plug} = \int_0^{R_c} u_{plug} \cdot 2\pi r \, dr$$

$$= 2\pi u_{plug} \left(\frac{1}{2}r^2\right)\Big]_0^{R_c} = 2\pi u_{plug}\left(\frac{1}{2}R_c^2\right)$$

$$\therefore Q_{plug} = -\frac{\pi R_c^2}{4\mu}\frac{dP}{dx}\left[R^2 - \frac{8}{3}\sqrt{R^3 R_c} + 2RR_c - \frac{1}{3}R_c^2\right]$$

$$= -\frac{\pi R^4}{8\mu}\frac{dP}{dx}\left[2\left(\frac{R_c}{R}\right)^2 - \frac{16}{3}\left(\frac{R_c}{R}\right)^{5/2} + 4\left(\frac{R_c}{R}\right)^3 - \frac{2}{3}\left(\frac{R_c}{R}\right)^4\right]$$

$$= -\frac{\pi R^4}{8\mu}\frac{dP}{dx}\left[2\zeta^2 - \frac{16}{3}\zeta^{5/2} + 4\zeta^3 - \frac{2}{3}\zeta^4\right]$$

where $\zeta = \frac{R_c}{R} = 0.30$

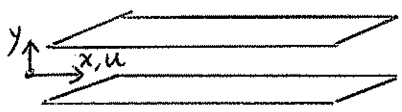
$$\therefore Q_{plug} = -\frac{\pi R^4}{8\mu}\frac{dP}{dx}\left(1.969 \times 10^{-2}\right)$$

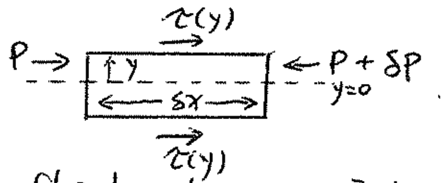
Also from the text,

$$Q_{TOT} = -\frac{\pi R^4}{8\mu}\frac{dP}{dx}\left(1 - \frac{16}{7}\sqrt{\zeta} + \frac{4}{3}\zeta - \frac{1}{21}\zeta^4\right)$$

$$= -\frac{\pi R^4}{8\mu}\frac{dP}{dx}\left(1.477 \times 10^{-1}\right)$$

$$\therefore \frac{Q_{plug}}{Q_{TOT}} = (1.969 \times 10^{-2})/(1.477 \times 10^{-1}) = 13.3\%$$





a fluid element with depth 1 into the page.

Force balance:

$$P(2y) - (P + \delta P)2y + 2\tau\delta x = 0$$

in the limit of $\delta x \to 0$, $\tau = y\frac{dP}{dx}$

For Casson fluid, $\sqrt{\tau} = \sqrt{\tau_y} + \sqrt{\mu\dot{\gamma}}$

where $\tau_y = \tau(y_c) = y_c\frac{dP}{dx}$ and $\dot{\gamma} = \frac{du}{dy}$

$$\therefore \sqrt{y\frac{dP}{dx}} = \sqrt{y_c\frac{dP}{dx}} + \sqrt{\mu\frac{du}{dy}}$$

$$\Rightarrow \mu\frac{du}{dy} = \frac{dP}{dx}\left(y - 2y^{1/2}y_c^{1/2} + y_c\right)$$

$$\Rightarrow \mu u(y) = \frac{dP}{dx}\left(\frac{1}{2}y^2 - \frac{4}{3}y^{3/2}y_c^{1/2} + y y_c\right) + \text{const.}$$

Boundary condition: $u(R) = 0$

$$\therefore \text{const.} = -\frac{dP}{dx}\left(\frac{1}{2}R^2 - \frac{4}{3}R^{3/2}y_c^{1/2} + Ry_c\right)$$

$$\therefore u(y) = \frac{1}{2\mu}\frac{dP}{dx}\left[(y^2 - R^2) - \frac{8}{3}y_c^{1/2}(y^{3/2} - R^{3/2}) + 2y_c(y - R)\right]$$

$$\Rightarrow u_{plug} = u(y_c) = \frac{1}{2\mu}\frac{dP}{dx}\left[y_c^2 - R^2 - \frac{8}{3}y_c^2 + \frac{8}{3}y_c^{1/2}R^{3/2} + 2y_c^2 - 2y_cR\right]$$

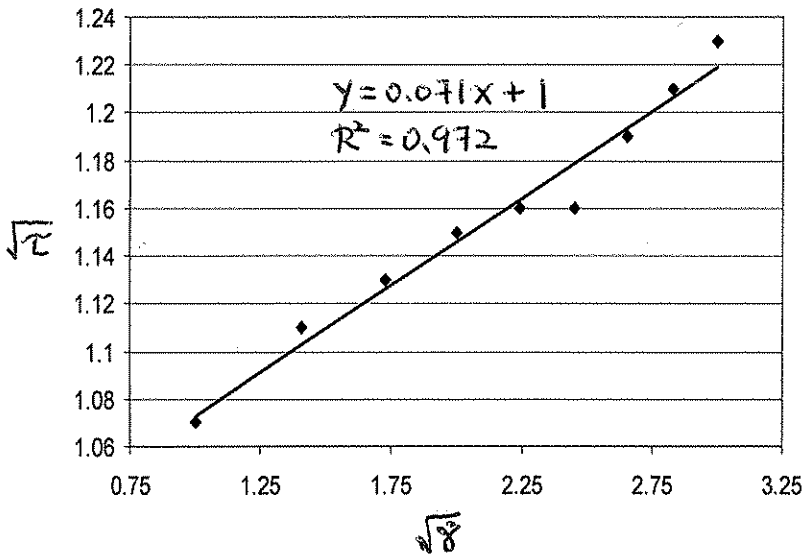
$$\therefore u_{plug} = \frac{-1}{2\mu}\frac{dP}{dx}\left[R^2 - \frac{y_c^2}{3} - \frac{8}{3}\sqrt{y_cR^3} + 2y_cR\right]$$

For Casson fluid,

$$\sqrt{\tau} = \sqrt{\tau_y} + \sqrt{\mu \dot{\gamma}}$$

Therefore, a plot of $\sqrt{\dot{\gamma}}$ vs. $\sqrt{\tau}$ should be a straight line with intercept $\sqrt{\tau_y}$ and slope $\sqrt{\mu}$.

shear rate (s^{-1})	shear stress (dynes/cm²)	$\sqrt{\dot{\gamma}}$ ($\sqrt{s^{-1}}$)	$\sqrt{\tau}$ ($\sqrt{\text{dynes/cm}^2}$)
1	1.14	1.00	1.07
2	1.24	1.41	1.11
3	1.28	1.73	1.13
4	1.32	2.00	1.15
5	1.34	2.24	1.16
6	1.35	2.45	1.16
7	1.42	2.65	1.19
8	1.46	2.83	1.21
9	1.50	3.00	1.23



intercept = 1.00
slope = 0.071

$\tau_y = 1.00$ dyne/cm²

$\mu = 5.04 \times 10^{-3}$ dyne·s/cm²

continued...

$$\therefore \xi = \frac{2\tau_y}{R\left|\frac{dp}{dx}\right|} = \frac{2 \cdot 1 \text{ dyne/cm}^2}{(2 \text{ cm})(50 \text{ dyne/cm}^2/10 \text{ cm})} = 0.20$$

$$\therefore F(\xi) = 1 - \frac{16}{7}\sqrt{\xi} + \frac{4}{3}\xi - \frac{1}{21}\xi^4 = 0.2444$$

$$\therefore Q = \frac{\pi R^4}{8\mu}\left|\frac{dP}{dx}\right| F(\xi)$$

$$= \frac{\pi (2 \text{ cm})^4}{8(5.04 \times 10^{-3} \text{ dyne}\cdot\text{s/cm}^2)} \left|\frac{50 \text{ dyne/cm}^2}{10 \text{ cm}}\right| \cdot 0.2444$$

$$= 1523 \text{ cm}^3/\text{s}$$

Given: $L = 10$ cm

$D = 0.6$ cm $\therefore R = 0.3$ cm

$\Delta P = 50$ dyne/cm^2 $\therefore \frac{dP}{dx} = 5$ dyne/cm^3

$\tau_y^A = 0.08$ dyne/cm^2 $\Rightarrow R_c^A = 2\tau_y^A / \frac{dP}{dx} = 0.032$ cm

$\tau_y^B = 0.12$ dyne/cm^2 $\Rightarrow R_c^B = 0.048$ cm

$$\therefore \xi_A = \frac{R_c^A}{R} = \frac{0.032 \text{ cm}}{0.3 \text{ cm}} = 0.107$$

$$\xi_B = \frac{R_c^B}{R} = 0.160$$

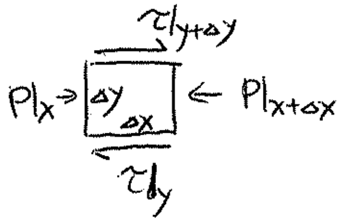
Since $Q = \frac{\pi R^4}{8\mu} \left| \frac{dP}{dx} \right| F(\xi)$

and R, μ, $\frac{dP}{dx}$ are the same for A and B

$$\therefore \frac{Q_A}{Q_B} = \frac{F(\xi_A)}{F(\xi_B)}, \text{ where } F(\xi) = 1 - \frac{16}{7}\xi^{1/2} + \frac{4}{3}\xi - \frac{1}{21}\xi^4$$

$$\therefore \frac{Q_A}{Q_B} = \frac{0.395}{0.299} = 1.32$$

a) Consider a fluid element:



by force balance:

$$P|_x \cdot \Delta y + \tau|_{y+\Delta y} \cdot \Delta x = P|_{x+\Delta x} \cdot \Delta y + \tau|_y \cdot \Delta x$$

$$\Rightarrow \frac{P|_{x+\Delta x} - P|_x}{\Delta x} = \frac{\tau|_{y+\Delta y} - \tau|_y}{\Delta y}$$

$$\therefore \frac{\partial P}{\partial x} = \frac{\partial \tau}{\partial y}$$

but $\frac{\partial P}{\partial x} = 0$ in this case $\therefore \frac{\partial \tau}{\partial y} = 0$

$\therefore \tau =$ constant at any location y.

but $\tau = F/A$ at $y = h$

$\therefore \tau = F/A$ at any location y

b) 2 cases:

i) $F/A < \tau_y \Rightarrow V = 0$, since there's no flow (i.e. $u = 0$)

ii) $F/A > \tau_y$

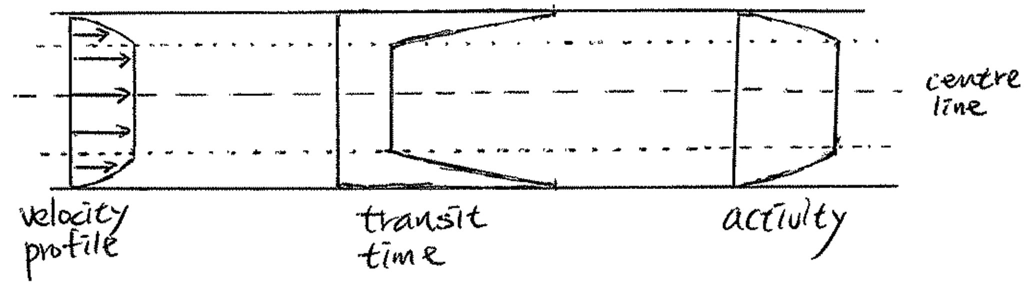
$$\sqrt{\tau} = \sqrt{\tau_y} + \sqrt{\mu \dot{\gamma}} \quad \text{or} \quad \sqrt{\frac{F}{A}} = \sqrt{\tau_y} + \sqrt{\mu \cdot \frac{du}{dy}}$$

$$\Rightarrow \left(\sqrt{\frac{F}{A}} - \sqrt{\tau_y}\right)^2 = \mu \frac{du}{dy}$$

Now integrate and then apply the boundary condition that $u = 0$ at $y = 0$ and $u = V$ at $y = h$ to derive the answer:

$$\therefore V = \frac{h}{\mu}\left(\frac{F}{A} + \tau_y - 2\sqrt{\frac{F\tau_y}{A}}\right)$$

a)



As velocity $\to 0$, transit time $\to \infty$, so $a \to 0$ at station B. In centre, velocity is maximum, so transitime is minimum and activity is maximum.

b) Given: $\frac{a(t)}{a_0} = 0.54 = e^{-t/\tau}$ $\Rightarrow t = -\tau \ln 0.54 = 4.31\ \text{s}$

$\tau = 7\ \text{s}$

For maximum activity, this corresponds to tube centre line region, i.e. u_{core}.

Also, blood travels 10 cm in 4.31 seconds,

$$\therefore u_{core} = \frac{10\ \text{cm}}{4.31\ \text{s}} = 2.32\ \text{cm/s}$$

From the text,

$$u_{core} = -\frac{R^2}{4\mu}\frac{dP}{dx}\left[1 - \frac{8}{3}\sqrt{\frac{R_c}{R}} + 2\cdot\frac{R_c}{R} - \frac{1}{3}\left(\frac{R_c}{R}\right)^2\right] \quad (*)$$

Call $\xi = \frac{R_c}{R}$, also $R_c = 2\tau_y / \left|\frac{dP}{dx}\right|$ $\Rightarrow -\frac{dP}{dx} = \frac{2\tau_y}{R\xi}$ $(**)$

plug the above into $(*)$,

$$u_{core} = \frac{2\tau_y}{4\mu}\frac{R}{\xi}\left[1 - \frac{8}{3}\xi^{1/2} + 2\xi - \frac{1}{3}\xi^2\right]$$

b) continued...

$$\therefore \xi^{-1} - \frac{8}{3}\xi^{-1/2} + 2 - \frac{1}{3}\xi = \frac{2\mu u_{core}}{R\tau_y}$$

$$= \frac{2(0.035\ g/cm\cdot s)(2.32\ cm/s)}{(1\ cm)(0.05\ dyne/cm^2)}$$

$$= 3.24$$

$$\therefore \xi^{-1} - \frac{8}{3}\xi^{-1/2} - \frac{1}{3}\xi^{2} = 1.248$$

solve numerically: (given $0.10 \leq \xi \leq 0.11$)

	ξ	L.H.S	
	0.100	1.564	
	0.105	1.291	
	0.110	1.407	
check	0.106	1.240	ok

From (**), $-\frac{dP}{dx} = \frac{2(0.05\ dyne/cm^2)}{(1\ cm)(0.106)} = 0.943\ dyne/cm^3$

$$\therefore Q = -\frac{\pi R^4}{8\mu}\frac{dP}{dx}\left(1 - \frac{16}{7}\xi^{1/2} + \frac{4}{3}\xi - \frac{1}{21}\xi^4\right)$$

$$= +\frac{\pi(1\ cm)^4}{8(0.035\ g/cm\cdot s)}(0.943\ dyne/cm^3)\left(1 - \frac{16}{7}(0.106)^{1/2} + \frac{4}{3}(0.106) - \frac{0.106^4}{21}\right)$$

$$= 4.204\ cm^3/s$$

Given : diameter, $D = 8$ cm

heart beat : 35 beats/min $\Rightarrow$ heart period $= \frac{60 \text{ s/min}}{35 \text{ beats/min}}$

$\therefore \quad T = 1.71$ s

$\therefore \omega = \frac{2\pi}{T} = 3.67 \text{ s}^{-1}$

Womersley parameter : $\alpha = \frac{D}{2}\sqrt{\frac{\omega}{\nu}} = \frac{8 \text{ cm}}{2} \cdot \sqrt{\frac{3.67 \text{ s}^{-1}}{0.035 \text{ cm}^2/\text{s}}} = 40.96$

Here we have assumed that $\nu = \frac{\mu}{\rho}$ for elephant blood is the same as for human blood.

In this range of α, velocity profiles are nearly flat, oscillatory back and forth. (Viscous layer is very small, so most fluid simply oscillates back and forth.)

We know $\tau = \mu \frac{du}{dy}$ for a Newtonian fluid.

The wall shear stress is given by

$$\tau_w = \mu \frac{du}{dy}\Big|_{y=R} = \frac{\mu}{R} \frac{du}{d\hat{y}}\Big|_{\hat{y}=1} \qquad \text{where } \hat{y} = y/R.$$

From the book, we have the solution for the velocity as

$$u(y,t) = \mathrm{Re}\left\{ \frac{i\Pi}{\rho\omega} \left[\frac{\cosh(\alpha \hat{y} \sqrt{i})}{\cosh(\alpha\sqrt{i})} - 1 \right] e^{i\omega t} \right\}$$

$$\therefore \tau_w = \mathrm{Re}\left\{ \frac{i\Pi\mu}{\rho\omega R} e^{i\omega t} \frac{d}{d\hat{y}} \left[\frac{\cosh(\alpha\hat{y}\sqrt{i})}{\cosh(\alpha\sqrt{i})} - 1 \right] \Big|_{\hat{y}=1} \right\}$$

$$= \mathrm{Re}\left\{ \frac{i\Pi\mu\, e^{i\omega t}}{\rho\omega R} \alpha\sqrt{i} \tanh(\alpha\sqrt{i}) \right\}$$

$$= \mathrm{Re}\left\{ \frac{\alpha\, i^{3/2}\, \Pi\mu}{\rho\omega R} \tanh(\alpha\sqrt{i})\, e^{i\omega t} \right\}$$

a) Momentum is the product of mass and velocity. Therefore, for two elements with equal mass, the element with the greater velocity will have more momentum. For Womersley flow, the fluid velocity is a maximum at the centre line of the tube (i.e. $r=0$) and a minimum at the wall (due to no-slip condition). Therefore, the fluid element near the centre line of the pipe will have greater momentum.

b) The fluid element with less "forward" momentum will change its direction first when acting upon by a pressure gradient in the "backward" direction. This can be shown through the impulse-momentum relationship, which is basically $F=ma$ written in terms of momentum:

$$F = ma = m\frac{dv}{dt} \Rightarrow \Delta t = \frac{m\Delta v}{F}$$

Therefore, an element with less momentum (i.e. the chunk near the wall) will require less time to change its velocity and therefore, direction

c) Recall $\alpha = R\sqrt{\frac{\rho\omega}{\mu}}$. The units of each parameter are:

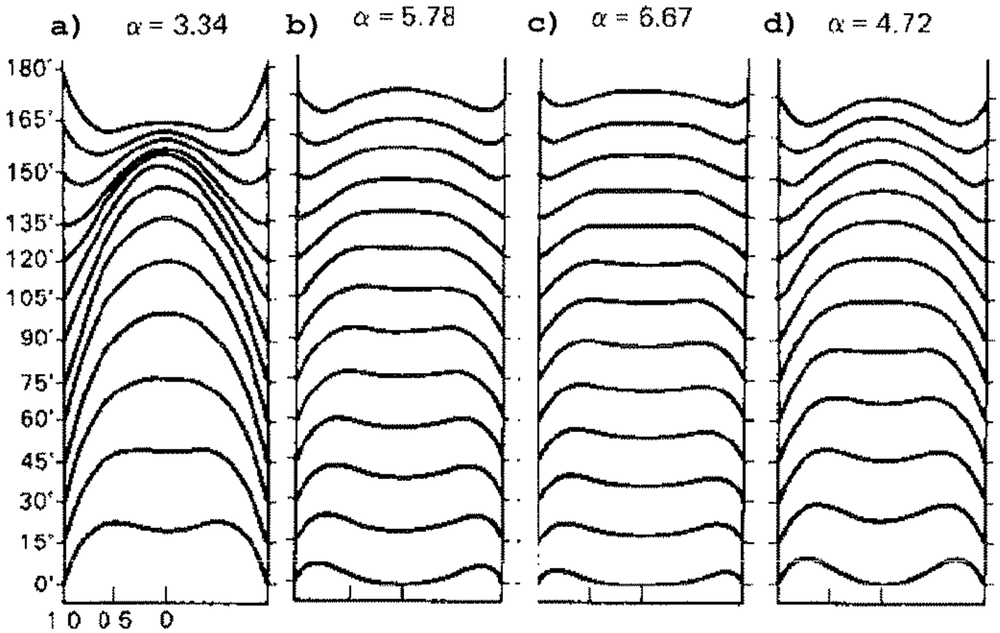
$$[R] \sim L \quad [\rho] \sim ML^{-3} \quad [\omega] \sim T^{-1} \quad [\mu] \sim ML^{-1}T^{-1}$$

Therefore, the units of the Womersley parameter are:

$$[\alpha] = L\,(ML^{-3})^{1/2}(T^{-1})^{1/2}(M^{-1}LT)^{1/2} = M^0L^0T^0$$

$\therefore$ α is dimensionless

d) The Womersley parameters for each flow profile are indicated in the figure below. Because $\alpha \propto \omega^{1/2}$, a greater frequency will produce greater unsteadiness (a higher α). However, a greater frequency means that fluid has less time to react to the fluctuating pressure changes, with the result being a "flatter" velocity profile for a more unsteady flow.



Given: - eq'n 3.29 $u(y,t) = \Re\left\{ \frac{i\Pi}{\rho\omega}\left[\frac{\cosh(\alpha\hat{y}\sqrt{i})}{\cosh(\alpha\sqrt{i})} - 1 \right] e^{i\omega t} \right\}$

- $\alpha = R\sqrt{\frac{\rho\omega}{\mu}}$

- $-\frac{\partial p}{\partial x} = \Pi\cos(\omega t) = \Re\left\{ \Pi e^{i\omega t} \right\}$

in the limit of $\alpha \to 0$

$$\cosh(\alpha\hat{y}\sqrt{i}) \to 1 + \frac{1}{2}(\alpha\hat{y}\sqrt{i})^2 = 1 + \frac{1}{2}\alpha^2\hat{y}^2 i$$

$$\cosh(\alpha\sqrt{i}) \to 1 + \frac{1}{2}(\alpha\sqrt{i})^2 = 1 + \frac{1}{2}\alpha^2 i$$

$$\Rightarrow u(y,t) \approx \Re\left\{ \frac{i\Pi}{\rho\omega}\left[\frac{1 + \frac{1}{2}\alpha^2\hat{y}^2 i}{1 + \frac{1}{2}\alpha^2 i} - 1 \right] e^{i\omega t} \right\}$$

Note: $$\frac{1 + \frac{1}{2}\alpha^2\hat{y}^2 i}{1 + \frac{1}{2}\alpha^2 i} - 1 = \frac{\frac{1}{2}\alpha^2 i(\hat{y}^2 - 1)}{1 + \frac{1}{2}\alpha^2 i}$$

$$= \frac{\frac{1}{2}\alpha^2 i(\hat{y}^2 - 1)(1 - \frac{1}{2}\alpha^2 i)}{1 - \frac{1}{4}\alpha^4} \quad ; \alpha^4 \approx 0$$

$$\approx \frac{1}{2}\alpha^2 i(\hat{y}^2 - 1)(1 - \frac{1}{2}\alpha^2 i)$$

$$\therefore u(y,t) \approx \Re\left\{ \frac{-\Pi}{\rho\omega} \cdot \frac{1}{2}\alpha^2(\hat{y}^2 - 1)(1 - \frac{1}{2}\alpha^2 i)(\cos\omega t + i\sin\omega t) \right\}$$

$$= -\frac{\Pi}{\rho\omega} \cdot \frac{1}{2}\alpha^2(\hat{y}^2 - 1)\,\Re\left\{ \cos\omega t + i\sin\omega t - \frac{1}{2}\alpha^2 i\cos\omega t + \frac{1}{2}\alpha^2\sin\omega t \right\}$$

$$= -\frac{\Pi}{\rho\omega} \cdot \frac{1}{2}\alpha^2(\hat{y}^2 - 1)(\cos\omega t + \frac{1}{2}\alpha^2\sin\omega t)$$

$$= -\frac{\Pi}{\rho\omega}(\hat{y}^2 - 1)\left(\frac{1}{2}\alpha^2\cos\omega t + \cancel{\frac{1}{4}\alpha^4\sin\omega t}\right) \quad \text{neglect}$$

$$= \frac{\Pi}{\rho\omega} \cdot \frac{1}{2} \cdot R^2 \cdot \frac{\rho\omega}{\mu}\left[1 - \left(\frac{y}{R}\right)^2 \right]\cos\omega t$$

$$= \Pi \cdot \frac{R^2}{2\mu}\left[1 - \left(\frac{y}{R}\right)^2 \right]\cos\omega t$$

continued...

Bonus:

eqn 3.30: $u(r,t) = \Re\left\{ \frac{i\Pi}{\rho\omega}\left[\frac{J_0(\alpha \hat{r} i^{3/2})}{J_0(\alpha i^{3/2})} - 1 \right] e^{i\omega t} \right\}$

In the limit of $\alpha \to 0$

$$J_0(\alpha \hat{r} i^{3/2}) \to 1 - \frac{1}{4}(\alpha \hat{r} i^{3/2})^2 = 1 + \frac{1}{4}\alpha^2 \hat{r}^2 i$$

$$J_0(\alpha i^{3/2}) \to 1 - \frac{1}{4}(\alpha i^{3/2})^2 = 1 + \frac{1}{4}\alpha^2 i$$

$$\therefore \frac{J_0(\alpha \hat{r} i^{3/2})}{J_0(\alpha i^{3/2})} - 1 \to \frac{1 + \frac{1}{4}\alpha^2 \hat{r}^2 i}{1 + \frac{1}{4}\alpha^2 i} - 1$$

$$= \frac{(1 + \frac{1}{4}\alpha^2 \hat{r}^2 i)(1 - \frac{1}{4}\alpha^2 i)}{1 - \cancel{\frac{1}{16}\alpha^4} \to \text{neglect}} - 1$$

$$= 1 - \frac{1}{4}\alpha^2 i + \frac{1}{4}\alpha^2 \hat{r}^2 i + \underbrace{\cancel{\frac{1}{16}\alpha^4 \hat{r}^2}}_{\text{neglect}} - 1$$

$$= \frac{1}{4}\alpha^2 i (\hat{r}^2 - 1)$$

$$\therefore u(r,t) \approx \Re\left\{ \frac{i\Pi}{\rho\omega} \cdot \frac{1}{4}\alpha^2 i (\hat{r}^2 - 1)(\cos\omega t + i\sin\omega t) \right\}$$

$$= -\frac{\Pi}{\rho\omega} \cdot \frac{1}{4} \cdot R^2 \cdot \frac{\rho\omega}{\mu}\left[\left(\frac{r}{R}\right)^2 - 1 \right] \cos\omega t$$

$$= \Pi \cdot \frac{R^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2 \right] \cos\omega t$$

a)

equation (3.29): $u(y,t) = \mathcal{R}\left\{ \frac{i\Pi}{\rho w}\left[\frac{\cosh(\alpha \hat{y}\sqrt{i})}{\cosh(\alpha\sqrt{i})} - 1 \right] e^{i w t} \right\}$

To find the flow rate in a 2D channel of half-height R

⇒ integrate (3.29) from $y=0$ to $y=R$ (i.e. $\hat{y}=0$ to $\hat{y}=1$)

$$Q(t) = 2\int_0^1 \mathcal{R}\left\{ \frac{i\Pi}{\rho w}\left[\frac{\cosh(\alpha\hat{y}\sqrt{i})}{\cosh(\alpha\sqrt{i})} - 1 \right] e^{iwt} \right\} R \cdot d\hat{y} \qquad \because \hat{y} = y/R, \quad dy = R\,d\hat{y}$$

$$= 2R \times \mathcal{R}\left\{ \frac{i\Pi}{\rho w}\left[\frac{1}{\alpha\sqrt{i}} \frac{\sinh(\alpha\hat{y}\sqrt{i})}{\cosh(\alpha\sqrt{i})} - \hat{y} \right] e^{iwt} \right\}\Bigg]_0^1$$

$$= 2R \times \mathcal{R}\left\{ \frac{i\Pi}{\rho w}\left[\frac{\tanh(\alpha\sqrt{i})}{\alpha\sqrt{i}} - 1 \right] e^{iwt} \right\}$$

$$= \mathcal{R}\left\{ \frac{2i\Pi R}{\rho w}\left[\frac{\tanh(\alpha\sqrt{i})}{\alpha\sqrt{i}} - 1 \right] e^{iwt} \right\}$$

b) If $Q(t) = \mathcal{R}\{\Lambda e^{iwt}\}$, by direct comparison to the expression in a), we know

$$\Lambda = \frac{i\Pi}{\rho w} \cdot 2R\left[\frac{\tanh(\alpha\sqrt{i})}{\alpha\sqrt{i}} - 1 \right]$$

$$= \frac{i\Pi}{\rho w} \cdot 2R\left[\frac{\sinh(\alpha\sqrt{i}) - \cosh(\alpha\sqrt{i})\cdot\alpha\sqrt{i}}{\cosh(\alpha\sqrt{i})\cdot\alpha\sqrt{i}} \right]$$

$$\therefore u(y,t) = \mathcal{R}\left\{ \frac{\Lambda}{2R}\left[\frac{\cosh(\alpha\sqrt{i})\cdot\alpha\sqrt{i}}{\sinh(\alpha\sqrt{i}) - \cosh(\alpha\sqrt{i})\cdot\alpha\sqrt{i}} \right]\left[\frac{\cosh(\alpha\hat{y}\sqrt{i}) - \cosh(\alpha\sqrt{i})}{\cosh(\alpha\sqrt{i})} \right] e^{iwt} \right\}$$

$$= \mathcal{R}\left\{ \frac{\alpha\sqrt{i}\Lambda}{2R}\left[\frac{\cosh(\alpha\hat{y}\sqrt{i}) - \cosh(\alpha\sqrt{i})}{\sinh(\alpha\sqrt{i}) - \cosh(\alpha\sqrt{i})\cdot\alpha\sqrt{i}} \right] e^{iwt} \right\}$$

c)

eq'n 3.30 : $u(r,t) = \mathcal{R}\left\{ \frac{i\Pi}{\rho\omega}\left[\frac{J_0(\alpha \hat{r} i^{3/2})}{J_0(\alpha i^{3/2})} - 1 \right] e^{i\omega t} \right\}$

$$\therefore Q = \int_0^1 \mathcal{R}\left\{ \frac{i\Pi}{\rho\omega}\left[\frac{J_0(\alpha \hat{r} i^{3/2})}{J_0(\alpha i^{3/2})} - 1 \right] e^{i\omega t} \cdot 2\pi \hat{r} \cdot R^2 d\hat{r} \right. \quad ; \quad \begin{matrix} \hat{r} = r/R \\ dr = R\,d\hat{r} \end{matrix}$$

$$= 2\pi R^2 \int_0^1 \mathcal{R}\left\{ \frac{i\Pi}{\rho\omega}\left[\frac{J_0(\alpha \hat{r} i^{3/2}) \cdot (\alpha \hat{r} i^{3/2})}{J_0(\alpha i^{3/2}) \cdot (\alpha i^{3/2})} - \hat{r} \right] e^{i\omega t} \right\} d\hat{r}$$

Note: $\int J_0(\alpha \hat{r} i^{3/2}) \cdot (\alpha \hat{r} i^{3/2}) \cdot d\hat{r} = (\alpha \hat{r} i^{3/2}) J_1(\alpha \hat{r} i^{3/2}) \cdot \frac{1}{\alpha i^{3/2}}$

$$\therefore Q = 2\pi R^2 \mathcal{R}\left\{ \frac{i\Pi}{\rho\omega}\left[\frac{\hat{r} J_1(\alpha \hat{r} i^{3/2})}{(\alpha i^{3/2}) J_0(\alpha i^{3/2})} - \frac{1}{2}\hat{r}^2 \right] e^{i\omega t} \right\} \Bigg]_0^1$$

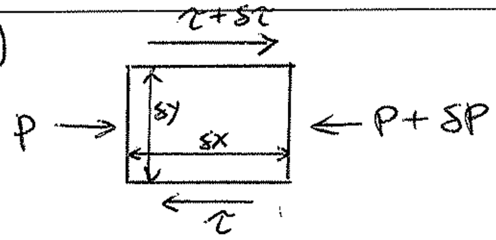
$$= \cancel{2}\pi R^2 \mathcal{R}\left\{ \frac{i\Pi}{\rho\omega}\left[\frac{2J_1(\alpha i^{3/2}) - (\alpha i^{3/2}) J_0(\alpha i^{3/2})}{\cancel{2} J_0(\alpha i^{3/2}) \cdot (\alpha i^{3/2})} \right] e^{i\omega t} \right\}$$

$$\therefore \Lambda = \pi R^2 \cdot \frac{i\Pi}{\rho\omega}\left[\frac{2J_1(\alpha i^{3/2}) - (\alpha i^{3/2}) \cdot J_0(\alpha i^{3/2})}{(\alpha i^{3/2}) \cdot J_0(\alpha i^{3/2})} \right]$$

$$\therefore u(r,t) = \mathcal{R}\left\{ \frac{\alpha i^{3/2} \Lambda}{\pi R^2}\left[\frac{J_0(\alpha i^{3/2})}{2J_1(\alpha i^{3/2}) - (\alpha i^{3/2}) J_0(\alpha i^{3/2})} \right]\left[\frac{J_0(\alpha \hat{r} i^{3/2}) - J_0(\alpha i^{3/2})}{J_0(\alpha i^{3/2})} \right] e^{i\omega t} \right\}$$

$$= \mathcal{R}\left\{ \frac{\alpha i^{3/2} \Lambda}{\pi R^2}\left[\frac{J_0(\alpha \hat{r} i^{3/2}) - J_0(\alpha i^{3/2})}{2J_1(\alpha i^{3/2}) - (\alpha i^{3/2}) J_0(\alpha i^{3/2})} \right] e^{i\omega t} \right\}$$

a)



Assuming unit width into the page.

Force balance:

$$p\delta y + (\tau+\delta\tau)\delta x - (p+\delta p)\delta y - \tau(\delta x) = 0$$

$$\delta\tau\,\delta x - \delta p\,\delta y = 0$$

In the limit of $\delta x, \delta y \to 0$

$$\Rightarrow \quad \frac{d\tau}{dy} = \frac{\partial p}{\partial x}$$

$$\therefore \ \tau(y) = \frac{\partial p}{\partial x} y + \text{const.}$$

but by symmetry, $\tau(H/2) = 0 = \frac{\partial p}{\partial x}\left(\frac{H}{2}\right) + \text{const.}$

$$\therefore \ \text{const.} = -\frac{\partial p}{\partial x}\left(\frac{H}{2}\right)$$

$$\therefore \ \tau(y) = (y - H/2)\frac{\partial p}{\partial x}$$

b) $\tau(y) = \left(y - \frac{H}{2}\right)\frac{dp}{dx} = \mu\frac{du}{dy}$

$$du = \frac{1}{\mu}\frac{dp}{dx}\left(y - \frac{H}{2}\right)dy$$

$$u(y) = \frac{1}{\mu}\frac{dp}{dx}\left(\frac{1}{2}y^2 - \frac{H}{2}y\right) + C \quad , \ C \text{ is a const.}$$

$$\Rightarrow u(y) = -\frac{1}{2\mu}\frac{dp}{dx}\left[y(H-y) + A\right] \quad , \ A \text{ is a const.}$$

which applies to both regions.

$$\therefore \ u_i(y) = -\frac{1}{2\mu_i}\frac{\partial p}{\partial x}\left[y(H-y) + A_i\right] \quad , \ i = 1, 2.$$

c)

$\because u_1(y=0)=0$

$\therefore 0 = -\frac{1}{2\mu_1}\frac{dP}{dx}\left[0 + A_1\right]$

since $\mu_1 \neq 0$, $\frac{dP}{dx} \neq 0$, A_1 must be 0

$\because u_1(y=\delta) = u_2(y=\delta)$

$\therefore -\frac{1}{2\mu_1}\frac{dP}{dx}\left[\delta(H-\delta)\right] = -\frac{1}{2\mu_2}\frac{dP}{dx}\left[\delta(H-\delta)+A_2\right]$

$\therefore A_2 = \delta(H-\delta)\left[\frac{\mu_2}{\mu_1} - 1\right]$

$A_1 = 0$

d)

Region 1 ($0<y<\delta$),

$$Q_1 = \int_0^\delta u_1(y)\,dy = \int_0^\delta \left(-\frac{1}{2\mu}\frac{dP}{dx}\right)(yH - y^2)\,dy$$

$$= \left(-\frac{1}{2\mu}\frac{dP}{dx}\right)\left(\frac{1}{2}H\delta^2 - \frac{1}{3}\delta^3\right)$$

$\because \delta^2$ and δ^3 are small (we will retain only 1st order terms of δ eventually)

$\therefore Q_1 \simeq 0$

Region 2 ($\delta<y<H/2$),

$$Q_2 = \int_\delta^{H/2} u_2(y)\,dy = \int_\delta^{H/2} -\frac{1}{2\mu_2}\frac{dP}{dx}\left[yH - y^2 + (\delta H - \delta^2)\left(\frac{\mu_2}{\mu_1}-1\right)\right]dy$$

$$= -\frac{1}{2\mu_2}\frac{dP}{dx}\left[\frac{H}{2}y^2 - \frac{1}{3}y^3 + \left(\frac{\mu_2}{\mu_1}-1\right)\delta H y - \left(\frac{\mu_2}{\mu_1}-1\right)\delta^2 y\right]_\delta^{H/2}$$

d) continued . . .

$$\therefore Q_2 = -\frac{1}{2\mu_2}\frac{dP}{dx}\left[\frac{H^3}{8} - \frac{H^3}{24} + \left(\frac{\mu_2}{\mu_1}-1\right)\delta\frac{H^2}{2} - \left(\frac{\mu_2}{\mu_1}-1\right)\delta^2\frac{H}{2} - \frac{H}{2}\delta^2 + \frac{1}{3}\delta^3 - \left(\frac{\mu_2}{\mu_1}-1\right)H\delta^2 + \left(\frac{\mu_2}{\mu_1}-1\right)\delta^3\right]$$

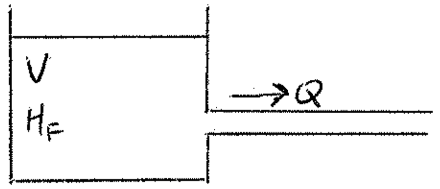
$$\simeq -\frac{1}{2\mu_2}\frac{dP}{dx}\left[\frac{H^3}{8} - \frac{H^3}{24} + \left(\frac{\mu_2}{\mu_1}-1\right)\delta\frac{H^2}{2}\right]$$ since we only retain 1^{st} and 0^{th} order of δ

$$= -\frac{H^3}{24}\frac{dP}{dx}\cdot\frac{1}{\mu_2}\left[\frac{3}{2} - \frac{1}{2} + \left(\frac{\mu_2}{\mu_1}-1\right)\delta\cdot\frac{6}{H}\right]$$

$\therefore$ by comparison,

$$\mu_{eff} = \frac{\mu_2}{1 + 6\delta\left(\frac{\mu_2}{\mu_1}-1\right)/H}$$

a)



$$V = V^{o} - Qt = V_{RBC} + V_p \quad \Rightarrow \frac{dV}{dt} = -Q \quad -①$$

$$\frac{dV_{RBC}}{dt} = -Q \times \text{tube hematocrit}$$

since $H_R = \dfrac{\text{tube hematocrit}}{H_F}$

$$\therefore \frac{dV_{RBC}}{dt} = -Q H_F H_R \quad -②$$

Now, we know $H_F = \dfrac{V_{RBC}}{V_{RBC} + V_p} = \dfrac{V_{RBC}}{V}$ or $V_{RBC} = V H_F$ $-③$

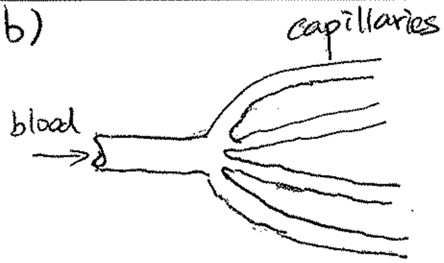
$$\therefore \frac{dH_F}{dt} = \frac{1}{V}\frac{dV_{RBC}}{dt} + \left(-\frac{1}{V^2}\right)\frac{dV}{dt} \cdot V_{RBC}$$

sub ①, ②, and ③ :

$$\frac{dH_F}{dt} = \frac{-QH_F H_R}{V} + \frac{QH_F}{V} = \frac{QH_F}{V}(1 - H_R)$$

$$\therefore \frac{dH_F}{dt} = \frac{QH_F}{V^{o} - Qt}(1 - H_R)$$

b)



Here the hematocrit does increase near the entry to the junction, but not in an unbalanced manner. It just has to increase enough so that each new red blood cell supplied by the blood is balanced by one red blood cell leaving. That means the local hematocrit at the entrance to the capillaries will increase to H/H_R, where H is the bulk blood hematocrit.

Given: R_c = radius of the core region = 40 μm
R = radius of the capillary = 50 μm
H = hematocrit in the core = 0.45

Velocity profile:

Blunt portion: $u(r) = \text{const.} = u_{core}$

Parabolic portion: $u(r) = u_{core}\left[1 - \left(\frac{r-R_c}{R-R_c}\right)^2\right]$, $R_c \le r \le R$

$= u_{core}\left[1 - \left(\frac{r-40}{10}\right)^2\right]$, $40 \le r \le 50$ μm

(Note that this form of $u(r)$ satisfies no-slip and joins smoothly with the core flow (velocity gradient = 0 at $r = R_c$))

Let V_{RBC} be the volume of red blood cells in the beaker
V_M be the total volume of the mixture in the beaker

$$\frac{dV_{RBC}}{dt} = H \cdot u_{core} \cdot \pi \cdot R_c^2 = u_{core}(0.45 \cdot 40^2)\pi = 2.26 \times 10^3 \cdot u_{core} \ \mu m^3/s$$

$$\frac{dV_M}{dt} = 2\pi \int_0^R u(r)\, r \cdot dr$$

$$= 2\pi \int_0^{40} u_{core} \cdot r\, dr + 2\pi \int_{40}^{50} u_{core}\left[1 - \frac{(r-40)^2}{10^2}\right] r\, dr$$

$$= u_{core} \cdot \pi \cdot 40^2 + 2\pi \cdot u_{core} \int_{40}^{50} \left[1 - \frac{(r-40)^2}{10^2}\right] r\, dr$$

$$= u_{core}\pi \left[40^2 + 2\int_0^1 (1-\hat{r}^2)(10\hat{r} + 40) \cdot 10\, d\hat{r}\right]$$

where $\hat{r} = \frac{r-40}{10}$

continued...

$$\therefore \frac{dV_M}{dt} = u_{core} \cdot \pi \left[40^2 + 200 \int_0^1 (\hat{r} - \hat{r}^3 + 4 - 4\hat{r}^2)\, d\hat{r} \right]$$

$$= u_{core} \pi \left[40^2 + 200 \left(\frac{1}{2} - \frac{1}{4} + 4 - \frac{4}{3} \right) \right]$$

$$= 6.86 \times 10^3 \cdot u_{core} \ \mu m^3/s$$

$\therefore$ Over a period of time T, the hematocrit in the beaker will be,

$$H = \frac{V_{RBC}}{V_M} = \frac{2.26 \times 10^3 u_{core} \cdot T}{6.86 \times 10^3 u_{core} \cdot T} = 0.33$$

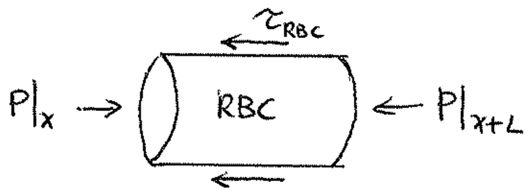
a)

$$\tau_{RBC} = \mu_p \frac{du}{dy}\bigg|_{y=-h}$$

$$\frac{du}{dy} = \frac{V_{RBC}}{2}\left(-\frac{1}{h}\right) + \frac{h^2}{2\mu_p}\frac{dP}{dx}\left[2 \cdot \frac{y}{h^2}\right]$$

$$\therefore \tau_{RBC} = \mu_p\left[-\frac{V_{RBC}}{2h} - \frac{h}{\mu_p}\frac{dP}{dx}\right]$$

$$\therefore \text{shear force} = 2\pi(R-2h)L\,\tau_{RBC}$$
$$= 2\pi(R-2h)Lh\left[-\frac{V_{RBC}}{2h^2}\mu_p - \frac{dP}{dx}\right]$$



Force balance:

$$P|_x \pi(R-2h)^2 - 2\pi(R-2h)Lh\left[\frac{V_{RBC}}{2h^2}\mu_p + \frac{dP}{dx}\right] - P|_{x+L}\pi(R-2h)^2 = 0$$

$$\Rightarrow -\frac{dP}{dx}\cdot L\cdot \pi(R-2h)^2 = 2\pi(R-2h)Lh\left[\frac{V_{RBC}}{2h^2}\mu_p + \frac{dP}{dx}\right]$$

$$\Rightarrow V_{RBC} = -\frac{dP}{dx}\frac{Rh}{\mu_p}$$

Here we have neglected end effects on the cell, so that the pressure varies linearly with distance along the cell, and $(P|_{x+L} - P|_x)/L$ is simply $\frac{dP}{dx}$

b) Matching V_{RBC} to V_{avg}.

$$\Rightarrow V_{RBC} = -\frac{dP}{dx}\frac{Rh}{\mu_p} = V_{avg} = -\frac{dP}{dx}\frac{R^2}{8\mu_{eff}}$$

$$\Rightarrow \mu_{eff} = \frac{R}{8h}\mu_p$$

a) Given: $d = 99\ \mu m = 99 \times 10^{-4}\ cm \Rightarrow R = 49.5 \times 10^{-4}\ cm$

$L = 0.1\ cm$

$\Delta P = 60000\ dynes/cm^2$

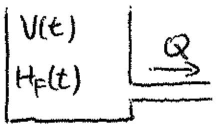
$\mu_{eff} = 2.3\ cP = 2.3 \times 10^{-2}\ g/cm \cdot s$

$$Q = \frac{\pi R^4}{8 \mu_{eff}} \cdot \frac{\Delta P}{L}$$

$$= \frac{\pi (49.5 \times 10^{-4}\ cm)^4}{8(2.3 \times 10^{-2}\ g/cm \cdot s)} \cdot \frac{60000\ dynes/cm^2}{0.1\ cm}$$

$$= 6.15 \times 10^{-3}\ cm^3/s$$

b)



$V = V_0 - Qt \Rightarrow \frac{dV}{dt} = -Q$

let V_{RBC} be the volume of red blood cell in the feed reservoir

$$\therefore H_F = \frac{V_{RBC}}{V} \quad \text{and} \quad \frac{dV_{RBC}}{dt} = -Q H_F H_R$$

$$\therefore \frac{dH_F}{dt} = \frac{1}{V}\frac{dV_{RBC}}{dt} - \frac{1}{V^2}\frac{dV}{dt} \cdot V_{RBC}$$

$$= \frac{-Q H_F H_R}{V} - \frac{1}{V^2}(-Q) \cdot (V H_F) = \frac{Q H_F}{V}(1 - H_R)$$

$$\therefore \frac{dH_F}{H_F} = \frac{Q}{V}(1 - H_R) dt = -\frac{Q}{V}(1 - H_R) \cdot \frac{1}{Q} dV$$

$$\Rightarrow \int_{H_{F,0}}^{H_F} \frac{dH_F}{H_F} = -(1 - H_R) \int_{V_0}^{V} \frac{dV}{V} \Rightarrow \ln \frac{H_F}{H_{F,0}} = -(1 - H_R) \ln \frac{V}{V_0}$$

$$\therefore \frac{H_F(t)}{H_{F,0}} = \left(\frac{V_0}{V_0 - Qt} \right)^{1 - H_R}$$

c) Given: $H_{F,0} = 30\%$, $V_0 = 5\,mL = 5\,cm^3$

$t = 3$ minute $= 180\,s$

$Q = 6.15 \times 10^{-3}\,cm^3/s$

$H_R \simeq 84\%$ according to Fig. 3-16

$$\therefore H_F(t=180s) = \left(\frac{5\,cm^3}{5\,cm^3 - (6.15\times10^{-3}\,cm^3/s)(180s)}\right)^{1-0.84} \cdot 0.30$$

$$= 31.2\%$$

d)

From part b), we know $\frac{dH_F}{dt} = \frac{QH_F}{V}(1-H_R)$

Now, $H_R = aH_F + b$

$$\therefore \frac{dH_F}{dt} = \frac{QH_F}{V}\left((1-b) - aH_F\right)$$

$$\therefore \frac{dH_F}{H_F[(1-b)-aH_F]} = \frac{Q}{V}dt = -\frac{dV}{V}$$

$$\Rightarrow \int_{H_{F_0}}^{H_F}\left[\frac{1/(1-b)}{H_F} + \frac{a/(1-b)}{[(1-b)-aH_F]}\right]dH_F = -\int_{V_0}^{V}\frac{dV}{V}$$

$$\Rightarrow \frac{1}{1-b}\ln\frac{H_F}{H_{F,0}} - \frac{1}{1-b}\ln\frac{(1-b)-aH_F}{(1-b)-aH_{F,0}} = -\ln\frac{V}{V_0}$$

$$\Rightarrow \frac{1}{1-b}\left[\ln\left(\frac{H_F}{H_{F,0}} \cdot \frac{(1-b)-aH_{F,0}}{(1-b)-aH_F}\right)\right] = -\ln\frac{V}{V_0}$$

$$\therefore \frac{H_F}{H_{F,0}} \cdot \frac{1-H_{R_0}}{1-H_R} = \left(\frac{V_0}{V_0-Qt}\right)^{1-b}$$

$$\therefore \frac{H_F}{H_{F,0}} = \left(\frac{V_0}{V_0-Qt}\right)^{1-b} \cdot \frac{1-H_R}{1-H_{R,0}} \quad \text{or} \quad \frac{H_F}{1-b-aH_F} = \frac{H_{F,0}}{1-b-aH_{F,0}}\left(\frac{V_0}{V_0-Qt}\right)^{1-b}$$

a) Given: $p = 120 \text{ mmHg} = 1.6 \times 10^4 \text{ N/m}^2$
$V = 100 \text{ cm/s} = 1 \text{ m/s}$
} typical pressure and velocity at peak systole in the proximal aorta.

$\rho_{blood} = 1060 \text{ kg/m}^3$ (from the text)

Pressure head:

$$h_p = \frac{p}{\rho_{blood} g} = \frac{1.6 \times 10^4 \text{ N/m}^2}{1060 \text{ kg/m}^3 \cdot 9.81 \text{ m/s}^2} = 1.54 \text{ m}$$

kinetic energy head:

$$h_k = \frac{V^2}{2g} = \frac{(1 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} = 0.051 \text{ m}$$

Sum of two heads: $h = h_p + h_k = 1.591 \text{ m}$

Kinetic energy head is $\frac{0.051}{1.591} = 3.2\%$

$\therefore$ We can safely neglect kinetic energy gains in calculating pump power.

b)

Given: $Q = 225 \text{ mL/min} = 225 \times \frac{10^{-3} \text{ L}}{60 \text{ s}} = 3.75 \times 10^{-3} \text{ L/s}$

O_2 capacity $= \frac{19.4 \text{ mL } O_2}{100 \text{ mL blood}} = 0.194 \text{ L } O_2/\text{L blood}$

Food energy released $= 4.83 \text{ kcal/L } O_2$

$= 4830 \text{ cal/L } O_2 \times \frac{4.186 \text{ J}}{1 \text{ cal}}$

$= 2.022 \times 10^4 \text{ J/L } O_2$

b) continued...

∴ the O_2 removal rate is

$$3.75\times10^{-3}\ L_{blood}/s \times 0.194\ L\,O_2/L_{blood} \times 65\%$$
$$= 4.73\times10^{-4}\ L\,O_2/s$$

∴ Energy consumed $= 2.022\times10^{4}\ J/L\,O_2 \times 4.73\times10^{-4}\ L\,O_2/s$
$$= 9.56\ J/s$$

$$\eta_{heart} = \frac{\text{power out}}{\text{power in}} = \frac{2\ W}{9.56\ W} = 20.9\%$$

The main assumption is that the food energy released goes towards muscular contractions.

c)

Metabolic rate $= 72\times10^{3}\ cal/hr$
$$= 72\times10^{3} \times \frac{4.186\ J}{3600\ s} = 83.72\ J/s$$

The heart consumes $\dfrac{9.56\ W}{83.72\ W} = 11.4\%$

a) Given $C = \frac{dV}{dp}$ and $V = \pi R^2 L$, where L is an effective length for the compliant arteries.

since L does not change, $\frac{dV}{dp} = 2\pi R L \frac{dR}{dp} = \frac{\pi}{2} D L \frac{dD}{dp} = C$

But $\beta = \frac{2}{D}\frac{\Delta D}{\Delta p} \approx \frac{2}{D}\frac{dD}{dp}$ $\therefore C = \frac{\pi}{2} DL \cdot \frac{D\beta}{2} = \frac{\pi}{4} D^2 L \beta = V\beta$

Also, $\beta = \frac{D}{Et}$, so $C = \frac{VD}{Et}$

b) Given $V = 700 \text{ cm}^3$

$Q = 5000 \text{ cm}^3/\text{min}$

$t/D = 0.07$, from the text

$E = 8 \text{ to } 20 \times 10^5 \text{ Pa}$ → take $E = 8\times 10^5 \text{ Pa}$.

$$\therefore C = \frac{700\text{cm}^3}{(8\times10^5 \text{ Pa})(0.07)} = 1.25 \times 10^{-2} \text{ cm}^3/\text{Pa}$$
$$= 1.25 \times 10^{-8} \text{ m}^3/\text{Pa}$$

$R = \frac{\Delta p}{Q}$ where Δp is the pressure drop across the systemic circulation

Assume $\Delta p = 90 - 15 \text{ mmHg}$
$$= 75 \text{ mmHg} \times \frac{101325 \text{ Pa}}{760 \text{ mmHg}}$$
$$= 10^4 \text{ Pa}$$

$$\therefore R = \frac{10^4 \text{ Pa}}{5000 \text{ cm}^3/\text{min}} = 2 \cdot \frac{\text{Pa}}{\text{cm}^3} \cdot \text{min}$$

$\therefore RC = 0.025 \text{ min} = 1.5 \text{ sec.}$

a) Pressure decreases with x due to frictional losses. Therefore D decreases with x.

b) We know $\beta = \frac{2}{D}\frac{dD}{dP}$ from the text

$$\therefore \beta = \frac{2}{R}\frac{dR}{dP}$$

Also, for a thin wall ($h \ll D$), $\beta = \frac{D}{Eh}$

$$\therefore \frac{D}{Eh} = \frac{2}{R}\frac{dR}{dP} \Rightarrow \frac{2R}{Eh} = \frac{2}{R}\frac{dR}{dP}$$

$$\therefore dR = \frac{R^2}{Eh}dP$$

c) Given: $Q = -\frac{\pi R^4}{8\mu}\frac{dP}{dx}$ and $dR = \frac{R^2}{Eh}dP$

$$\Rightarrow dp = -Q \cdot \frac{8\mu}{\pi R^4}dx$$

$$\therefore dR = \frac{R^2}{Eh}\left(-Q\frac{8\mu}{\pi R^4}dx\right) = -Q \cdot \frac{8\mu}{Eh} \cdot \frac{1}{\pi R^2}dx$$

$$\therefore R^2 dR = -\frac{Q}{Eh} \cdot \frac{8\mu}{\pi}dx \Rightarrow \int_{R_0}^{R} R^2 dR = -\int_0^x \frac{8\mu Q}{Eh\pi}dx$$

$$\therefore \frac{1}{3}\left(R^3 - R_0^3\right) = -\frac{8\mu Q}{Eh\pi}x$$

$$\therefore R = \left(R_0^3 - \frac{24\mu Q}{Eh\pi}x\right)^{1/3}$$

d) Given: $\mu = 3.5\ cP = 3.5 \times 10^{-2}\ g/cm \cdot s$

$E = 100\ dynes/cm^2$

$h = 1\ mm = 0.1\ cm$

$Q = 100\ mL/min = 1.67\ cm^3/s$

$R_0 = 1\ cm$

$$R(x = 20cm) = \left[(1\ cm)^3 - \frac{24(3.5 \times 10^{-2}\ g/cm \cdot s)(1.67\ cm^3/s)}{(100\ dynes/cm^2)(0.1\ cm)\pi} \cdot 20cm \right]^{1/3}$$

$$= 0.47\ cm$$

a) Given $E = \frac{2\Delta p R_i^2 (1-\nu^2)}{R_o^2 - R_i^2} \cdot \frac{R_o}{\Delta R_o}$ — ①

Also, from the text, $\beta = \frac{2}{D_o}\frac{\Delta D_o}{\Delta p} = \frac{2}{R_o} \cdot \frac{\Delta R_o}{\Delta p}$ — ②

From ①, $\frac{\Delta R_o}{\Delta p} = \frac{2 R_o R_i^2 (1-\nu^2)}{E(R_o^2 - R_i^2)}$

sub this into ②,

$$\beta = \frac{2}{R_o} \cdot \frac{2 R_o R_i^2 (1-\nu^2)}{E(R_o^2 - R_i^2)} = \frac{4 R_i^2 (1-\nu^2)}{E(R_o^2 - R_i^2)}$$

$$\therefore c = \frac{1}{\sqrt{\rho\beta}} = \left[\frac{E(R_o^2 - R_i^2)}{4\rho R_i^2 (1-\nu^2)}\right]^{1/2}$$

b)

$R_o = R_i + t$

$\therefore R_o^2 - R_i^2 = 2tR_i + t^2 \simeq 2tR_i$, if t is small.

$$\therefore c = \left[\frac{2tR_i E}{4\rho R_i^2 (1-\nu^2)}\right]^{1/2} = \left[\frac{Et}{2\rho R_i (1-\nu^2)}\right]^{1/2}$$

$$\Rightarrow c = \frac{c_o}{\sqrt{1-\nu^2}}$$

where $c_o = \sqrt{\frac{Et}{\rho D}}$ is the Korteweg-Moens wave speed.

For $\nu = 0.5$, this corresponds to an increase in c of $1/\sqrt{3/4}$ i.e. by a factor of 1.16.

a) $\mathbf{T}$ = tension $[\mathbf{T}] \sim [MLT^{-2}]$

$\mathbf{M}$ = mass per unit length $[\mathbf{M}] \sim [ML^{-1}]$

c_0 = speed of an elastic wave $[c_0] \sim [LT^{-1}]$

h = distance $[h] \sim [L]$

From the theory of π-groups, we can form one group. Choose T, M, and h as the core group of variables:

π group : $[\mathbf{T}]^a [\mathbf{M}]^b [h]^c [c_0] = 1$

$$\Rightarrow [MLT^{-2}]^a [ML^{-1}]^b [L]^c [LT^{-1}] = M^0 L^0 T^0$$

For L : $a - b + c = -1$

M : $a + b = 0$

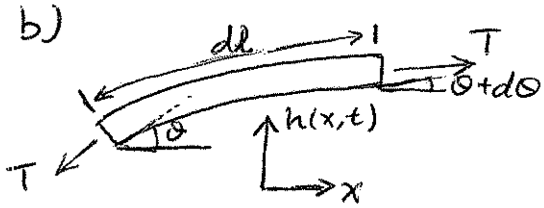
T : $-2a = 1$

$\therefore a = -\frac{1}{2}, \; b = \frac{1}{2}, \; c = 0$

$\therefore \pi = c_0 \left[\frac{\mathbf{M}}{\mathbf{T}}\right]^{1/2}$ = constant, since this π-group is not a function of any other variables.

$$\Rightarrow c_0 = \text{const.} \sqrt{\mathbf{T}/\mathbf{M}}$$

b)



Assume constant tension T at both ends.

$$\sum F_y = m a_y$$

$$\Rightarrow -T\sin\theta + T\sin(\theta + d\theta) = M\, dl \frac{\partial^2 h}{\partial t^2}$$

c) Approximations:

$$\sin(\theta + d\theta) = \sin\theta + d\theta$$

$$dl = dx$$

$$\theta = \frac{\partial h}{\partial x}$$

$\therefore$ the resulting equation in part b) becomes

$$-T\sin\theta + T\sin\theta + Td\theta = M\,dx\,\frac{\partial^2 h}{\partial t^2}$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = \frac{M}{T}\frac{\partial^2 h}{\partial t^2}$$

but since $\theta = \frac{\partial h}{\partial x} \Rightarrow \frac{\partial \theta}{\partial x} = \frac{\partial^2 h}{\partial x^2}$

$$\therefore \frac{\partial^2 h}{\partial x^2} = \frac{M}{T}\frac{\partial^2 h}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 h}{\partial x^2} - \frac{M}{T}\frac{\partial^2 h}{\partial t^2} = 0 \qquad (*)$$

d) Show $h(x,t) = f(x \pm c_0 t)$ satisfies $(*)$

let $\phi = x \pm c_0 t$ so $f(x \pm c_0 t) = f(\phi)$

By chain rule, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{\partial f}{\partial \phi}$

$$\therefore \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \phi}\right) = \frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial \phi}\right) = \frac{\partial^2 f}{\partial \phi^2}$$

Also by chain rule, $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial t} = \frac{\partial f}{\partial \phi}(\pm c_0)$

$$\therefore \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial \phi}(\pm c_0)\right) = (\pm c_0)\frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial t}\right) = c_0^2\frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial \phi}\right)$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial \phi^2} \quad \text{and} \quad \frac{\partial^2 f}{\partial t^2} = c_0^2\frac{\partial^2 f}{\partial \phi^2}$$

d) continued...

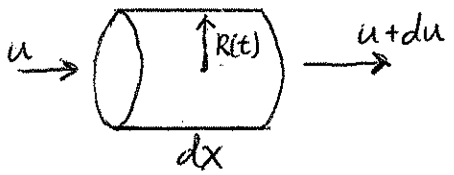
plug the expressions derived above for $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial t^2}$ into the governing equation (*):

$$\frac{\partial^2 f}{\partial \phi^2} - \frac{M}{T} \cdot C_0^2 \frac{\partial^2 f}{\partial \phi^2} = 0$$

$$\Rightarrow \left(1 - \frac{M}{T} C_0^2\right) \frac{\partial^2 f}{\partial \phi^2} = 0$$

From this, we deduce that the given expression is a solution of the governing equation if $C_0 = \sqrt{\frac{T}{M}}$

a)



let ρ = density of fluid

Mass balance:

$$\frac{\partial}{\partial t}(\rho \pi R^2 dx) = u R^2 \pi \cdot \rho - (u+du) R^2 \pi \cdot \rho$$

$$\Rightarrow 2R\frac{\partial R}{\partial t}dx + R^2 du = 0$$

$$\therefore R\frac{\partial u}{\partial x} + 2\frac{\partial R}{\partial t} = 0$$

b) Given : $\beta dp = \frac{2}{R}dR$ -①, $\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}$ -②

$$\therefore \frac{\partial p}{\partial x} = \frac{2}{\beta R}\frac{\partial R}{\partial x}$$

sub this into ②

$$\Rightarrow \rho\frac{\partial u}{\partial t} = -\frac{2}{\beta R}\frac{\partial R}{\partial x} \Rightarrow R\frac{\partial u}{\partial t} + \frac{2}{\rho\beta}\frac{\partial R}{\partial x} = 0$$

But we know $c_0^2 = \frac{1}{\rho\beta}$ from the text

$$\therefore R\frac{\partial u}{\partial t} + 2c_0^2\frac{\partial R}{\partial x} = 0$$

c)

$$\begin{cases} R\frac{\partial u}{\partial x} + 2\frac{\partial R}{\partial t} = 0 \\ R\frac{\partial u}{\partial t} + 2c_0^2\frac{\partial R}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} \frac{\partial R}{\partial t}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial t\partial x}\cdot R + 2\frac{\partial^2 R}{\partial t^2} = 0 \\ \frac{\partial R}{\partial x}\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t\partial x}\cdot R + 2c_0^2\frac{\partial^2 R}{\partial x^2} = 0 \end{cases}$$

Assume products of first order derivatives are small compared to second order derivatives, which is valid for small amplitude waves, to write:

$$\therefore \begin{cases} \frac{\partial^2 u}{\partial t\partial x}\cdot R + 2\frac{\partial^2 R}{\partial t^2} = 0 \\ \frac{\partial^2 u}{\partial t\partial x}\cdot R + 2c_0^2\frac{\partial^2 R}{\partial x^2} = 0 \end{cases} \Rightarrow \frac{\partial^2 R}{\partial t^2} - c_0^2\frac{\partial^2 R}{\partial x^2} = 0 \quad (*)$$

d) Show $R(x,t) = f(x \pm c_0 t)$ is a solution to the resulting equation in part c), i.e. ⊛

let $\phi = x \pm c_0 t$ so $f(x \pm c_0 t) = f(\phi)$

By chain rule, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{\partial f}{\partial \phi}$

$\therefore \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \phi}\right) = \frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial \phi}\right)$

Also by chain rule, $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial t} = \pm c_0 \frac{\partial f}{\partial \phi}$

$\therefore \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\left(\pm c_0 \frac{\partial f}{\partial \phi}\right) = \pm c_0 \frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial t}\right) = c_0^2 \frac{\partial}{\partial \phi}\left(\frac{\partial f}{\partial \phi}\right)$

$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial \phi^2}$ and $\frac{\partial^2 f}{\partial t^2} = c_0^2 \frac{\partial^2 f}{\partial \phi^2}$

sub these two into ⊛

$\Rightarrow c_0^2 \frac{\partial^2 f}{\partial \phi^2} - c_0^2 \frac{\partial^2 f}{\partial \phi^2} = 0$

$\therefore 0 = 0$

$\therefore R(x,t) = f(x \pm c_0 t)$ is a solution

a) Given $\Psi = \frac{\sigma^2 V}{2E} \Rightarrow \frac{\Psi}{V} = \frac{\sigma^2}{2E}$

But we know $\sigma = \frac{pD}{2t}$ from the text,

$$\therefore \frac{\Psi}{V} = \frac{p^2 D^2}{8Et^2}$$

b) The pressure distribution in a travelling pressure wave is given by:

$p = p_0 \cos(\omega t - \frac{2\pi x}{\lambda}) = p_0 \cos(\frac{2\pi x}{\lambda} + \phi)$ at a fixed time, t

where ϕ is a constant phase offset.

Consider a length of artery dx. Then the volume is $V = \pi D t dx$ (here t is wall thickness, $t \ll D$).

The incremental energy stored in the length is

$$d\Psi = \frac{p_0^2 D^2}{8Et^2} \pi D t \cos^2\left(\frac{2\pi x}{\lambda} + \phi\right) dx$$

Then over one λ, $\Psi = \int_0^\lambda d\Psi$

$$\therefore \Psi = \frac{\pi p_0^2 D^3}{8Et} \int_0^\lambda \cos^2\left(\frac{2\pi x}{\lambda} + \phi\right) dx$$

$$= \frac{\pi p_0^2 D^3}{8Et} \cdot \frac{\lambda}{2}$$

$$= \frac{\pi \lambda p_0^2 D^3}{16Et}$$

This assumes small amplitude waves so that variations in the diameter D can be neglected to first order in the above integral.

c) The time required for the wave to travel one λ is $\frac{2\pi}{\omega}$. Thus the rate of energy transport, E_r, is

$$E_r = \frac{\Psi \text{ in one } \lambda}{\text{time to travel } \lambda} = \frac{\pi \lambda p_o^2 D^3}{16 E t} \cdot \frac{\omega}{2\pi}$$

But $\frac{\omega}{c} = \frac{2\pi}{\lambda}$, from equation (4.38)

$$\therefore E_r = \frac{\pi c p_o^2 D^3}{16 E t}$$

Also, we know $Et = c^2 \rho D$ from equation (4.27)

$$\therefore E_r = \frac{\pi p_o^2 D^2}{16 c \rho}$$

$$= \frac{p_o^2 A}{4 \rho c}, \quad \text{since } \frac{\pi D^2}{4} = A$$

$$= \frac{p_o^2}{4 Z_o}, \quad \text{where } Z_o = \frac{\rho c}{A} \text{ from equation (4.46)}$$

d) By conservation of energy,

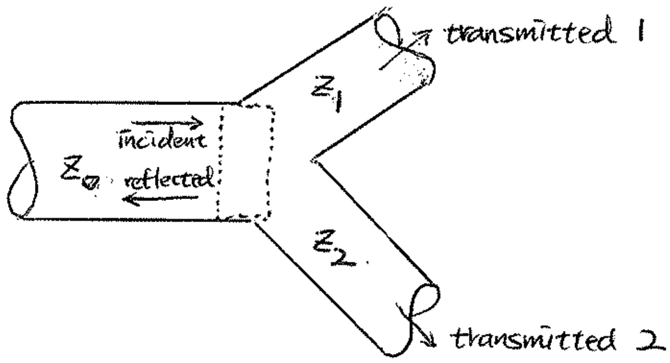
Rate at which energy enters junction
= Rate at which energy leaves junction

$$\therefore \frac{p_{o,i}^2}{4 Z_{o,p}} = \frac{p_{o,r}^2}{4 Z_{o,p}} + 2 \frac{p_{o,t}^2}{4 Z_{o,d}} \Rightarrow 1 = \left(\frac{p_{o,r}}{p_{o,i}}\right)^2 + 2\left(\frac{p_{o,t}}{p_{o,i}}\right)^2 \frac{Z_{o,p}}{Z_{o,d}}$$

$$\therefore 1 = R^2 + 2 \frac{Z_{o,p}}{Z_{o,d}} T^2$$ where R = reflected pulse ratio, T = pulse transmission ratio

A hardened artery will resist expansion / collapse due to the externally imposed cuff pressure more effectively than the normal one. Therefore, the hardened artery will open at a higher pressure when the systolic pressure is recorded, and will stay open at a higher pressure when the diastolic pressure is recorded!

This will lead to an over-estimation of blood pressure.



Balancing forces and flows for the small fluid element:

force: $|p_i| + |p_r| = |p_{t,1}| = |p_{t,2}|$

flow: $|Q_i| - |Q_r| = |Q_{t,1}| + |Q_{t,2}|$

Also, $Z_0 = \dfrac{|p_i|}{|Q_i|} = \dfrac{|p_r|}{|Q_r|}$; $Z_1 = \left|\dfrac{p_{t,1}}{Q_{t,1}}\right|$; $Z_2 = \left|\dfrac{p_{t,2}}{Q_{t,2}}\right|$

$\Rightarrow |p_i| = Z_0 \times |Q_i|$ and $|Q_{t,1}| = \dfrac{|p_{t,1}|}{Z_1}$, $|Q_{t,2}| = \dfrac{|p_{t,2}|}{Z_2}$

$|p_r| = Z_0 \times |Q_r|$

$$\therefore |p_i| - |p_r| = Z_0\left(|Q_i| - |Q_r|\right) = Z_0\left(|Q_{t,1}| + |Q_{t,2}|\right)$$

$$= \frac{Z_0}{Z_1}|p_{t,1}| + \frac{Z_0}{Z_2}|p_{t,2}|$$

$$= \left(\frac{Z_0}{Z_1} + \frac{Z_0}{Z_2}\right)\left(|p_i| + |p_r|\right) \quad (*)$$

continued...

$$\therefore R = \frac{|p_r|}{|p_i|} = \frac{1-\left(\frac{Z_0}{Z_1}+\frac{Z_0}{Z_2}\right)}{1+\left(\frac{Z_0}{Z_1}+\frac{Z_0}{Z_2}\right)} = \frac{Z_1 Z_2 - Z_0(Z_1+Z_2)}{Z_1 Z_2 + Z_0(Z_1+Z_2)}$$

$$\therefore |p_r| = |p_i| \cdot \frac{1-\left(\frac{Z_0}{Z_1}+\frac{Z_0}{Z_2}\right)}{1+\left(\frac{Z_0}{Z_1}+\frac{Z_0}{Z_2}\right)}$$

Also, $|p_i| + |p_r| = |p_{t,1}| = |p_{t,2}| = |p_t|$

$\therefore$ (*) becomes

$$|p_i|\left[1 - \frac{1-\left(\frac{Z_0}{Z_1}+\frac{Z_0}{Z_2}\right)}{1+\left(\frac{Z_0}{Z_1}+\frac{Z_0}{Z_2}\right)}\right] = \left(\frac{Z_0}{Z_1}+\frac{Z_0}{Z_2}\right)|p_t|$$

$$\therefore T = \frac{|p_t|}{|p_i|} = \frac{2\left(\frac{Z_0}{Z_1}+\frac{Z_0}{Z_2}\right)}{1+\left(\frac{Z_0}{Z_1}+\frac{Z_0}{Z_2}\right)} \times \frac{1}{\left(\frac{Z_0}{Z_1}+\frac{Z_0}{Z_2}\right)}$$

$$\therefore T = \frac{2}{1+\left(\frac{Z_0}{Z_1}+\frac{Z_0}{Z_2}\right)} = \frac{2 Z_1 Z_2}{Z_1 Z_2 + Z_0(Z_1+Z_2)}$$

(a) Note that the pressure traces are consistent with a (larger) distally propagating wave an a (smaller) proximally propagating wave. From looking at the times when the two peaks reach the transducers, we deduce that the travel time for the wave between the 2 transducers is 0.0075 seconds. Since the distance between transducer taps is 4 cm, we have

$$c_p = \frac{4\,\text{cm}}{0.0075\,\text{sec}} = 533.3\ \text{cm/s}$$

(b) From the graph $p_{o,i} = 50$ mmHg, $p_{o,r} = 30$ mmHg } This is the amplitude of the time-varying part of the pressure wave

$$\therefore R = \frac{p_{o,r}}{p_{o,i}} = 0.60 = \frac{Z_{o,d} - 2Z_{o,p}}{Z_{o,d} + 2Z_{o,p}}$$

$$\therefore Z_{o,d} - 2Z_{o,p} = 0.60\,(Z_{o,d} + 2Z_{o,p})$$

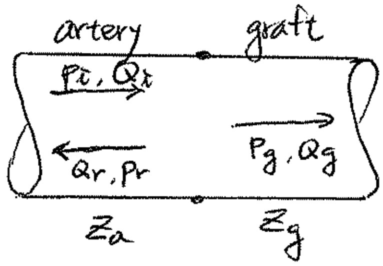
$$0.40\, Z_{o,d} = 3.2\, Z_{o,p}$$

$$Z_{o,d} = 8\, Z_{o,p}$$

But $Z = \frac{\rho c}{A}$ & ρ is the same for daughters & parent, so

$$\frac{c_d}{A_d} = 8\,\frac{c_p}{A_p}$$

But since $A_p = 3A_d$, we have $c_d = \frac{8}{3} c_p = \frac{8}{3}(533.3\ \text{cm/s}) = 1422\ \text{cm/s}$



Mass balance : $Q_i - Q_r = Q_g$ ①

Force balance : $p_i + p_r = p_g$ ②

Also, we know $Z = \frac{p}{Q} \Rightarrow p = ZQ$ by definition

∴ ② can be written as

$$Z_a(Q_i + Q_r) = Z_g Q_g$$

$$\therefore \quad Q_i + Q_r = \frac{Z_g}{Z_a} Q_g \qquad ③$$

combine ① and ③

$$\Rightarrow \quad 2Q_i = Q_g\left(1 + \frac{Z_g}{Z_a}\right)$$

$$\therefore \quad \frac{Q_g}{Q_i} = \frac{2Z_a}{Z_a + Z_g}$$

$$\therefore \quad T = \frac{p_g}{p_i} = \frac{Z_g Q_g}{Z_a Q_i} = \frac{2Z_g}{Z_a + Z_g}$$

T is not equal to one because the graft typically has a different stiffness than the artery, which translates into an impedance difference. This means that the flow pulse travelling down the artery causes a larger pressure pulse when it enters the graft. and thus there is a pressure mismatch at the interface.

a) Given: $C = \frac{D_s - D_d}{P_s - P_d} \frac{1}{D_d} = \frac{1}{D_d} \frac{\Delta D}{\Delta P}$

But $\beta = \frac{2}{D_d} \frac{\Delta D}{\Delta P} = \frac{D_d}{Et}$ assuming thin wall (from the text.)

$\therefore \quad C = \frac{\beta}{2} = \frac{D_d}{2Et}$

Given: $E_{Teflon} = 40 E_{artery}$

$D_{Teflon} = 5.4$ mm

$D_{artery} = 6.8$ mm

$C_{Teflon} = 1.2 \times 10^{-4}$ mmHg^{-1}

$C_{artery} = 8.0 \times 10^{-4}$ mmHg^{-1}

$$\therefore \quad \frac{C_{Teflon}}{C_{artery}} = \frac{D_{Teflon}}{D_{artery}} \frac{E_{artery}}{E_{Teflon}} \frac{t_{artery}}{t_{Teflon}}$$

$$\Rightarrow \quad \frac{1.2 \times 10^{-4}}{8.0 \times 10^{-4}} = \frac{5.4}{6.8} \cdot \frac{1}{40} \cdot \frac{t_{artery}}{t_{Teflon}}$$

$$\therefore \quad \frac{t_{Teflon}}{t_{artery}} = 0.132$$

b)

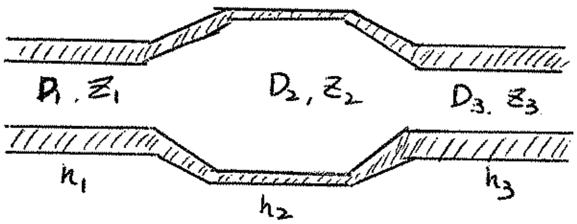
Pros: may help prolong graft patency

Cons: wall may be too thin to maintain graft integrity. This is a concern for suturing and also may be important for burst strength of graft.

c)

Endothelial cells on a graft that is too stiff will be exposed to lower than normal levels of stretch, due to wall pulsations. This may lead then to release factors that disturb the downstream artery wall.

Also, there is some evidence that compliance mismatch at the graft-artery junction may lead to localized elevations of wall strain, which may also disturb the mechanobiology of the resident endothelial cells.



Given: $D_1 = D_3 = 2\text{ cm}$ $\quad h_1 = h_3 = h = 2\text{ mm} = 0.2\text{ cm}$

$D_2 = 4\text{ cm}$

Since artery wall cross-section stays constant,

$$(D_1 + 2h_1)^2 - D_1^2 = (D_2 + 2h_2)^2 - D_2^2$$

$$\therefore (2 + 2(0.2))^2 - 2^2 = (4 + 2h_2)^2 - 4^2$$

$$\therefore h_2 = 0.107\text{ cm}$$

Since $Z = \dfrac{\rho c_0}{A}$, $c_0 = \left(\dfrac{Eh}{\rho D}\right)^{1/2}$

$$\therefore \frac{Z_1}{Z_2} = \left(\frac{\rho c_0}{A}\right)_1 \cdot \left(\frac{A}{\rho c_0}\right)_2 = \frac{c_{0,1}}{c_{0,2}} \left(\frac{D_2}{D_1}\right)^2 = \left(\frac{h_1}{h_2}\right)^{1/2} \left(\frac{D_2}{D_1}\right)^{5/2}$$

$$\therefore \frac{Z_1}{Z_2} = 7.734$$

Similarly, $\dfrac{Z_2}{Z_3} = \left(\dfrac{h_2}{h_3}\right)^{1/2} \left(\dfrac{D_3}{D_2}\right)^{5/2} = 0.129$

$$\therefore T_{1,2} = \frac{2}{1 + \frac{Z_1}{Z_2}} \quad \text{and} \quad T_{2,3} = \frac{2}{1 + \frac{Z_2}{Z_3}}$$

$$\therefore T_{net} = T_{1,2} \cdot T_{2,3} = \frac{2}{1 + 7.734} \cdot \frac{2}{1 + 0.129}$$

$$\therefore T_{net} = 0.406$$

Given: $c_p = c_d$

$$\therefore \frac{Z_p}{Z_d} = \frac{\rho c_p}{A_p} \frac{A_d}{\rho c_d} = \frac{A_d}{A_p} = 0.4$$

$$\therefore R = \frac{Z_d - 2Z_p}{Z_d + 2Z_p} = \frac{1-2(0.4)}{1+2(0.4)} = 0.111$$

$$T = \frac{2Z_d}{Z_d + 2Z_p} = \frac{2}{1+2(0.4)} = 1.111$$

At $t = 0.6\,s$, the incident waveform amplitude is

$$p(t=0.6) = 45\cos[(6.2)(0.6)] + 16\cos[(12.4)(0.6)] + 9\cos[(18.6)(0.6)]$$
$$= -29.77\ Pa$$

$$\therefore p_r(t=0.6) = R\,p = -3.31\ Pa$$

$$p_t(t=0.6) = T\,p = -33.1\ Pa$$

a)

$$Z = \frac{\rho c}{A} \Rightarrow Z = \frac{\rho c}{\frac{D^2}{4}\pi}$$

$c = \left(\frac{Et}{\rho D}\right)^{1/2}$ from the text

$$\therefore Z = \rho\left(\frac{Et}{\rho D}\right)^{1/2} \cdot \frac{4}{D^2 \pi}$$

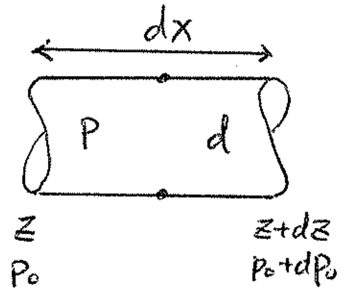
$$= \rho\left(\frac{Et}{\rho}\right)^{1/2} \frac{4}{\pi} D^{-5/2}$$

$$\therefore \frac{dZ}{dx} = \rho\left(\frac{Et}{\rho}\right)^{1/2} \frac{4}{\pi}\left(-\frac{5}{2}\right) D^{-7/2} \frac{dD}{dx}$$

$$= \rho\left(\frac{Et}{\rho D}\right)^{1/2} \frac{4}{\pi D^2}\left(-\frac{5}{2}\right) \cdot \frac{1}{D}(-\alpha)$$

$$\therefore \frac{dZ}{dx} = \frac{5\alpha Z}{2D}$$

b)



$$T = \frac{2Z_d}{Z_p + Z_d} = \frac{2Z + 2dZ}{2Z + dZ}$$

$$\therefore T = \frac{2Z + dZ}{2Z + dZ} + \frac{dZ}{2Z + dZ} \simeq 1 + \frac{1}{2}\frac{dZ}{Z}$$

c)

$$T = \frac{P_t}{P_i} = \frac{P_o + dP_o}{P_o}$$

$$\therefore \ 1 + \frac{1}{2}\frac{dz}{z} = 1 + \frac{dP_o}{P_o} \quad \Rightarrow \quad \frac{dP_o}{P_o} = \frac{1}{2}\frac{dz}{z} = \frac{1}{2}\frac{dz}{dx}\cdot\frac{dx}{z}$$

since $\frac{dz}{dx} = \frac{5\alpha z}{2D}$ and $z = D_1 - \alpha x$

$$\therefore \ \frac{dP_o}{P_o} = \frac{1}{2}\left(\frac{5\alpha z}{2D}\right)\frac{dx}{z} = \frac{5\alpha}{4}\frac{dx}{D_1 - \alpha x}$$

$$\therefore \ \ln p_o = -\frac{5}{4}\ln(D_1 - \alpha x) + \text{const.}$$

$$\therefore \ p_o = \text{const}\,(D_1 - \alpha x)^{-5/4}$$

a)

In arteries	Water waves
distension due to pressure	free surface elevation
flow rate	flow rate per unit breadth (into page)
$Z = \frac{P}{Q}$	$Z_o = \frac{\eta}{Q} = \frac{1}{c} = \frac{1}{\sqrt{gh}}$

b)

$$T = \frac{2}{1 + Z_u/Z_d}, \quad Z_u = \frac{1}{\sqrt{gh_u}} \text{ and } Z_d = \frac{1}{\sqrt{gh_d}}$$

$$\therefore T = \frac{2}{1+\sqrt{h_d/h_u}} = \frac{2}{1+\sqrt{0.6/1.0}} = 1.13$$

$$\therefore \eta_{max} \text{ downstream} = (1.13)(10\text{ cm}) = 11.3\text{ cm}$$

a) From continuity, $q_k = Q/N_k$, where q_k is the flow rate in a single artery in level k.

For Poiseuille flow, $\Delta P_k = \dfrac{8\mu L_k q_k}{\pi R_k^4} = \dfrac{8\mu L_k}{\pi R_k^4 N_k} Q$

$$\therefore \text{Power} = \sum_{k=0}^{N} N_k q_k \Delta P_k = Q^2 \sum_{k=0}^{N} \frac{8\mu L_k}{\pi R_k^4 N_k}$$

b)

Energy expenditure rate is

$$\dot{E} = c_1 \sum_{k=0}^{N} N_k L_k \pi R_k^2 + c_2 \frac{8\mu Q^2}{\pi} \sum_{k=0}^{N} \frac{L_k}{R_k^4 N_k}$$, c_1, c_2 are constants

$$\frac{d\dot{E}}{dR_j} = 2\pi c_1 N_j L_j R_j - 4c_2 \cdot \frac{8\mu Q^2}{\pi} \cdot \frac{L_j}{R_j^5 N_j} = 0$$, by the hint

$$\therefore N_j R_j N_j R_j^5 = \frac{16 c_2 \mu Q^2}{c_1 \pi^2} = \text{const.}$$

$$\therefore (N_j R_j^3)^2 = \text{const.} \Rightarrow N_j R_j^3 = \text{const.}$$

c) For Poiseuille flow, $u(r) = 2u_{avg}[1 - (r/R)^2]$

$$\therefore |\tau_w| = \left|\mu \frac{du}{dr}\right|_{r=R} = 2\mu u_{avg} \cdot \frac{2}{R} = \frac{4\mu}{R} \frac{q}{\pi R^2}$$, since $u_{avg} = \dfrac{q}{\pi R^2}$

This holds for any tube $\Rightarrow q_j = Q/N_j$

$$\therefore |\tau_w|_j = \frac{4\mu Q}{\pi N_j R_j^3}$$

$\therefore$ if $|\tau_w|_j$ is constant, then $N_j R_j^3$ is constant.

d) Murray's law implies $N_j R_j^3 = N_{j-1} R_{j-1}^3$.

For a bifurcation, $N_j / N_{j-1} = 2$, so $2^{1/3} = R_{j-1}/R_j$

Note that for the given assumptions, $c_0 = \sqrt{\frac{Et}{\rho D}}$ = constant.

$$\therefore Z_j = \frac{\rho c_0}{A_j} \sim \frac{1}{R_j^2}$$

$$\therefore R = \frac{Z_{o,d} - 2Z_{o,p}}{Z_{o,d} + 2Z_{o,p}} = \frac{\frac{1}{R_j^2} - \frac{2}{R_{j-1}^2}}{\frac{1}{R_j^2} + \frac{2}{R_{j-1}^2}} = \frac{\left(\frac{R_{j-1}}{R_j}\right)^2 - 2}{\left(\frac{R_{j-1}}{R_j}\right)^2 + 2} = \frac{2^{2/3} - 2}{2^{2/3} + 2}$$

$$\therefore R = -0.115$$

In chamber 1 : $c = m_1/V$

$p = 0$ gauge

In chamber 2 : $c = m_2/V$

$p = \rho_2 g h_2$

Here we assume that since riser tube 1 has a large cross-sectional area the height changes in riser tube 1 are small.

$$Q = A_m J_H = A_m L_p(\Delta P - \Delta\pi)$$
$$= A_m L_p\left(-\rho_2 g h_2 - \frac{(m_1 - m_2)}{V} RT\right)$$

where we assume ideal solution behaviour. Here J_H is defined as positive going from chamber 1 to 2.

Note that the fluid entering chamber 2 goes into the riser tube, so that $Q = A_2 \frac{dh_2}{dt}$

$$\therefore \quad A_2 \frac{dh_2}{dt} = A_m L_p\left(\frac{(m_2 - m_1)}{V} RT - \rho_2 g h_2\right)$$

$$\therefore \quad \frac{dh_2}{dt} + \frac{A_m L_p \rho_2 g}{A_2} h_2 = \frac{A_m L_p}{A_2} \cdot \frac{(m_2 - m_1) RT}{V}$$

Solving this equation and using the initial condition that $h_2 = 0$ at $t = 0$ gives:

$$\therefore \quad h_2 = \frac{(m_2 - m_1) RT}{\rho_2 g V}\left(1 - e^{-\tau}\right), \text{ where } \tau = \frac{t}{A_2/(A_m L_p \rho_2 g)}$$

Here we have neglected the dilution of the solute in chamber 2 and the concentration of solute in chamber 1, which is acceptable if the volume of solvent crossing the membrane is small compared to V.

a)

Assume the solute has constant density.

$$\rho_s = \frac{MW}{\frac{4}{3}\pi r_s^3} = \text{const.}$$

$$\therefore\ MW \sim r_s^3 \quad \text{or} \quad r_s \sim (MW)^{1/3}$$

b)

As $r_s \to 0$, $\sigma \to 0$ and there are no osmotic effects. As $r_s \to \infty$, $MW \to \infty$, and the molar cocentration in chamber 2 approaches 0 (for a fixed mass of solute). Therefore the osmotic effect gets small.

Somewhere in between there is a non-zero osmotic effect and therefore a max in Q_{12}.

c)

$$\sigma = 1 - 2(1-\eta)^2$$ neglecting the fourth order term

$$Q = AL_p(\Delta p - \sigma\Delta\pi)$$

$$= \text{const.} - \text{const.}\,\sigma\Delta\pi$$, but $\Delta\pi = \pi_1 - \pi_2$

$$= -RTC_2 = \frac{-RT(\text{mass}_2)}{V_2\,(MW)}$$

$$\therefore\ Q = \text{const} + \text{const}\cdot\frac{\sigma}{MW}$$

where here 'constant' means that the quantity does not depend on solute radius.

For max Q_{12} $\Rightarrow \dfrac{dQ_{12}}{d\eta} = 0 \Rightarrow \dfrac{d}{d\eta}\left(\dfrac{\sigma}{MW}\right) = 0$

But $MW \sim r_s^3 \sim \eta^3 \Rightarrow \dfrac{d}{d\eta}\left(\dfrac{1-2(1-\eta)^2}{\eta^3}\right) = 0.$

c) continued ...

$$\therefore \frac{-3}{\eta^4}\left(1-2(1-\eta)^2\right) + \frac{2(1-\eta)\cdot 2}{\eta^3} = 0$$

$$\Rightarrow 1-2(1-2\eta+\eta^2) = \frac{4}{3}\eta(1-\eta)$$

$$\Rightarrow \frac{2}{3}\eta^2 - \frac{8}{3}\eta + 1 = 0$$

$$\Rightarrow \eta^2 - 4\eta + \frac{3}{2} = 0$$

$$\therefore \eta = \frac{4 \pm \sqrt{16 - 12/2}}{2} = 2 \pm \sqrt{5/2}$$

Since $\eta \leq 1$, $\eta = 2 - \sqrt{5/2}$

a) Assumptions:

- No ion leakage through RBC membrane
- Membrane area A = constant, L_p = constant

Internal ionic concentration: $C_{int} = \dfrac{V_i C_i}{V}$

External ionic concentration: C_o = const. $< C_{int}$

$$\Delta\pi = \pi_i - \pi_o = RT(C_{int} - C_o) = RT\left(\frac{V_i C_i}{V} - C_o\right)$$

Flow rate into cell: $Q = L_p A \Delta\pi = \dfrac{dV}{dt}$, since $\Delta p = 0$

$$\therefore \frac{dV}{dt} = L_p ART C_o \left(\frac{V_i C_i}{C_o V} - 1\right)$$

Call $V_\infty = \dfrac{V_i C_i}{C_o}$; $\hat{V} = \dfrac{V}{V_\infty}$; $\tau = \left(\dfrac{L_p ART C_o}{V_\infty}\right)t$

Then $$\frac{d\hat{V}}{d\tau} = \frac{1}{\hat{V}} - 1 = -\left(\frac{\hat{V}-1}{\hat{V}}\right)$$

$$\Rightarrow \frac{\hat{V}}{(\hat{V}-1)}d\hat{V} = -d\tau \Rightarrow \left[\frac{1}{\hat{V}-1} + 1\right]d\hat{V} = -d\tau$$

$$\Rightarrow -\tau = \int d\hat{V} + \int \frac{d\hat{V}}{\hat{V}-1} = \hat{V} + \ln(\hat{V}-1) + \text{const.}$$

Initial condition: $\hat{V} = \hat{V}_i = \dfrac{V_i}{V_\infty}$ at $\tau = 0$

$$\therefore \hat{V}_i - \hat{V} + \ln\left(\frac{\hat{V}_i - 1}{\hat{V}-1}\right) = \tau$$

or $$V_i - V + \ln\left(\frac{V_\infty - V_i}{V_\infty - V}\right) = (L_p ART C_o)t$$

b)

The RBC will swell to a sphere and then burst
The volume at this point will be

$$V_{\infty} = \frac{4}{3}\pi R^3$$

And the surface area will be $A = 4\pi R^2$.

Given : $A = 130\ \mu m^2 \Rightarrow R = 3.22\ \mu m$

$$\therefore V_{\infty} = 139.8\ \mu m^3$$

$$\therefore C_o = \frac{V_i C_i}{V_{\infty}} = \frac{(98\mu m^3)(300mM)}{(139.8\mu m^3)} = 210.3\ mM.$$

a) Given: $(p - \pi) = -5$ cmH_2O for the interstitium

$J_H = 0$ when $(p-\pi)_{capillary} = (p-\pi)_{interstitium}$

From the graph, when $J_H = 0 \Rightarrow P_{capillary} \simeq 11.5$ cmH_2O

$\therefore 11.5 - \pi_{capillary} = -5$

$\therefore \pi_{capillary} = 16.5$ cmH_2O

b) $$L_p = \frac{J_H}{\Delta p - \Delta \pi}$$

Assuming $\pi_{capillary} \simeq$ constant, then L_p is just the slope of the line.

$\therefore L_p \simeq 5.69 \times 10^{-3}$ $\mu m/cmH_2O \cdot s$

c) Net flow rate out of the capillary is

$$Q = \int J_H \, dA \quad , \quad J_H = L_p \Delta(p-\pi)$$

where $\Delta(p-\pi) = (p-\pi)_{cap.} - (p-\pi)_{int.} = P_{cap} - \pi_{cap} - (-5)$

and $dA = \pi D dx$

$$\therefore Q = L_p \pi D \int_0^L (P_{cap} - \pi_{cap} + 5)\, dx \quad , \text{ where } X \text{ is in } \mu m$$

But P_{cap} decreases linearly from 25 to 5 cmH_2O

$$\Rightarrow P_{cap} = 25 - 20\frac{X}{L}$$

where here and below pressures are in cmH_2O

c) continued . . .

$$\therefore Q = L_p \pi D \int_0^L \left[25 - 20\frac{x}{L} - \pi_{cap.} + 5 \right] dx$$

$$= \left(5.69 \times 10^{-3} \frac{\mu m}{cmH_2O \cdot s}\right) \pi (8 \mu m) \int_0^{500} \left[25 - 20\frac{x}{500} - 11.5 \right] dx$$

$$= 0.143 \int_0^{500} (13.5 - 0.04x)\, dx$$

$$= 0.143 (13.5x - 0.02x^2) \Big]_0^{500}$$

$$\therefore Q = 250.26 \ \mu m^3/s$$

a) At the balance point, $J_H = 0$.

For $J_H = 0$, $\Delta p - \Delta\pi = 0$

From the previous question, $\Delta p - \Delta\pi = 0$ is given by

$$\left(25 - 20\frac{x}{L}\right) - \pi_{cap.} - (-5) = 0 \quad \text{, pressures in } cmH_2O$$

where $p_{cap.} = \left(25 - 20\frac{x}{L}\right)\ cmH_2O \qquad L = 500\ \mu m$

$\pi_{cap} = 16.5\ cmH_2O$

$(p - \pi)_{int.} = -5\ cmH_2O$.

$\Rightarrow x_b = 337.5\ \mu m$ is the balance point.

b) From the previous question part c), we know the flow rate is

$$Q(x) = 0.143\int_0^x (13.5 - 0.04x)\,dx \quad , \quad Q \text{ in } \mu m^3/s, \; x \text{ in } \mu m$$

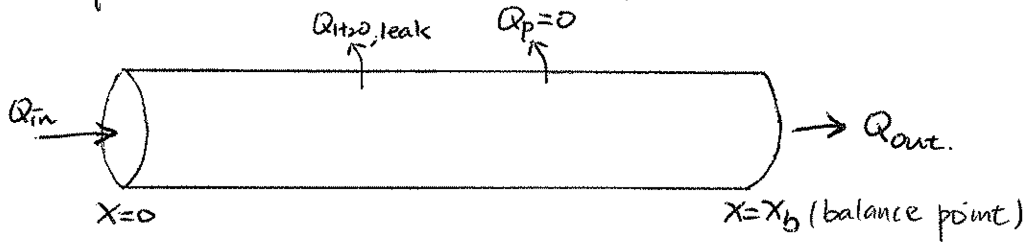
$\therefore$ between the inlet and the balance point,

$$Q = 0.143\int_0^{337.5} (13.5 - 0.04x)\,dx$$

$\Rightarrow Q = 325.8\ \mu m^3/s$

c) Do a control volume analysis on the capillary. Call the total flow rate into the capillary Q_{in}. This consists of 3 parts: RBCs, proteins, and water. Since there is no protein leakage, we always have

$$\text{protein volume flow rate, } Q_p = 0.04\,Q_{in} \quad (1)$$



Balancing water gives: $\quad Q_{H_2O,in} = Q_{H_2O,\text{leak}} + Q_{H_2O,\text{out}}$.

$$\Rightarrow 0.54\,Q_{in} - Q_{H_2O,\text{leak}} = Q_{H_2O,\text{out}}. \quad (2)$$

Now, the percentage change in π is given by

$$\left[\frac{\pi(x_b)-\pi(0)}{\pi(0)}\right]\times 100 = 100\left[\frac{\pi(x_b)}{\pi(0)}-1\right]$$

$$= 100\left[\frac{c(x_b)}{c(0)}-1\right]$$

where c is the protein concentration, and $\pi = RTc$ (van't Hoff's law)

But $c = \dfrac{Q_p}{Q_p + Q_{H_2O}}$, where the flow rates Q are evaluated at the location where the concentration c is to be determined.

$$\therefore \frac{c(x_b)}{c(0)} - 1 = \left[\frac{Q_p(0)+Q_{H_2O}(0)}{Q_p(0)}\cdot\frac{Q_p(x_b)}{Q_p(x_b)+Q_{H_2O}(x_b)}\right]-1$$

c) continued...

use ① and ②,

$$\frac{C(x_b)}{C(0)} - 1 = \frac{Q_p(0) + Q_{H_2O}(0)}{Q_p(x_b) + Q_{H_2O}(x_b)} - 1$$

$$= \frac{0.04\,Q_{in} + 0.54\,Q_{in}}{0.04\,Q_{in} + 0.54\,Q_{in} - Q_{H_2O,leak}} - 1$$

$$\simeq \frac{0.58}{0.58 - \frac{Q_{H_2O,leak}}{Q_{in}}} - 1$$

But $Q_{H_2O,leak} = 325.8\ \mu m^3/s$ (from part b))

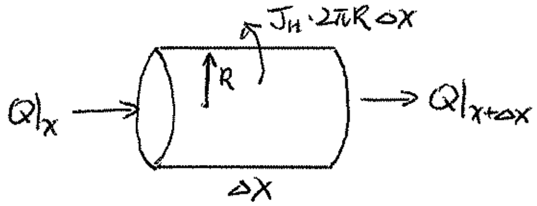
$Q_{in} = 4.2 \times 10^4\ \mu m^3/s$ (given)

$\therefore$ percentage change in π $= 100\left[\frac{0.58}{0.58 - \frac{325.8}{4.2\times10^4}} - 1\right]$

$= 1.36\%$

$\therefore$ Our assumption is a good one

Consider a fluid element in the capillary.



mass balance : $Q|_x = J_H \cdot 2\pi R \Delta x + Q|_{x+\Delta x}$

$$\Rightarrow \frac{Q|_{x+\Delta x} - Q|_x}{\Delta x} = - J_H \cdot 2\pi R .$$

In the limit of $\Delta x \to 0$,

$$\frac{dQ}{dx} = -2\pi R \cdot J_H = -2\pi R L_p (p - B)$$

where L_p = constant , $B = p_{ex} + \Delta\pi$ = constant

Assume flow is everywhere locally Poiseuille $\Rightarrow Q = -\frac{\pi R^4}{8\mu}\frac{dP}{dx}$

$$\therefore \frac{dQ}{dx} = -\frac{\pi R^4}{8\mu}\frac{d^2P}{dx^2} = -2\pi R L_p (P - B)$$

$$\therefore \frac{d^2}{dx^2}(P - B) - \frac{16\mu L_p}{R^3}(P - B) = 0$$

Let $L_{char} = \left(\frac{R^3}{16\mu L_p}\right)^{1/2} \Rightarrow \frac{d^2}{dx^2}(P - B) - \frac{(P-B)}{L_{char}^2} = 0$

solving the differential equation,

$$\Rightarrow p - B = A_1 \sinh\left(\frac{x}{L_{char}}\right) + A_2 \cosh\left(\frac{x}{L_{char}}\right)$$

where A_1 , A_2 are constants.

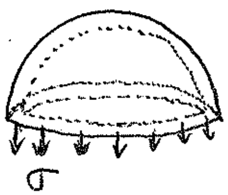
a) From the table, the total intracellular concentration = 255.5 mM
the total extracellular concentration = 246.3 mM

$$\therefore \Delta\pi = RT\Delta C$$

$$= 8.314\ \frac{J}{mol \cdot K} \cdot 310\,K\,(255.5 - 246.3) \times 10^{-3}\ \frac{mol}{L} \times \frac{L}{10^{-3}\,m^3}$$

$$= 2.37 \times 10^{4}\ Pa = 23.7\ kPa.$$

b)



let wall thickness = $t = 0.1\,\mu m$
pressure difference across membrane = Δp

Force balance, $\uparrow \Sigma F = 0$:

$$\Delta p(\pi R^2) - 2\pi R \sigma t = 0$$

$$\therefore \sigma = \frac{\Delta p R}{2t}$$

In this case, $\Delta\pi$ (osmotic pressure) replaces the pressure difference, Δp.

$$\therefore \sigma = \frac{R\Delta\pi}{2t} = \frac{(12\,\mu m)(2.37 \times 10^{4}\,Pa)}{2\,(0.1\,\mu m)} = 1.42 \times 10^{6}\ Pa$$

c)

$$\varepsilon = \frac{\sigma}{E} = \frac{1.42 \times 10^{6}\ Pa}{300 \times 10^{-12}\ N/10^{-12}\,m^2} = 4.73 \times 10^{3}$$

d) The calculated strain is too large. Factors neglected are:
- cell membrane is not completely impermeable to ions
- water will quickly flow across cell membrane to reduce $\Delta\pi$
- cytoskeleton takes some stress.

a) Assuming Poiseuille's law holds in each pore:

$$q = \frac{\pi R^4}{8\mu}\frac{\Delta p}{\ell}$$

If we assume all pores are identical then for N pores,

$$Q = \frac{N\pi R^4}{8\mu}\frac{\Delta p}{\ell}$$

b) Darcy's law gives:

$$K = \frac{Q}{A}\cdot\frac{\mu L}{\Delta p} = \left(\frac{N\pi R^4}{8\mu}\cdot\frac{\Delta p}{\ell}\right)\cdot\frac{1}{A}\cdot\frac{\mu L}{\Delta p}$$

$$= \frac{N\pi R^4 L}{8A\ell}$$

$$= \frac{n\pi R^4}{8}\cdot\frac{L}{\ell}, \quad \text{where } n = \frac{N}{A}$$

Now define:

$$\text{tortuosity}, \ \tau = \frac{\text{pore length}}{\text{block length}} = \frac{\ell}{L}$$

$$\text{porosity}, \ \varepsilon = \frac{\text{pore volume}}{\text{block volume}} = \frac{N\pi R^2\ell}{AL} = n\tau\pi R^2$$

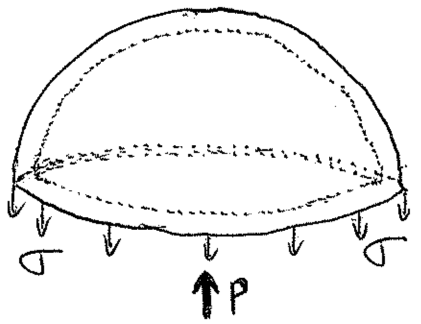
$$\text{specific surface}, \ S = \frac{\text{pore wetted area}}{\text{block volume}} = \frac{2\pi R\ell N}{LA}$$

$$= 2n\tau\pi R$$

$$\Rightarrow \frac{\varepsilon}{\tau} = n\pi R^2 \quad \text{and} \quad \frac{\varepsilon}{S} = \frac{R}{2}$$

$$\therefore K = \frac{n\pi R^4}{8}\cdot\frac{L}{\ell} = \frac{\varepsilon}{\tau^2}\frac{\varepsilon^2}{2S^2} = \frac{\varepsilon^3}{2\tau^2 S^2}$$

For other shaped pores: $K = \dfrac{\varepsilon^3}{\kappa S^2}$, κ = Kozeny constant.



wall thickness = t
diameter = D
hoop stress = σ

Force balance: $\sigma t \pi D = p \cdot \dfrac{\pi D^2}{4}$

$$\therefore \sigma = \frac{pD}{4t} = E\varepsilon$$

$$\therefore \varepsilon = p\left(\frac{D}{4Et}\right)$$

This is the strain due to a pressure p. Now we consider the additional strain, $\Delta\varepsilon$, due to an incremental change in pressure, Δp. This creates an incremental change in diameter, ΔD.

$$\Rightarrow \Delta\varepsilon = \frac{\Delta D}{D} = \Delta p\left(\frac{D}{4Et}\right)$$

$$\Rightarrow \beta = \frac{1}{V}\frac{dV}{dp} \approx \frac{1}{V}\frac{\Delta V}{\Delta p} = \frac{3}{D}\cdot\frac{\Delta D}{\Delta p} = \frac{3D}{4Et}$$

where we have used the fact that $\Delta V \sim 3D^2 \Delta D$

a) mass balance: $\frac{dV}{dt} = Q_{in} - Q_{out}$ — ①

Given: $V = Cp$, $Q_{in} = P_{ss}/R$, $Q_{out} = P/R$

$\therefore$ ① becomes $C\frac{dP}{dt} = \frac{P_{ss}}{R} - \frac{P}{R}$

$\therefore \frac{dP}{dt} + \frac{P - P_{ss}}{RC} = 0$

b) The mass balance equation derived in a) still applies in this case.

Now solve the differential equation:

$$\frac{d(P-P_{ss})}{dt} = -\frac{(P-P_{ss})}{RC} \Rightarrow \frac{d(P-P_{ss})}{(P-P_{ss})} = -\frac{dt}{RC}$$

$\therefore \ln(P-P_{ss}) = -\frac{t}{RC} + \text{const.}$

$\therefore (P-P_{ss}) = ce^{-t/RC} \Rightarrow P = P_{ss} + ce^{-t/RC}$, $c = \text{const.}$

Initial condition: $P(0) = P_{ss} + \delta P$

$\therefore P = P_{ss} + \delta P e^{-t/RC}$

Return to within 5% of its steady state $\Rightarrow P(t) = 1.05 P_{ss}$

$\therefore 1.05 P_{ss} = P_{ss} + \delta P e^{-t/RC}$

Solve for t:

$t = RC \ln\left(\frac{\delta P}{0.05 P_{ss}}\right)$ where $\delta P = 10$ mmHg, $R = 4$ mmHg·min/μL, $C = 3$ μL/mmHg, $P_{ss} = Q_{in}R = 8$ mmHg.

$\therefore t = 12 \ln\left(\frac{\delta P}{0.4}\right)$ minutes $= 38.6$ minutes

a) Ions inside the cornea must be conserved

$\therefore h_{dry} C_{dry} = hC$, where C = the concentration of 'excess' positive ions

$$\therefore C = \frac{h_{dry} C_{dry}}{h} = \frac{(220\,\mu m)(0.8\times10^{-3} M)}{h} = \frac{0.176\,\mu m \cdot M}{h} \quad -①$$

b) at equilibrium : $(p-\pi)_{saline} = (p-\pi)_{cornea}$

$\Rightarrow \Delta p - \Delta\pi = 0$, where the osmotic pressure is due to the 'excess' positive ions.

$\therefore k(h-h_o) - RTC = 0$

$\Rightarrow h - h_o - \frac{RT}{k}C = 0$, substitution of ① gives

$$h^2 - hh_o - \frac{RT}{k}(0.176\,\mu m M) = 0$$

Given: $h_o = 345\,\mu m$ $\quad R = 8.314\ J/mol\cdot K$

$k = 5.5\ Pa/\mu m$ $\quad T = 310K$

Note: $$\frac{RT}{k}(0.176\,\mu m M) = \frac{(8.314\ N\cdot m/mol\cdot k)(310\,k)}{5.5\ N\cdot/m^2\cdot\mu m}\cdot 0.176\,\mu m\cdot\frac{mol}{10^{-3}m^3}$$

$$= 8.247\times10^4\ \mu m^2$$

$\therefore h^2 - 345h - 8.247\times10^4 = 0$, $[h] = \mu m$

$\Rightarrow$ solve for h , $h>0$

$\therefore h = 508\,\mu m$

From the text we have $\frac{h_o - h(x)}{h_o} = 1 - \tilde{h}(x) = \frac{IOP - p(x)}{E}$ — ①

where $\tilde{h} = h/h_o$. Note that equation ① implies that

$$\frac{d\tilde{h}}{dx} = \frac{1}{E}\frac{dp}{dx} \quad -②$$

Also, for the solution for flow in a thin channel,

$$\frac{dp}{dx} = \frac{12\mu\, \varphi(x)}{w h^3(x)}$$ where μ and w are constant

Rearranging this equation and differentiating with respect to x gives:

$$\frac{d\varphi}{dx} = \frac{w}{12\mu}\frac{d}{dx}\left(h^3\frac{dp}{dx}\right)$$

$$= \frac{w}{12\mu}\frac{d}{dx}\left(h^3 E\frac{d\tilde{h}}{dx}\right)$$

$$= \frac{Ewh_o^3}{12\mu}\frac{d}{dx}\left(\tilde{h}^3\frac{d\tilde{h}}{dx}\right)$$

$$= \frac{Ewh_o^3}{12\mu}\left[3\tilde{h}^2\left(\frac{d\tilde{h}}{dx}\right)^2 + \tilde{h}^3\frac{d^2\tilde{h}}{dx^2}\right]$$

By conservation of mass

$$\frac{d\varphi}{dx} = \frac{IOP - p(x)}{R_{iw}} = \frac{E(1-\tilde{h})}{R_{iw}}$$

equating these two equations gives

$$1-\tilde{h} = \frac{wR_{iw}h_o^3}{12\mu}\left[3\tilde{h}^2\left(\frac{d\tilde{h}}{dx}\right)^2 + \tilde{h}^3\frac{d^2\tilde{h}}{dx^2}\right]$$

Finally, noting that $x = \tilde{x}s$, where s is the half-distance between collector channels, we have

$$\frac{12\mu s^2}{wR_{iw}h_o^3}(1-\tilde{h}) = 3\tilde{h}^2\left(\frac{d\tilde{h}}{d\tilde{x}}\right)^2 + \tilde{h}^3\frac{d^2\tilde{h}}{d\tilde{x}^2} \quad — (*)$$

as required

Physically, $\gamma^2 = \dfrac{12\mu s^2}{wR_{iw}h_o^3}$

$$\sim \frac{\text{pressure drop along Schlemm's canal from } x=0 \text{ to } s \text{ when the canal is open}}{\text{pressure drop across the inner wall / trabecular meshwork from } x=0 \text{ to } s.}$$

When $\gamma^2 \ll 1$, then the left side of (*) can be ignored and the governing equation becomes

$$0 = 3\tilde{h}^2\left(\frac{d\tilde{h}}{d\tilde{x}}\right)^2 + \tilde{h}^3\frac{d^2\tilde{h}}{d\tilde{x}^2} = \frac{d}{d\tilde{x}}\left(\tilde{h}^3\frac{d\tilde{h}}{d\tilde{x}}\right)$$

$$\therefore \tilde{h}^3\frac{d\tilde{h}}{d\tilde{x}} = \text{constant}$$

$$\therefore h^3\frac{dh}{dx} = \text{constant}.$$

a) $dE = p\,dV$

where E = energy, p = gauge pressure of the spere, V = volume

Assume p is constant since the change in the size of the sphere is so small.

$\therefore dE = \frac{2\sigma}{R} dV$, by Laplace's law

$\therefore \Delta E = \frac{2\sigma}{R} \Delta V$

b) Let N = # of alveolar sacs in lung

ΔV = increase in volume of a single alveolar sac with a normal breath

V_T = tidal volume

= average of 500 mL or 500 cm^3 $= N\Delta V$

$$\text{breathing rate} = \frac{12 \text{ breaths}}{\text{minute}} = \frac{1 \text{ inspiration}}{5 \text{ seconds}}$$

$$\text{Power, } \mathbb{P} = \frac{\Delta E}{t} = \left(\frac{2\sigma}{Rt}\Delta V\right)N = \frac{2\sigma}{Rt} V_T$$

$$= \frac{2(25 \text{ dyne/cm})}{(0.015 \text{ cm})(5 \text{ s})}(500 \text{ cm}^3) = 3.3 \times 10^5 \text{ dyne}\cdot\text{cm/s}$$

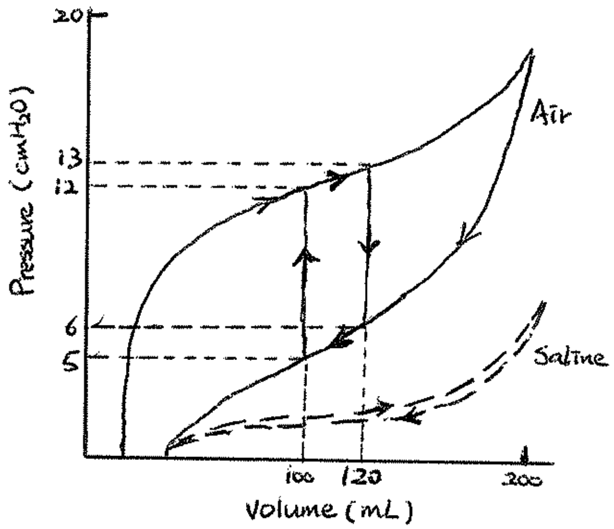
$$\therefore \mathbb{P} = 0.033 \text{ W}$$

But expiration is passive, therefore the power is expened only during inspiration.

Assume 1/2 time per breath (2.5 s) goes for inspiration

$$\therefore \mathbb{P} = \frac{2(25 \text{ dyne/cm})}{(0.015 \text{ cm})(2.5 \text{ s})}(500 \text{ cm}^3) = 0.067 \text{ W}$$

c)



Assume cat is breathing at a rate of 1 breath/5 seconds and inspiration takes half the time of one breath (2.5s).

$$\mathbb{P} = \frac{2(25 \text{ dyne/cm})}{(0.005\text{cm})(2.5\text{s})}(20 \text{ cm}^3) = 8\times10^4 \text{ dyne}\cdot\text{cm/s} = 0.008 \text{ W}$$

Rough estimate of $\mathbb{P}$ using graph above starting at 100 cm^3

Approximate area under curve to be a parallelogram

$$\text{Work} = \text{area} = (7 \text{ cmH}_2\text{O})(20 \text{ cm}^3)$$

$$= (6865 \text{ dyne/cm})(20 \text{ cm}^3) = 1.373\times10^5 \text{ dyne}\cdot\text{cm}$$

$$\mathbb{P} = \frac{\text{work}}{t} = \frac{1.373\times10^5 \text{ dyne}\cdot\text{cm}}{2.5\text{s}} = 5.49\times10^5 \text{ dyne}\cdot\text{cm/s} = 0.0055 \text{ W}$$

Comparison of the two power estimates suggests that surface tension is the dominant restoring force in the cat lung. Note, however, these are very rough estimates, and as a result there is significant discrepancy between the two values.

a)

$$\text{Energy} = \int_{R_1}^{R_2} \Delta p \, dV$$

where $\Delta p = \frac{2\sigma}{R}$, $dV = d\left(\frac{4}{3}\pi R^3\right) = 4\pi R^2 dR$

$$\therefore \text{Energy} = \int_{R_1}^{R_2} 2\sigma \cdot 4\pi R \, dR = \frac{8\pi\sigma}{2}\left(R_2^2 - R_1^2\right)$$

$$\therefore \text{Energy} = 4\pi\sigma\left(R_2^2 - R_1^2\right)$$

b) Given:

$N = 300 \times 10^6$

$V_1 = 2500 \text{ cm}^3 = N\pi \cdot \frac{4}{3} R_1^3 \Rightarrow R_1 = 1.258 \times 10^{-2} \text{ cm}$

$V_2 = 3000 \text{ cm}^3 = N\pi \frac{4}{3} R_2^3 \Rightarrow R_2 = 1.337 \times 10^{-2} \text{ cm}$

$\sigma = 35 \text{ dynes/cm}$

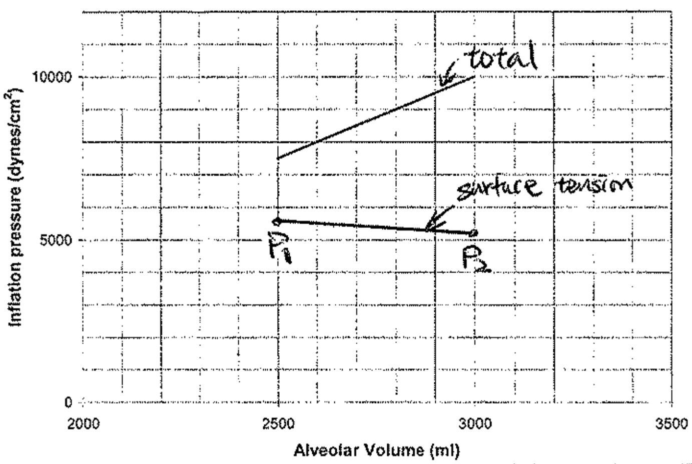
$$\therefore \text{Energy} = 4\pi\sigma \cdot N\left(R_2^2 - R_1^2\right)$$

$$= 4\pi (35 \text{ dyne/cm})(300 \times 10^6)\left[(1.337 \times 10^{-2} \text{ cm})^2 - (1.258 \times 10^{-2} \text{ cm})^2\right]$$

$$= 2.705 \times 10^6 \text{ dyne} \cdot \text{cm}$$

$$= 0.2705 \text{ J}$$

c)



c) continued...

from the graph above:

$$P_1 = \frac{2\sigma}{R_1} = \frac{2(35\ \text{dyne/cm})}{1.258\times10^{-2}\ \text{cm}} = 5564\ \text{dyne/cm}^2$$

$$P_2 = \frac{2\sigma}{R_2} = 5236\ \text{dyne/cm}^2$$

$$\text{Total work} = \int p\,dV = \text{area under curve}$$

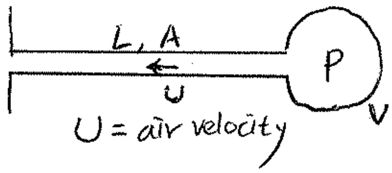
$$= (500\ \text{mL})\cdot\frac{(7500+10000)\ \text{dyne/cm}^2}{2}$$

$$= 4.375\times10^6\ \text{dyne}\cdot\text{cm}$$

$$= 0.4375\ \text{J}$$

$\therefore$ surface tension effects account for $\frac{0.2705}{0.4375} = 61.8\%$

a) For balloon



$$\Delta p = \frac{1}{C}\Delta V$$

or $p = \frac{1}{C}(V - V_0)$ — ①

V = balloon volume, V_0 = balloon volume for $p=0$

In airway, mass in $= \rho A L$, but $F = m\frac{dU}{dt}$

$\therefore$ Force is $(p_{balloon} - p_{atm})A = \rho A L \frac{dU}{dt}$

$$\Rightarrow p = \rho L \frac{dU}{dt}$$, p is balloon gauge pressure.

But $U = Q/A$ and $Q = -\frac{dV}{dt}$ $\Rightarrow \frac{dU}{dt} = -\frac{1}{A}\frac{d^2V}{dt^2}$

$$\Rightarrow p = -\frac{\rho L}{A}\frac{d^2V}{dt^2} \quad - ②$$

combining ① and ② gives

$$\frac{d^2V}{dt^2} + \frac{A}{\rho L C}V = \frac{A}{\rho L C}V_0$$

$$\Rightarrow V = c_1\cos\omega t + c_2\sin\omega t + V_0$$

where $\omega = \sqrt{\frac{A}{\rho L C}}$ is the natural frequency, c_1, c_2 are const.

b) $\frac{A}{L} = 5.6\times10^{-4}\,m$, $\rho = 1.2\ kg/m^3$, $C = 0.0296$ Liter/cmH$_2$O $= 2.96\times10^{-7}\ m^5/N$

$$\therefore \omega = \sqrt{\frac{5.6\times10^{-4}}{1.2\ kg/m^3 \cdot 2.9\times10^{-7}\ m^5/N}} = 39.7\ s^{-1}$$

frequency $f = \frac{\omega}{2\pi} = 6.3$ Hz

b) continued...

Despite the model being relatively simple, the calculated natural frequency is a reasonable approximation of the measured value. Differences between the calculated and measured natural frequencies may be due to inaccuracies in scaling A/L from humans to dogs and in the model assumptions.

Point	Volume (mL)	R (cm)	P_{air} (dyne/cm²)	P_{saline} (dyne/cm²)	ΔP (dyne/cm²)	σ (dyne/cm)
A	5000	0.0189	15500	10000	5500	52.0
B	1500	0.0127	10000	3500	6500	41.3
C	50	0.0041	9000	2200	6800	13.9
D	1500	0.0127	8000	3500	4500	28.6

Note:

$\Delta P = P_{air} - P_{saline}$: this is the pressure needed to overcome surface tension. P_{air} and P_{saline} are read from the graph.

Radius, R, is calculated using

$$0.85V = \frac{4}{3}\pi R^3 N$$

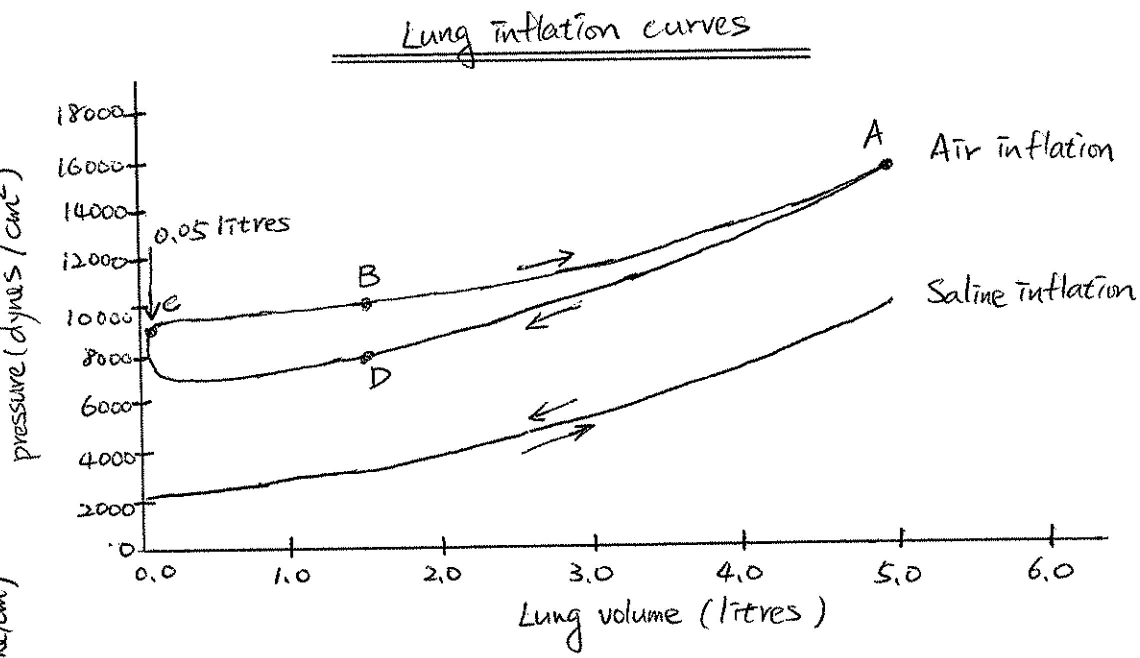
where V is lung volume
N is number of alveoli

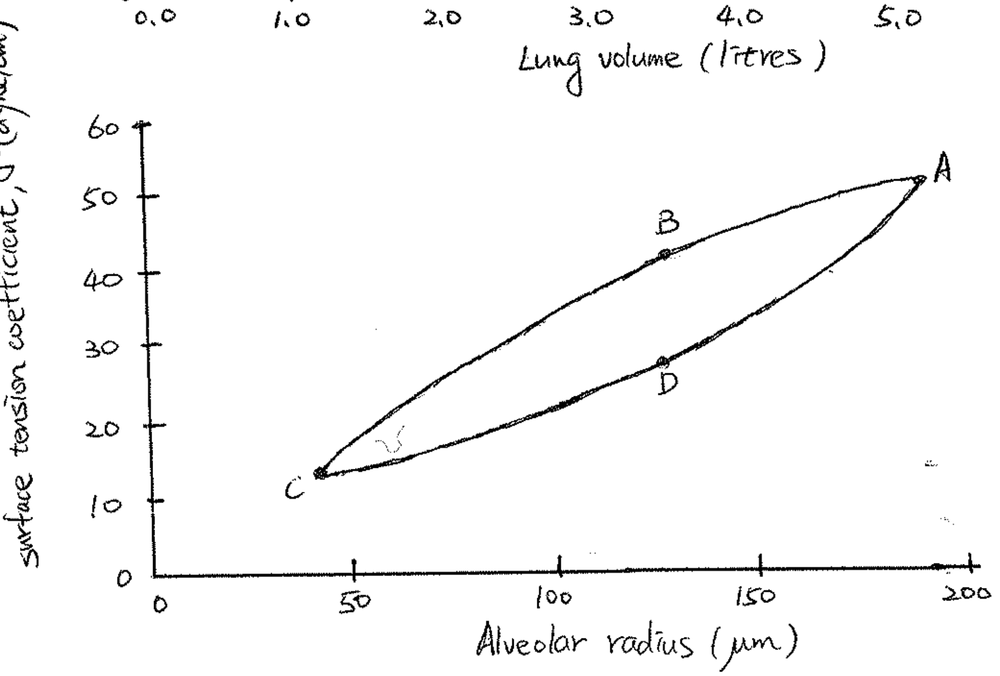
$$\Rightarrow R = \left(\frac{2.55V}{4\pi N}\right)^{1/3} = 1.106\times10^{-3}\, V^{1/3}$$

Surface tension balance:

$$\Delta P = \frac{2\sigma}{R}$$

$$\therefore \sigma = \frac{\Delta P R}{2}$$


Lung inflation curves
pressure (dynes / cm²)
18000
16000
14000
12000
10000
8000
6000
4000
2000
0
0.05 litres
A
B
C
D
Air inflation
Saline inflation
0.0
1.0
2.0
3.0
4.0
5.0
6.0
Lung volume (litres)


surface tension coefficient, σ (dyne/cm)
60
50
40
30
20
10
0
A
B
C
D
0
50
100
150
200
Alveolar radius (μm)

a) $W = \int p\,dV$,

But under stated assumptions, $p = \frac{2\sigma}{R}$

$\therefore W = \int \frac{2\sigma}{R} dV = \int \frac{2\sigma}{R} 4\pi R^2 dR = \int \sigma \, 8\pi R\, dR = \int \sigma\, dS$ for a single alveolus

$\therefore$ summing over all alveoli, $W = \int \sigma\, dA$

(This assumes that all alveoli have the same radius at all times.)

b) Energy disipated is the area under curve on σ-A graph

$$\text{Energy} = \Delta\sigma \cdot \Delta A = (40\ \text{dyne/cm})(2\ \text{m}^2)$$
$$= 8 \times 10^5\ \text{dyne}\cdot\text{cm}$$
$$= 0.08\ \text{J}$$

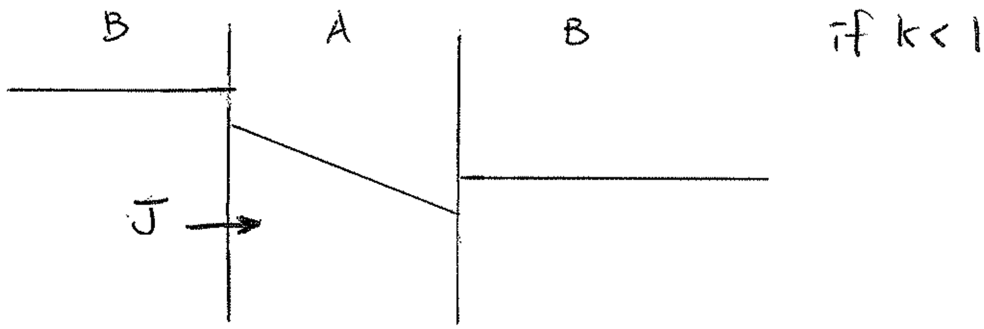
By Fick's law: $J = -D_A \frac{\Delta C}{\Delta y_A}$

where $\Delta C = C_2^A - C_1^A = (kC_2^B - kC_1^B) = k(C_2^B - C_1^B)$

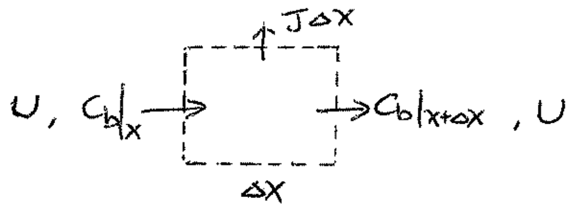
$$\therefore \quad J = -D_A k \frac{C_2^B - C_1^B}{\Delta y_A}$$

$$= -D_{eff} \frac{C_2^B - C_1^B}{\Delta y_A}$$, where $D_{eff} = D_A k$

concentration profile:



a) Draw a control volume around a blood element



mass balance : $2R_oU\left(C_b|_{x+\Delta x} - C_b|_x\right) + J\Delta x = 0$ (Note this is valid for a steady process)

in the limit of $\Delta x \to 0$, $\dfrac{dC_b}{dx} = -\dfrac{J}{2R_oU}$ — ①

Similarly for gas, $\dfrac{dC_g}{dx} = \dfrac{J}{2R_oU}$ — ②

Combining ① and ② $\Rightarrow$ $\dfrac{dC_b}{dx} + \dfrac{dC_g}{dx} = 0$

$\Rightarrow$ $C_b + C_g = \text{const.}$

Also, $J = \dfrac{D_{eff}}{\Delta y}\left(C_b - C_g\right)$

$\therefore$ $\dfrac{dC_b}{dx} = -\dfrac{D_{eff}}{2R_oU\Delta y}\left(C_b - C_g\right)$

Define $L_{char} = \left(\dfrac{D_{eff}}{2R_oU\Delta y}\right)^{-1}$

$\therefore$ $\dfrac{dC_b}{dx} = -\dfrac{1}{L_{char}}\left(C_b - C_g\right)$

b) Given $C_g(x=0)=0$, $C_b(x=0)=C_b^0$

since $C_b+C_g=\text{const}$, $C_g+C_b=C_b^0$

$\therefore C_g=C_b^0-C_b$

$\therefore$ the result in part a) becomes $\frac{dC_b}{dx}=-\frac{1}{L_{char}}(2C_b-C_b^0)$

Solve the differential equation:

$$\frac{1}{2}\frac{d(2C_b-C_b^0)}{dx}=-\frac{1}{L_{char}}(2C_b-C_b^0)$$

$$\Rightarrow \int_{C_b^0}^{C_b}\frac{d(2C_b-C_b^0)}{(2C_b-C_b^0)}=-\int_0^x\frac{2}{L_{char}}dx$$

$$\Rightarrow \ln\left(\frac{2C_b-C_b^0}{C_b^0}\right)=-\frac{2x}{L_{char}}$$

$$\therefore C_b=\frac{1}{2}C_b^0\left(1+e^{-2x/L_{char}}\right)$$

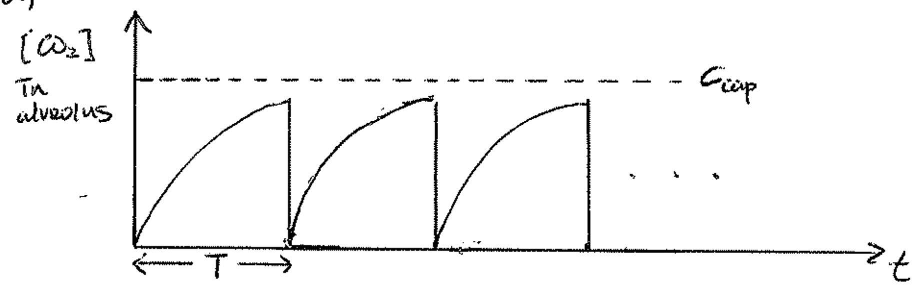
To reduce C_b to $0.6\,C_b^0$, we need

$$0.6=\frac{1}{2}\left(1+e^{-2x/L_{char}}\right)$$

$$\ln 0.2=-\frac{2x}{L_{char}}$$

$$\therefore \frac{x}{L_{char}}=0.805$$

a)



b) CO_2 concentration in the alveolus is time-varying. Call this quantity C. Then the total mass of CO_2 in the alveolus is CV.

If J is the flux of CO_2 across the blood/gas barrier into the alveolus, then

$$\frac{d}{dt}(CV) = JA$$

, assuming well-mixed alveolus, no communication with bronchioles.

By Fick's law, $J = \frac{D_{eff}(C_{cap} - C)}{\Delta y}$, Δy = wall thickness

Assume V = const. , $V\frac{dc}{dt} = \frac{AD_{eff}}{\Delta y}(C_{cap} - C)$ — ①

Define $\hat{C} = 1 - \frac{C}{C_{cap}}$. Note that $\hat{c} = 1$ at $t=0$ (fresh air).

Then ① can be written as,

$$-\frac{d\hat{c}}{dt} = \left(\frac{A D_{eff}}{V \Delta y}\right)\hat{c}$$

continued...

Solve the differential equation:

$$\int_1^{\hat{c}} \frac{d\hat{c}}{\hat{c}} = -\int_0^t \left(\frac{A D_{eff}}{V \delta y}\right) dt$$

$$\therefore \ln \hat{c} = -\frac{A D_{eff}}{V \delta y} t$$

$$\therefore \hat{c} = \exp\left(-\frac{A D_{eff}}{V \delta y} t\right) \qquad 0 \le t \le T$$

or

$$C = C_{cap}\left[1 - \exp\left(-\frac{A D_{eff}}{V \delta y} t\right)\right]$$

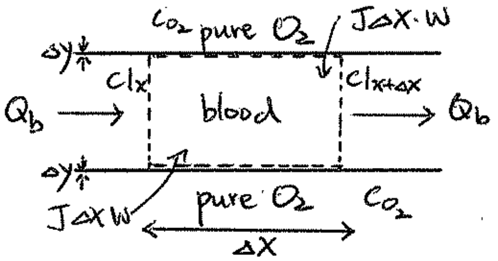
a) Draw a control volume around the entire oxygenator:

$$C_{in} \cdot Q_{blood} + 200 \text{ mL/min} = Q_{blood} \cdot C_{out}$$

$$\therefore C_{out} = C_{in} + \frac{200 \text{ mL/min}}{Q_{blood}} = 0.1 \frac{\text{mL } O_2}{\text{mL blood}} + \frac{200 \text{ mL } O_2\text{/min}}{5000 \text{ mL blood/min}}$$

$$\therefore C_{out} = 0.14 \text{ mL } O_2 \text{ / mL blood.}$$

b) Draw a control volume in the blood channel:



C: O_2 concentration in blood
Q_b: blood flow rate in a channel
J: O_2 flux
C_{O_2}: pure O_2 concentration
Δy: membrane thickness
W: membrane width (into the page)

mass balance: $Q_b C|_x + 2 J \Delta x W = Q_b C|_{x+\Delta x}$

By Fick's law, $J = D_{eff} \dfrac{C_{O_2} - C}{\Delta y}$

$$\therefore \text{ in the limit of } \Delta x \to 0 \Rightarrow \frac{dC}{dx} = 2 \cdot \frac{D_{eff}}{Q_b} \cdot \frac{C_{O_2} - C}{\Delta y} \cdot W$$

$$\Rightarrow -\int_{C_{in}}^{C_{out}} \frac{d(C - C_{O_2})}{(C - C_{O_2})} = \int_0^L \frac{2 D_{eff} W}{Q_b \Delta y} dx$$

$$\therefore -\ln \frac{C_{out} - C_{O_2}}{C_{in} - C_{O_2}} = \frac{2 D_{eff} W}{Q_b \Delta y} L$$

b) continued...

Given:

$C_{in} = 0.1\ cm^3\ O_2 / cm^3$ blood ; $C_{O_2} = 0.204\ cm^3\ O_2 / cm^3$ blood

$C_{out} = 0.14\ cm^3\ O_2 / cm^3$ blood (from part a))

$D_{eff} = 10^{-6}\ cm^2/s$

$W = 10\ cm$

$L = 10\ cm$

$\Delta y = 5\times10^{-4}\ cm$

$Q_b = Q/N = \left(\frac{5000\ cm^3}{60\ s}\right)/N = \frac{83.3}{N}\ cm^3/s$

where N is the number of membrane units.

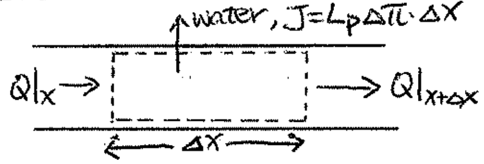
$$\therefore -\ln\frac{0.14-0.204}{0.1-0.204} = \frac{2(10^{-6}\ cm^2/s)(10\ cm)}{\left(\frac{83.3}{N}\ cm^3/s\right)(5\times10^{-4}\ cm)}\cdot 10\ cm$$

$\therefore N = 101.2$

$\therefore$ 102 units are needed.

a) Water leaving the blood reduces Q. Since the osmotically active components cannot leave, c must increase.

b) Draw a control volume in the blood stream, where $\Delta\pi$ is defined to be the difference in osmotic pressure between the dialysate and the blood:



By van't Hoff's law:

$$\Delta\pi = RT(c_d - c(x))$$

mass balance on water in the blood: $Q|_x = -L_p RT(c_d - c(x))$

in the limit of $\Delta x \to 0 \Rightarrow$ $$\frac{dQ}{dx} = -L_p RT(c_d - c(x))$$

or $$\frac{dQ}{dx} = L_p RT[c(x) - c_d]$$

c) Use the same control volume as in b), but do a mass balance on the osmotically active components in the blood:

$$Q|_x c|_x = Q|_{x+\Delta x} c|_{x+\Delta x} = Q_o c_o \quad \text{for all } x$$

where Q_o and c_o are the values of $Q(x)$ and $c(x)$ at the inlet.

$\therefore Q(x)c(x) = \text{constant}$

d) $$\frac{dQ}{dx} = -L_p RT\left(c_d - \frac{c_o Q_o}{Q(x)}\right) = -L_p RT c_o Q_o\left[\frac{1}{Q_r} - \frac{1}{Q}\right]$$

$$\therefore \frac{dQ}{dx} = -L_p RT \frac{c_o Q_o}{Q_r}\left[\frac{Q - Q_r}{Q}\right]$$

By hint: $$-Q_r \ln\left(\frac{Q_r - Q}{Q_r}\right) - Q + \text{const} = \frac{L_p RT c_o Q_o}{Q_r} x$$

At $x = 0$, $Q = Q_o$: $$\text{const} = Q_o + Q_r \ln\left(\frac{Q_r - Q_o}{Q_r}\right)$$

d) continued...

$$\therefore Q_r \ln\left(\frac{Q_r - Q}{Q_r - Q_o}\right) + Q - Q_o = -\frac{L_p R T c_o Q_o}{Q_r} x$$

$$\Rightarrow \ln\left(\frac{Q_r - Q}{Q_r - Q_o}\right) + \frac{Q - Q_o}{Q_r} = -\frac{L_p R T c_o Q_o x}{Q_r^2}$$

$$\therefore \ln\left(\frac{Q_r - Q_o}{Q_r - Q}\right) + \frac{Q_o - Q(x)}{Q_r} = \frac{L_p R T c_o Q_o}{Q_r^2} x$$

e) Given:

$Q_o = 25/60 \text{ cm}^2/\text{s} = 0.417 \text{ cm}^2/\text{s}$

$Q(L) = 24/60 \text{ cm}^2/\text{s} = 0.4 \text{ cm}^2/\text{s}$

$c_d = 3.2 \times 10^{2} \text{ mol}/\text{m}^3$

$c_o = 2.85 \times 10^{2} \text{ mol}/\text{m}^3$

$L_p = 10^{-8} \text{ cm}/\text{s}\cdot\text{Pa}$

$R = 8.314 \text{ J}/\text{mol}\cdot\text{K}$

$T = 310 \text{ K}$

$$\therefore Q_r = \frac{c_o Q_o}{c_d} = 0.371 \text{ cm}^2/\text{s}$$

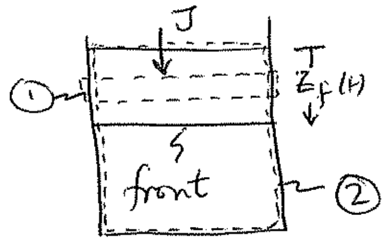
$$\therefore \frac{L_p R T c_o Q_o}{Q_r^2} = \frac{\left(10^{-8} \frac{\text{cm}}{\text{s}\cdot\text{kg}/\text{m}\cdot\text{s}^2}\right)\left(8.314 \frac{\text{kg}\cdot\text{m}^2/\text{s}^2}{\text{mol}\cdot\text{K}}\right)(310\text{K})\left(2.85\times10^{2} \text{mol}/\text{m}^3\right)}{\left(0.371 \text{ cm}^2/\text{s}\right)^2}$$

$$= 2.22 \times 10^{-2} \text{ cm}^{-1}$$

$$\therefore \ln\left(\frac{0.371 - 0.417}{0.371 - 0.4}\right) + \frac{0.417 - 0.4}{0.371} = 2.22 \times 10^{-2} \cdot L$$

$$\Rightarrow L = 22.9 \text{ cm}$$

(a) If the front is slowly moving compared to the diffusive time scale, then the problem can be treated as quasi-steady.



Use the control volume labelled as ① in the figure at right. Since the problem is being treated as quasi-steady, by conservation of mass we must have J = constant. But by Fick's law, $J = -D\dfrac{dc}{dz}$, so $\therefore \dfrac{dc}{dz}$ must be constant above the front, i.e. the concentration profile above the front is linear. At the free surface, $c = c_0$, while at the front we have $c = 0$ (no free oxygen since it is binding to the hemoglobin). Therefore

$$-\frac{dc}{dz} = \frac{c_0}{z_f}, \quad \text{and} \quad J = \frac{D c_0}{z_f}$$

(b) Use the control volume labelled ② in the figure. We know the rate of oxygen entry at the upper interface is $\dfrac{D c_0 A}{z_f}$, where A is the area of the interface.

The total mass of oxygen in the control volume is given by:

$$\underbrace{4 c_{Hb} z_f A}_{\substack{\text{Oxygen bound} \\ \text{to hemoglobin} \\ \text{above the front,} \\ 4\,O_2 \text{ per Hb}}} + \underbrace{\frac{c_0}{2} z_f A}_{\substack{\text{free oxygen in} \\ \text{solution above the} \\ \text{front, average} \\ \text{Concentration} = c_0/2}} \quad \text{—}(*)$$

Mass balance: $J = \frac{d}{dt}(\text{mass of } O_2 \text{ in c.v.})$

$$\frac{D c_0 A}{z_f} = \frac{d}{dt}\left(4 c_{Hb} z_f A + \frac{c_0}{2} z_f A\right)$$

$$\frac{D c_0}{z_f} = \left(4 c_{Hb} + \frac{c_0}{2}\right)\frac{dz_f}{dt}$$

$$\frac{D c_0}{4 c_{Hb} + 1/2\, c_0} = \frac{1}{2}\frac{d}{dt}\left(z_f^2\right)$$

Integrate, noting $z_f = 0$ at $t = 0$ to obtain

$$z_f = \sqrt{\frac{2 D c_0 t}{4 c_{Hb} + c_0/2}}$$

Note: including the $\frac{d}{dt}\left(\frac{c_0}{2} z_f A\right)$ is inconsistent with the quasi-steady approximation in part (a). For blood, the second term in (*) is small compared with the first term, so the final answer is not materially affected by whether or not this second term is included or not.

From equation (7.6), (7.9), (7.11), (7.12), we know

$$C_{out} - C_{in} = (C_{alv} - C_{in})(1 - e^{-L/L_{char}})$$

where C = CO_2 concentration

L = length of capillary

$L_{char} = U R_0 \frac{\Delta y}{D}$

Define mass transfer resistance, $R = \frac{\Delta y}{D}$ $\Rightarrow$ $L_{char} = U R_0 R$

Given in the text,

$D_{tissue} = 1.92 \times 10^{-5}\ cm^2/s$

$\Delta y_{tissue} = 0.6 \times 10^{-4}\ cm$

$\Delta y_{scar} = 1 \times 10^{-4}\ cm$

$D_{scar} = 0.7 \times 10^{-6}\ cm^2/s$

$R_0 = 4 \times 10^{-4}\ cm$

$U = 0.1\ cm/s$

$L = 50 \times 10^{-4}\ cm$

$$R_{tissue} = \frac{0.6 \times 10^{-4}\ cm}{1.92 \times 10^{-5}\ cm^2/s} = 3.13\ s/cm$$

$$R_{tissue+scar} = R_{tissue} + R_{scar} = 3.13\ s/cm + \frac{1 \times 10^{-4}\ cm}{0.7 \times 10^{-6}\ cm^2/s} = 145.99\ s/cm$$

$\therefore$ For normal tissue,

$$C_{out} - C_{in} = (C_{alv} - C_{in})\left[1 - \exp\left(-\frac{50 \times 10^{-4}\ cm}{0.1\ cm/s \cdot 4 \times 10^{-4}\ cm \cdot 3.13\ s/cm}\right)\right]$$

$$\simeq C_{alv} - C_{in}$$

For scarred tissue

$$C_{out} - C_{in} = (C_{alv} - C_{in})\left[1 - \exp\left(-\frac{50 \times 10^{-4}\ cm}{0.1\ cm/s \cdot 4 \times 10^{-4}\ cm \cdot 145.99\ s/cm}\right)\right]$$

$$= 0.58(C_{alv} - C_{in})$$

$\therefore$ there is 42% reduction due to scar tissue

a) To get the minimum number of breaths, we should exchange as much Xenon as possible with each breath, i.e. exchange the maximum ventilatory volume.

let vital capacity (VC) = 4800 mL ; residual volume (RV) = 1200 mL

volume of dead space (V_D) = 150 mL

$\therefore$ Air into lungs $= VC - V_D = 4800 - 150 = 4650$ mL

1st breath: Xe into lungs mixes with RV of air

Assuming complete mixing, % Xe in lungs is:

$$\% Xe = \frac{4650\text{ mL}}{1200\text{ mL} + 4650\text{ mL}} = 0.79$$

2nd breath: Xe into lungs mixes with RV of 79% Xe.

$$\% Xe = \frac{(1.0)(4650\text{ mL}) + (0.79)(1200\text{ mL})}{1200\text{ mL} + 4650\text{ mL}}$$

$$= 0.957$$

3rd breath:

$$\% Xe = \frac{(1.0)(4650) + 0.957(1200)}{1200 + 4650} = 0.991$$

4th breath:

$$\% Xe = \frac{(1.0)(4650) + 0.991(1200)}{1200 + 4650} = 0.998$$

b) Average tidal volume (TV) = 500 mL

alveolar air volume (V_A) $= TV - V_D = 350$ mL $\rightarrow$ air into lungs

1st breath: TV of 100% air mixes with RV (1200 mL) + ERV (1200 mL)

$$\% Xe = \frac{0.998(2400\text{ mL})}{350\text{ mL} + 2400\text{ mL}} = 0.871$$

let $\frac{2400}{350+2400} = Q \Rightarrow n^{th}$ breath : $\% Xe = 0.998\, Q^n$

$$\Rightarrow 0.001 = 0.998\left(\frac{2400}{350+2400}\right)^n \Rightarrow n \simeq 51$$

After the first breath, the composition is as listed in Table 7-2, last column.

There are 9 breaths left:

N_2 : unchanged $\longrightarrow$ 393.1 mL

O_2 : all used up $\longrightarrow$ 0 mL

CO_2 : added by breathing $\longrightarrow$ $19.1 + 9 \times \frac{235}{12} = 195.4\,mL$

$\Rightarrow$ total volume of dry air = 588.5 mL

Water is added:

$$\frac{\text{volume water}}{\text{volume dry gases}} = \frac{\text{partial pressure water}}{\text{partial pressure dry gases}}$$

$$\therefore \quad \frac{V_{H_2O}}{588.5\,mL} = \frac{47\,mmHg}{(760-47)\,mmHg} \quad \Rightarrow V_{H_2O} = 38.8\,mL$$

$\therefore$ Total volume = 627.3 mL

$$\therefore \; CO_2 \text{ concentration} = \frac{195.4}{627.3} = 31.2\,\%$$

a) Treating air as an ideal gas.

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}$$

with ① Ambient $P_1 = 235$ mmHg $T_1 = 273$ K $V_1 = 1000$ mL

② BTP $P_2 = 760$ mmHg $T_2 = 310$ K

$$\therefore V_2 = \frac{P_1}{P_2}\frac{T_2}{T_1}V_1 = \frac{235\text{ mmHg}}{760\text{ mmHg}}\frac{310\text{ K}}{273\text{ K}} \cdot 1000\text{ mL} = 351.1\text{ mL}$$

This is the tidal volume at BTP with 20.8% O_2

$\Rightarrow$ 73.0 mL O_2

Can only use 30% of this O_2 $\Rightarrow$ 21.9 mL usable O_2

Requirement is 284 mL O_2/min

$$\therefore \text{need } \frac{284\text{ mL }O_2/\text{min}}{21.9\text{ mL }O_2/\text{breath}} \simeq 13\text{ breaths/min}$$

b) Tidal volume into lungs = 351.1 mL at BTP

$$N_{2_{in}} = (0.786)(351.1) = 276.0\text{ mL}$$
$$O_{2_{in}} = (0.208)(351.1) = 73.0\text{ mL}$$
$$CO_{2_{in}} = (0.0004)(351.1) = 6.14\text{ mL}$$
$$H_2O_{in} = (0.005)(351.1) = 1.8\text{ mL}$$

For volumes out:

$$N_{2_{out}} = N_{2_{in}} = 276.0\text{ mL}$$
$$O_{2_{out}} = O_{2_{in}} - O_{2_{used}} = (73.0 - 21.9)\text{ mL} = 51.1\text{ mL}$$
$$CO_{2_{out}} = CO_{2_{in}} + CO_{2_{made}} = 0.14\text{ mL} + \frac{227\text{ mL/min}}{13\text{ breaths/min}} = 17.6\text{ mL}$$

Volume of dry air $= (276.0 + 51.1 + 17.6)\,mL = 344.7\,mL$

$$H_2O_{out} = \frac{47\,mmHg}{(760-47)\,mmHg} \cdot 344.7\,mL = 22.7\,mL$$

$\therefore$ Total $V_{out} = 344.7\,mL + 22.7\,mL = 367.4\,mL$

For compositions (at BTP conditions):

$$N_2 : \frac{276.0}{367.4} = 75.1\%$$

$$O_2 : \frac{51.1}{367.4} = 13.9\%$$

$$CO_2 : \frac{17.6}{367.4} = 4.8\%$$

$$H_2O : \frac{22.7}{367.4} = 6.2\%$$

First do alveolar air, for which TV (tidal volume) = 380 mL effectively.

Note: $(\text{volume of gas})_{in} = \frac{\text{partial pressure}}{760\ \text{mmHg}} \cdot TV$

Gas	Partial Pressure (mmHg)	V_{in} (mL)	V_{out} (mL)	Composition_{out}
N_2	594	297	297	75.3%
O_2	156	78	53.4	13.5%
CO_2	0.3	0.15	19.8	5.0%
H_2O	9.7	4.85	24.4	6.2%

Note: $V_{H_2O,out} = V_{dry} \cdot \frac{P_{H_2O}}{P_{dry}} = (297 + 53.4 + 19.7)\,mL \cdot \frac{47\,mmHg}{(760-47)\,mmHg} = 24.4\,mL$

Note: O_2 removed per breath $= 295/12 = 24.6\,mL$

$\Rightarrow V_{O_2,out} = (78 - 24.6)\,mL = 53.4\,mL$

CO_2 added per breath $= 235/12 = 19.6\,mL$

$\Rightarrow V_{CO_2,out} = (0.15 + 19.6)\,mL = 19.8\,mL$

Note: Total $V_{in} = TV = 380\,mL$; Total $V_{out} = 394.5\,mL$

Now do the same for dead space, but no CO_2 or O_2 change.

Gas	Partial Pressure (mmHg)	V_{in} (mL)	V_{out} (mL)	Composition_{out}
N_2	594	117.2	117.2	74.2%
O_2	156	30.8	30.8	19.5%
CO_2	0.3	0.06	0.06	~0%
H_2O	9.7	1.91	9.76	6.2%

Note: Total $V_{in} = 150\,mL$; Total $V_{out} = 157.9\,mL$

Assume :

- Tidal volume in = 500 mL
- O_2 consumption = 284 mL/min @ BTP
- CO_2 production = 227 mL/min @ BTP
- Breathing rate = 12 min^{-1}

Gas	Molar fraction in ambient air	Volume$_{in}$ (mL)	Volume$_{out}$ (mL
N_2	75.85%	379.25	379.25
O_2	20.11%	100.55	76.9
CO_2	0.04%	0.2	19.1
H_2O	4.00%	20.0	31.3
Total		500.0	

Calculations :

Volume$_{in}$ = (500mL) x molar fraction

$N_{2\,out} = N_{2\,in}$

$O_{2\,out} = (100.55 - 284/12)\,mL = 76.9\,mL$

$CO_{2\,out} = (0.2 + 227/12)\,mL = 19.1\,mL$

$$\frac{V_{H_2O}}{V_{dry\,gas}} = \frac{V_{H_2O}}{(379.25 + 76.9 + 19.1)\,mL} = \frac{47\,mmHg}{(760 - 47)\,mmHg} \Rightarrow V_{H_2O} = 31.3\,mL$$

Difference in H_2O volume out-in = $(31.3 - 20.0)\,mL = 11.3\,mL$

$$\text{Mass } H_2O \text{ per breath} = \frac{pV}{RT} MW_{H_2O} = \frac{(1\,atm)(11.3 \times 10^{-3}\,L)}{(82.05 \times 10^{-3}\,\frac{L \cdot atm}{K \cdot mol})(310\,K)} \cdot 18\,g/mol$$

$$= 8.01 \times 10^{-3}\,g/\text{breath}$$

$$\text{Number of breaths} = \frac{1.299\,g}{8.01 \times 10^{-3}\,g/\text{breath}} \approx 162 \text{ breaths}$$

O_2 consumption rate at BTP

$= (208.4 - 158.4)$ mL/breath $\times$ 25 breath/min

$= 1250$ mL/min

Convert to STP : multiply by $\frac{273}{310}$ (temperature ratio)

$\therefore$ O_2 consumption rate at STP

$= 1250$ mL/min $\times \frac{273}{310} = 1101$ mL/min

CO_2 production rate at BTP

$= (37.3 - 0.4)$ mL/breath $\times$ 25 breath/min

$= 922.5$ mL/min

$\therefore$ CO_2 production rate at STP

$= 922.5$ mL/min $\times \frac{273}{310} = 812$ mL/min

Gas	Molar fraction	Partial pressure (mmHg)	Volume (μL)		Molar fraction out
			In	Out	
N_2	**78.62%**	597	117.93	117.93	74.66%
O_2	**20.84%**	159	31.26	31.26-5.28 = 25.98	16.45%
CO_2	**0.04%**	0.3	0.06	0.06+4.22 = 4.28	2.71%
H_2O	**0.50%**	3.7	0.75	9.77	6.19%
Total	**100.00%**	760	150	157.96	100.00%

Note:

Tidal volume = 150 μL

O_2 consumption at STP

$$= 45.5 \text{ mL/hr} \times \frac{1 \text{ hr}}{60 \text{ min}} \times \frac{1 \text{ min}}{163 \text{ breath}} = 4.65 \text{ μL/breath}$$

⇒ O_2 consumption rate at BTP

$$= 4.65 \text{ μL/breath} \times \frac{310}{273} = 5.28 \text{ μL/breath}$$

From the text, the CO_2 production / O_2 consumption ratio is $\frac{227}{284}$

∴ CO_2 production rate at BTP

$$= 5.28 \text{ μL/breath} \times \frac{227}{284} = 4.22 \text{ μL/breath}$$

The expired dry gas volume = 148.19 μL (calculated from the table above)

∴ expired water volume

$$= 148.19 \text{ μL} \times \frac{47}{760-47} = 9.77 \text{ μL}$$

a) Call V_{CO_2} = volume of CO_2 added to lung per breath

Assuming constant pressure and temperature, and ideal gas

⇒ volume of CO_2 is proportional to number of moles of CO_2

⇒ CO_2 volume added to a lung is proportional to the blood perfusion to that lung.

Mass balance gives: $TV \times C_E = V_{CO_2L} + V_{CO_2R}$; TV = tidal volume, C_E = expelled CO_2 molar fraction(%)

$\therefore V_{CO_2L} + V_{CO_2R} = 3V_{CO_2L} = (540\text{ mL})(3.5\%) \Rightarrow V_{CO_2L} = 6.3\text{ mL}$

$V_{CO_2R} = 12.6\text{ mL}$

Now, TV = dead space + $V_L + V_R$ and $V_R = 1.6V_L$

$\therefore 540\text{ mL} = 160\text{ mL} + V_L + 1.6V_L$

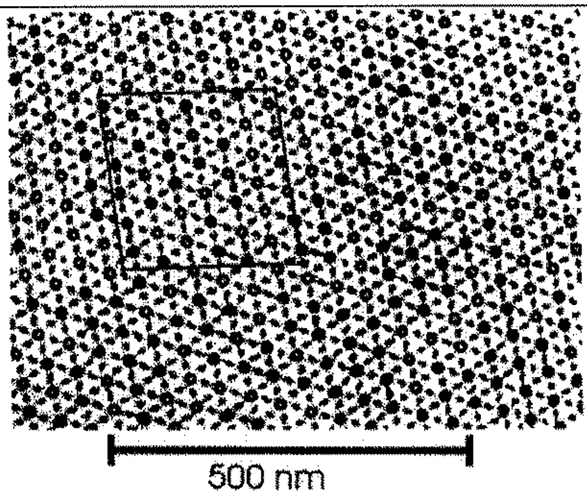
$\Rightarrow V_L = 146.2\text{ mL}$ and $V_R = 233.8\text{ mL}$

$\therefore$ left lung : $[CO_2]_L = \dfrac{V_{CO_2L}}{V_L} = \dfrac{6.3\text{mL}}{146.2\text{mL}} = 4.31\%$

right lung : $[CO_2]_R = \dfrac{V_{CO_2R}}{V_R} = \dfrac{12.6\text{mL}}{233.8\text{mL}} = 5.39\%$

b)

$\dfrac{\text{blood perfusion}}{\text{tidal volume entering}}$ ratio is equal for 2 lungs.



In a square 250 nm X 250 nm, there are 40 myosin filaments. Call F the force generated by one filament. Then;

$$\frac{40F}{(250\times10^{-7}\,cm)^2} = 20\ N/cm^2$$

$$\therefore\ F = (20\,N/cm^2)(250\times10^{-7}\,cm)^2\cdot\frac{1}{40}$$

$$= 3.13\times10^{-10}\ N$$

Tension-length graph can be fit by $T = c(L - L_o)$,
where L_o is 65% of L_{max}.

Max tension occurs at $R = 3\text{ cm}$ $\Rightarrow$ $L_{max} = 2\pi \cdot 3\text{ cm} = 6\pi\text{ cm}$

$L_o = 0.65 L_{max} = 3.9\pi\text{ cm}$

$$\therefore\ c = \frac{10^6\ \text{dynes/cm}^2}{(6-3.9)\pi\ \text{cm}} = 1.516\times10^5\ \text{dynes/cm}^3$$

Pressure-stress relation for a thin-walled cylinder:

$2pR = 2Th$; h is wall thickness and the muscle tension, T, is generated in the "hoop" direction

$$\therefore\ p = \frac{Th}{R} = \frac{h}{R}c(L-L_o) = \frac{h}{R}c(2\pi)(R-R_o)$$

$$\therefore\ p = 2\pi hc\left(1 - \frac{R_o}{R}\right)$$

$V = \pi R^2 H$ or $R = \sqrt{\frac{V}{\pi H}}$. Therefore, $\frac{R_o}{R} = \sqrt{\frac{V_o}{V}}$, and

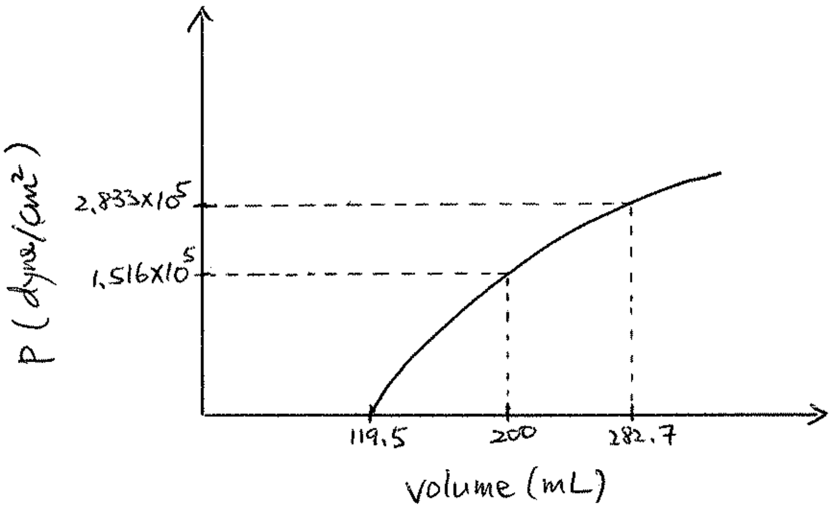
$$p = 2\pi hc\left[1 - \sqrt{\frac{V_o}{V}}\right]$$

$V_o = \pi R_o^2 H = \pi[(0.65)(3)]^2 10 = 119.5\text{ cm}^3$

$V_{max} = \pi R_{max}^2 H = \pi \cdot 3^2 \cdot 10 = 282.7\text{ cm}^3$

$$p_{max} = (2\pi)(0.7)(1.516\times10^5)\left[1 - \sqrt{\frac{119.5}{282.7}}\right]$$
$$= 2.333\times10^5\ \text{dynes/cm}^2$$

continued...



$$p = 6.668 \times 10^5 \left[1 - \sqrt{\frac{V_0}{V}} \right]$$

Since the muscle follows the 3 element model,

$$\frac{T}{T_0} = 1 - e^{-k_0' t/\eta_0} \quad \text{or} \quad e^{-k_0' t/\eta_0} = 1 - T/T_0$$

Given: $\eta_0 = 2.5\,\text{Ns/m}$, $T/T_0 = 0.8$, $t = 0.04\,\text{s}$

$$-\frac{k_0' t}{\eta_0} = \ln(1 - T/T_0)$$

$\Rightarrow$ solve for k_0', $k_0' = -\frac{2.5\,\text{Ns/m}}{0.04\,\text{s}} \cdot \ln(1-0.8) = 100.59\ \text{N/m}$

Putting a spring in series with the muscle changes the spring constant, k, of the system.

$$k = k_0' + k_0 = 100.59 + 200 = 300.59\ \text{N/m}$$

where k_0' = spring constant of the muscle from above

k_0 = spring constant of the spring.

$$\therefore \frac{T}{T_0} = 1 - e^{-kt/\eta_0} = 1 - \exp\left(\frac{-(300.59\,\text{N/m})(0.02\,\text{s})}{2.5\ \text{Ns/m}}\right)$$

$$\Rightarrow \frac{T}{T_0} = 91.0\%$$

$\therefore$ tension is 91% of the maximum

a)

$$+\downarrow \sum F_y = Mg - T = M\frac{d^2x}{dt^2}$$

Also, $T = T_0 + \eta_0 \frac{dx}{dt} + k_0(x - \bar{x})$, $\bar{x}$ = length of unstretched spring

At rest ($\frac{dx}{dt} = 0$, $T_0 = 0$) :

$$Mg = T = k_0(x_0 - \bar{x})$$

$$\therefore \quad \bar{x} = x_0 - \frac{Mg}{k_0}$$

When muscle contracts:

$$T = T_0 + \eta_0\frac{dx}{dt} + k_0(x - x_0) + Mg$$

$$\therefore Mg - T = -\left[T_0 + \eta_0\frac{dx}{dt} + k_0(x - x_0)\right] = M\frac{d^2x}{dt^2}$$

$$\therefore \frac{d^2x}{dt^2} + \frac{\eta_0}{M}\frac{dx}{dt} + \frac{k_0}{M}(x - x_0) = -\frac{T_0}{M} \qquad (*)$$

Initial Conditions:

$$x - x_0 = 0 \quad \text{at } t = 0 \qquad (1)$$

$$\frac{d}{dt}(x - x_0) = 0 \quad \text{at } t = 0 \qquad (2)$$

Solve for (*) :

$$x - x_0 = -\frac{T_0}{k_0}\left[1 + c_1 e^{r_1 t} + c_2 e^{r_2 t}\right]$$

where $r_1 = -\frac{\eta_0}{2M}\left[1 + \sqrt{1 - 4k_0 M/\eta_0^2}\right]$; $r_2 = -\frac{\eta_0}{2M}\left[1 - \sqrt{1 - 4k_0 M/\eta_0^2}\right]$

c_1 , c_2 are constants

a) continued . . .

Apply initial conditions :

① $\Rightarrow$ $1 + c_1 + c_2 = 0$

② $\Rightarrow$ $c_1 r_1 + c_2 r_2 = 0$

$$\therefore \quad c_1 = \frac{-r_2}{r_2 - r_1} \quad , \quad c_2 = \frac{r_1}{r_2 - r_1}$$

$$\therefore \quad x - x_0 = -\frac{T_0}{k_0}\left[1 + \frac{r_1 e^{r_2 t} - r_2 e^{r_1 t}}{r_2 - r_1}\right]$$

b) Given:

$T_0 = 15\,N$ $\quad$ $M = 1\,kg$

$k_0 = 500\,N/m$ $\quad$ $\eta_0 = 100\,Ns/m$

$$r_1 = -\frac{100\,Ns/m}{2(1\,kg)}\left[1 + \left(1 - \frac{4(500\,N/m)(1\,kg)}{(100\,Ns/m)^2}\right)^{1/2}\right] = -94.72\ s^{-1}$$

$$r_2 = -5.28\ s^{-1}$$

At $t = C = 0.1\,s$

$$x - x_0 = -\frac{15N}{500\,N/m}\left[1 + \frac{(-94.72\,s^{-1})e^{-5.28(0.1)} - (-5.28\,s^{-1})e^{-94.72(0.1)}}{(-5.28\,s^{-1}) - (-94.72\,s^{-1})}\right]$$

$$= -1.126\ cm$$

$\therefore$ the mass moves up 1.126 cm at the end of contraction

Q6

For box 1: $\frac{T_1}{T_0} = 1 - e^{-k_0 t/\eta_0}$ for $t \geq 0$

For box 2: $\frac{T_2}{T_0} = 1 - e^{-k_0(t - \frac{1}{2}C)/\eta_0}$ for $t \geq C/2$

Assuming independent action, during the contraction phase of both muscles, we have

$$\frac{T_{tot}}{T_0} = \frac{T_1 + T_2}{T_0} = 2 - e^{-k_0 t/\eta_0} - e^{-k_0(t - \frac{1}{2}C)/\eta_0} \quad \text{for } t \geq C/2$$

At $t = C$: $T_{tot} = T_0 \left\{ 2 - e^{-k_0 C/\eta_0} - e^{-k_0 C/2\eta_0} \right\}$

here, $k_0 C/\eta_0 = \frac{(0.3\ \text{dyne/cm})(0.4\ \text{s})}{(0.06\ \text{dyne}\cdot\text{s/cm})} = 2$

$\therefore$ $T_{tot} = (4\ \text{dyne})(2 - e^{-2} - e^{-1}) = 5.987\ \text{dyne}$

$T_1 = 4(1 - e^{-2}) = 3.459\ \text{dyne}$

$T_2 = 4(1 - e^{-1}) = 2.529\ \text{dyne}$

Since it is isotonic, T_1 is fixed. A free body diagram shows that the spring is under constant tension, hence its length does not change. $L-L_0$ will therefore be determined only by the dashpot and force generator.

Call $x_1 = L - L_0$. Then

for $0 \le t \le c$, $T_0 + \eta_0 \frac{dx_1}{dt} = T_1$

$$x_1 = \left(\frac{T_1 - T_0}{\eta_0}\right)t + \text{const.}$$

At $t=0$, $x_1 = 0$, $\therefore$ const $= 0$

$$\therefore\ L = L_0 + \left(\frac{T_1 - T_0}{\eta_0}\right)t \quad \text{or} \quad L = L_0 - \left(\frac{T_0 - T_1}{\eta_0}\right)t$$

At $t = c$, $x_1 = L_1 = \left(\frac{T_1 - T_0}{\eta_0}\right)c$

At $t \ge c$, $\eta_0 \frac{dx_1}{dt} = T_1 \qquad \therefore\ x_1 = \frac{T_1}{\eta_0}t + \text{const.}$

since $x_1 = L_1$ at $t = c$, const $= L_1 - \frac{T_1}{\eta_0}c$

$\therefore\ x_1 - L_1 = \frac{T_1}{\eta_0}(t-c)$

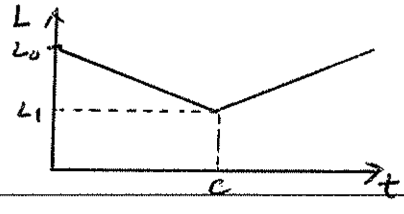
valid until $x_1 = 0$, or until $t - c = -\frac{L_1 \eta_0}{T_1}$

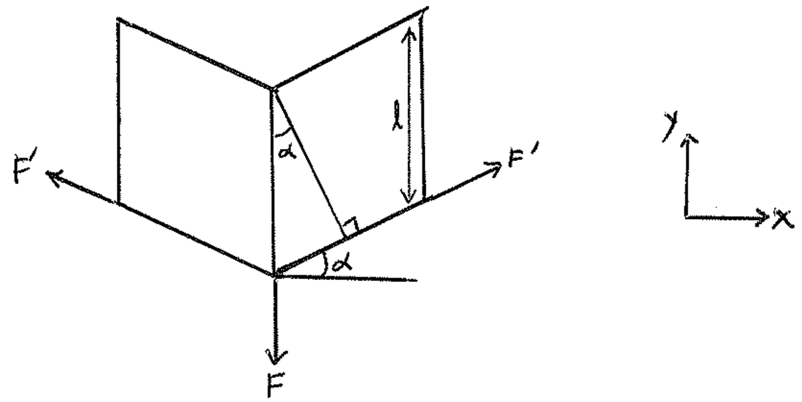
i.e. $t \le c + \left(\frac{T_0 - T_1}{T_1}\right)c = \frac{c}{T_1} \cdot T_0$

So for $c \le t \le \frac{cT_0}{T_1}$

$$L - L_0 = L_1 + \frac{T_1}{\eta_0}(t-c) = \left(\frac{T_1 - T_0}{\eta_0}\right)c + \frac{T_1}{\eta_0}(t-c)$$

$$\therefore\ L = L_0 + \frac{T_1 t - T_0 c}{\eta_0}$$





Cross-sectional area perpendicular to muscle fibers per unit depth:

$$A = l\cos\alpha$$

Force created along muscle fibres per unit depth:

$$F' = f(l\cos\alpha)$$

Balancing the forces in y direction:

$$F = 2F'\sin\alpha$$

$$\therefore F = 2fl\sin\alpha\cos\alpha$$

From Question 8, we know the force generated per unit width is

$$F = 2fl\sin\alpha\cos\alpha \quad \text{for one portion of the muscle}$$

Given:

$f = 20\,N/cm^2$

$l = 5\,cm$

$\alpha = 25°$

W = width of muscle = 1 cm

∴ The total force generated in one muscle portion is

$$FW = 2fl\sin\alpha\cos\alpha \cdot W$$

$$= 2(20\,N/cm^2)(5\,cm)(\sin 25° \cos 25°)(1\,cm)$$

$$= 766\ N$$

For two muscle units in parallel, the total force is the sum of the two:

$$\text{Total force} = 2 \cdot 766\,N = 1532\,N.$$

a)

From the text, we know $F_g = \frac{l}{h} F_m$ (equation 8.14).

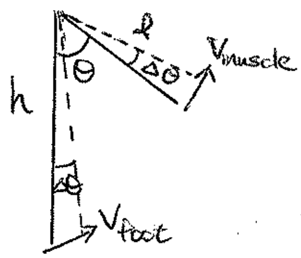
And from the question we know $F_m = f k_2 L^2$

$\therefore \; F = F_g = \frac{l}{h} F_m = \frac{l}{h} f k_2 L^2$

$\therefore$ acceleration is $a = \frac{F}{m} = \frac{l/h \cdot f k_2 L^2}{k_1 L^3} = f \cdot \frac{k_2}{k_1} \cdot \frac{l}{h} \cdot \frac{1}{L}$

$$\therefore \; a \sim \frac{l}{h} \frac{f}{L}$$

b) Top speed will be proportional to maximum velocity of foot



$$\frac{l \Delta\theta}{V_{muscle}} = \frac{h \Delta\theta}{V_{foot}}$$

$$\Rightarrow V_{foot} = \frac{h}{l} V_{muscle}$$

$$\therefore \; V_{max} \sim V_{foot} \sim \left(\frac{h}{l}\right) V_{muscle} \sim \left(\frac{h}{l}\right) k_3 L \;\Rightarrow\; V_{max} \sim L\left(\frac{h}{l}\right)$$

c) If acceleration is uniform,

$$\Delta t = \frac{V_{max} - 0}{a} = \text{const.} \frac{L(h/l)}{\frac{1}{L}(l/h)} = \text{const.} \left(\frac{h}{l}\right)^2 L^2$$

where const. is the same for all species since it involves only k_1, k_2, k_3 and f.

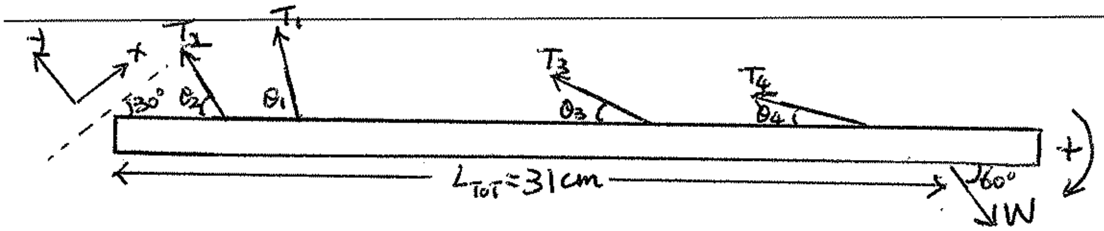
$$\therefore \; \frac{\Delta t_{horse}}{\Delta t_{armadillo}} = \left[\frac{L_{horse}}{L_{armadillo}} \cdot \frac{(l/h)_{armadillo}}{(l/h)_{horse}} \right]^2 = \left(8 \cdot \frac{1/4}{1/13} \right)^2 = 676$$

d)

This seems too big. If it takes an armadillo 1 second to accelerate to top speed, it takes a horse 11 minutes, which is too long.

Possible problems:

- k_1, k_2, k_3, f not the same for both species
- neglects hind leg effects which are important and many differ between species
- neglects force needed to rotate legs

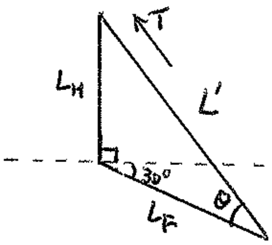


		L_H (cm)	L_F (cm)	PCSA (cm²)	T (N)	θ
1	Biceps	31	8	12.3	246	48.8°
2	Brachialis	10	5	13.0	260	41.0°
3	Brachioradialis	8	24	2.9	58	13.9°
4	ECRL	3	25	3.6	72	5.6°

Note:

- $T = (PCSA) \cdot f$, $f = 20\ \text{N/cm}^2$

- To find θ's :



1. Find L' : $L'^2 = L_H^2 + L_F^2 - 2L_H L_F \cos 120°$
2. Find θ :

$$\frac{L_H}{\sin\theta} = \frac{L'}{\sin 120°} \Rightarrow \theta = \sin^{-1}\left(\frac{L_H}{L'}\sin 120°\right)$$

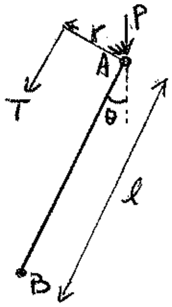
Moment balance about the elbow:

$$W \sin 60° L_{TOT} - \left(T_1 \sin\theta_1 L_{F_1} + T_2 \sin\theta_2 L_{F_2} + T_3 \sin\theta_3 L_{F_3} + T_4 \sin\theta_4 L_{F_4}\right) = 0$$

$$\therefore W = \frac{(246)(\sin 48.8°)(8) + (260)(\sin 41.0°)(5) + (58)(\sin 13.9°)(24) + (72)(\sin 5.6°)(25)}{(\sin 60°)(31)}$$

$\therefore W = 105.9\ \text{N}$ or $W = 10.8\ \text{kg}$.

a)



Moment balance about point B:

$$P l \sin\theta = T r$$

$$\therefore \quad T = \frac{P l \sin\theta}{r}$$

b) Given: $\delta T = \beta A \dfrac{\delta L}{L}$

but from geometry, $\delta L = r\,\delta\theta$ (arc length)

$$\therefore \quad \delta T = \beta A \frac{\delta L}{L} = \beta A \frac{r\,\delta\theta}{L}$$

After the person squats a little further, a moment balance about point B gives:

$$T + \delta T = \frac{P l \sin(\theta + \delta\theta)}{r}$$

$$\Rightarrow \frac{P l \sin\theta}{r} + \beta A \frac{r\,\delta\theta}{L} = \frac{P l}{r}\left[\sin\theta \cancelto{1}{\cos\delta\theta} + \cos\theta \cancelto{\delta\theta}{\sin\delta\theta}\right], \text{ for small } \delta\theta$$

$$\therefore \quad \beta = \frac{L l P}{A r^2} \cdot \frac{1}{\delta\theta} \cdot \left[\sin\theta + \delta\theta\cos\theta - \sin\theta\right]$$

$$\therefore \quad \beta = \frac{L l P}{A r^2}\cos\theta$$

Assuming no extensibility in patellar tendon, the length of the muscle depends linearly on θ. Specifically

$$L - L_0 = r(\theta - \theta_0)$$

where r = distance from C to P, L_0 and θ_0 are constants. θ in radians.

Given: $L_0 = 30$ cm, $\theta_0 = 45° = \pi/4$

From Figure 8-14, greater than or equal to 80% of maximal isotonic tetanic tension will occur if the muscle length is roughly in the range of

$$0.75 L_0 \le L \le 1.2 L_0$$

For the lower end of the range:

$$L - L_0 = -0.25 L_0 = -7.5 \text{ cm} = r(\theta - \theta_0)$$

Using $r = 10$ cm, $\theta_0 = \pi/4$ gives.

$$\theta - \theta_0 = -0.75 \text{ rad} \doteq -43°$$

$$\therefore \theta = 45° - 43° = 2°$$

For the upper end of the range:

$$L - L_0 = 0.2 L_0 = 6 \text{ cm} = r(\theta - \theta_0)$$

$$\therefore \theta - \theta_0 = 0.6 \text{ rad} = 34.4°$$

$$\therefore \theta = 45° + 34.4° = 79.4°$$

$$\therefore \quad 2° \le \theta \le 79.4°$$

From Figure 9-36, the following parameters can be determined:

	Cortical Bone	Trabecular Bone (ρ = 0.9 g/cm^3)	Trabecular Bone (ρ = 0.3 g/cm^3)
Yield Strength (MPa)	165	35	5
Ultimate Strength (MPa)	180	60	5
Yield Strain (m/m)	0.01	0.03	0.04
Ultimate Strain (m/m)	0.025	0.235	0.23
Elastic Modulus (MPa)	16.5	1.2	0.125
Anelastic Modulus (GPa)	N/A	120	0
Strain Energy Density (J/cm^3)	3.4	10.3	1.05

Note:

- $E_{anelastic} = \dfrac{S_u - S_y}{\varepsilon_u - \varepsilon_y} = \dfrac{60-35}{0.235-0.03} = 122 \text{ MPa}$

  for trabecular bone with $\rho = 0.9 \text{ g/cm}^3$

- There are several methods could be used to approximate the strain energy density from the graph. Generally $U = \int_0^{\varepsilon_u} \sigma \, d\varepsilon$

For cortical bone, based on area of triangle for the elastic region plus area of trapezoid for the plastic region:

$$U_c = \int_0^{\varepsilon_u} \sigma \, d\varepsilon \approx \frac{1}{2} S_y \varepsilon_y + \frac{1}{2}(S_y + S_u)(\varepsilon_u - \varepsilon_y)$$

$$\therefore U_c = \frac{1}{2}(165 \text{ MPa})(0.01) + \frac{1}{2}(165 \text{ MPa} + 180 \text{ MPa})(0.025 - 0.01)$$

$$= 3.4 \text{ MPa} = 3.4 \text{ J/cm}^3$$

Similarly, for trabecular bone, the strain energy can also be approximated by the area under the $\sigma - \varepsilon$ curve.

continued . . .

$\therefore$ for trabecular bone with $\rho = 0.9\ g/cm^3$

$$U_t \approx \frac{1}{2} S_y \varepsilon_y + \frac{1}{2}(S_u + S_y)(\varepsilon_u - \varepsilon_y)$$
$$= \frac{1}{2}(35\,MPa)(0.03) + \frac{1}{2}(60\,MPa + 35\,MPa)(0.235 - 0.03)$$
$$= 10.3\ MPa = 10.3\ J/cm^3$$

for trabecular bone with $\rho = 0.3\ g/cm^3$

$$U_t \approx \frac{1}{2} S_y \varepsilon_y + \frac{1}{2}(S_u + S_y)(\varepsilon_u - \varepsilon_y)$$
$$= \frac{1}{2}(5\,MPa)(0.04) + \frac{1}{2}(5\,MPa + 5\,MPa)(0.23 - 0.04)$$
$$= 1.05\ MPa = 1.05\ J/cm^3$$

The strain energy density at failure is much greater for dense trabecular bone than for cortical bone. This implies that the trabecular bone can absorb a significantly greater amount of energy before it fails than can cortical bone. Further more, for a given level of energy absorption, trabecular bone will generate a lower peak force. Thus, trabecular bone acts in a manner similar to packing foam in that it absorbs energy from impacts.

Loss of trabecular bone density, as occurs in osteoporosis, reduces the energy that can be absorbed prior to failure. The result is a higher risk of failure.

Trabecular bone mechanical properties can be approximated using equation (9.4) and (9.5):

$$\frac{E^*}{E_s} = c_1\left(\frac{\rho^*}{\rho_s}\right)^2 \quad ; \quad \frac{\sigma_y^*}{\sigma_{ys}} = c_2\left(\frac{\rho^*}{\rho_s}\right)^2$$

where E^*, ρ^*, σ_y^* are the apparent properties
E_s, ρ_s, σ_{ys} are the properties of the tissue matrix itself.
c_1, c_2 are constants.

Assuming Hooke's law is valid for trabecular bone up to yield strain.

$\therefore \sigma_y^* = E^* \varepsilon_y^*$ where ε_y^* is the apparent yield strain.

$$\Rightarrow \varepsilon_y^* = \frac{\sigma_y^*}{E^*} = \frac{\sigma_{ys}}{E_s}\frac{c_2}{c_1}$$

Since σ_{ys} and E_s are tissue matrix properties, they can be treated as constants.

Therefore, $\varepsilon_y^* = \text{const}$ and independent of apparent density.

a)
$$\frac{\rho^*}{\rho_s} = \frac{\text{mass}/V_{cube}}{\text{mass}/V_{struts}} = \frac{V_{struts}}{V_{cube}} \propto \frac{t^2 l}{l^3} = \left(\frac{t}{l}\right)^2$$

b) For a beam with rectangular cross-section, the moment of inertia $I = 1/12\, b h^3$, where b and h are the width and height, respectively. For a square cross-section, $b = h = t$, and therefore, $I \propto t^4$

c)
$$F_{crit} \propto \sigma^* l^2 \propto \frac{E_s I}{l^2}.$$

Substituting $I \propto t^4$ and rearranging gives
$$\frac{\sigma^*}{E_s} \propto \frac{t^4}{l^4} \propto \left(\frac{\rho^*}{\rho_s}\right)^2$$

E_s and σ_{ys} are constants (based on the properties of cortical bone), so substituting one for another to obtain equation (9.4) simply changes the constant of proportionality.

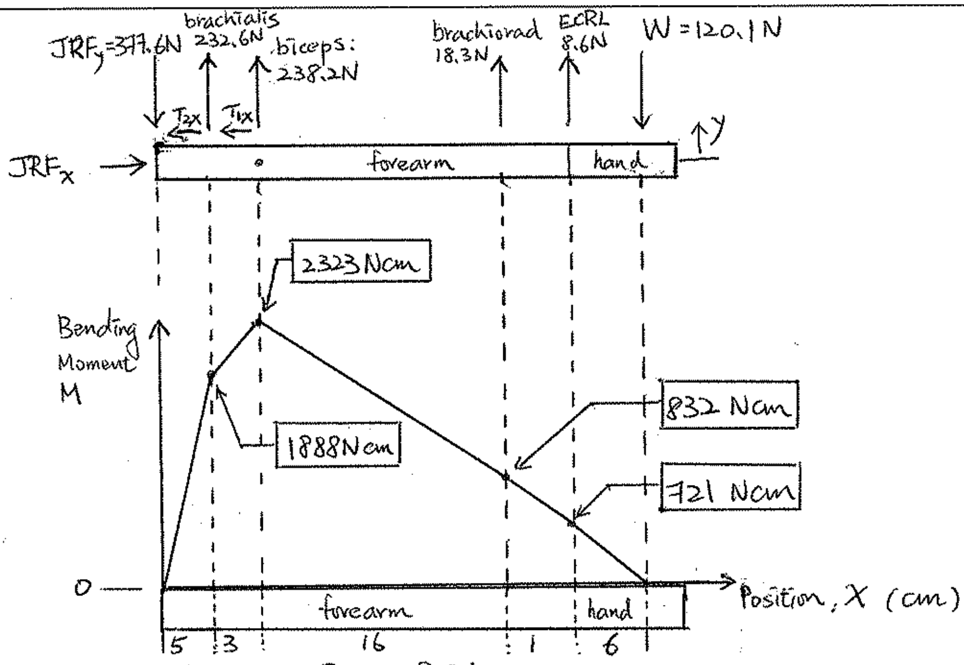


Figure 8-25

The axial stress due to the bending moment, $M(x)$, and compressive internal force, $F_x(x)$ is given by

$$\sigma(x,y) = \frac{M(x)\cdot y}{I_z} + \frac{F_x(x)}{A} \qquad (\text{eq'n } 8.17)$$

From Figure 8.25, we know $M(8\text{ cm}) = 2323\text{ Ncm}$

$$F_x(8\text{ cm}) = T_{1x} + T_{2x} - JRF_x$$

With Table 8-1 and Figure 8-24, T_{1x} and T_{2x} can be calculated:

$$T_{1x} = T_1\cos\theta_1 = (T_{1y}/\sin\theta_1)\cos\theta_1 = 238.2\text{N}\cdot\cot(76°) = 59.4\text{N}$$

$$T_{2x} = T_2\cos\theta_2 = (T_{2y}/\sin\theta_2)\cos\theta_2 = 232.6\text{N}\cdot\cot(63°) = 118.5\text{N}$$

From the text, we know $JRF_x = 304.3\text{ N}$

$$\therefore F_x(8\text{cm}) = 59.4\text{N} + 118.5\text{ N} - 304.3\text{ N} = -126.4\text{ N}$$

continued...

Also from the text:

$I_z = 0.177\ cm^4$

$A = 1.15\ cm^2$

$D_o = 1.4\ cm \Rightarrow R_o = 0.7\ cm$

Assume the 0.01 mm defect is under maximum tensile stress (that is, on the top surface of the bone).

$\therefore y = 0.7\ cm$.

$$\therefore \sigma_{max}(8, 0.7) = \frac{2323\ N\,cm}{0.177\ cm^4} \cdot 0.7\ cm - \frac{126.4\ N}{1.15\ cm^2}$$

$$= 9077.1\ N/cm^2 = 9.08 \times 10^7\ Pa$$

where we have accounted for the slight reduction in the tensile stress due to the compressive axial force, F_x.

$\therefore$ the magnitude of the maximum tensile stress 8 cm away from the elbow is 9.08×10^7 Pa.

Now, when $K_{max} = K_c$, fast fracture will occur.

Let's take the lower bound of K_c value to be the K_{max},

$\therefore K_{max} = 2.2\ MN/m^{3/2}$

$\therefore K_{max} = \sigma_{max}\sqrt{\pi a} = 2.2\ MN/m^{3/2}$, where a = crack length just before fast fracture

$\therefore (9.08 \times 10^7\ Pa) \cdot \sqrt{\pi a} = 2.2 \times 10^6\ N/m^{3/2}$

$\Rightarrow a = 0.187 \times 10^{-3}\ m$

Use equation 9.20 $$N_f = \frac{1}{C(\Delta\sigma)^m \pi^{m/2}} \left. \frac{a^{1-m/2}}{1-m/2} \right]_{a_o}^{a_f}$$

continued...

given $m = 2.5$; $C = 2.5 \times 10^{-6}\, m(MN/m^{3/2})^{-2.5}$, $a_0 = 0.01$ mm

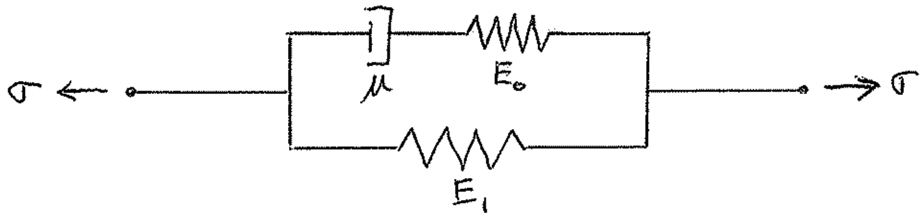
and $\Delta\sigma = \sigma_{max} - 0 = 9.08 \times 10^{7}$ Pa $= 90.8$ MPa

$$\therefore N_f = \frac{1}{2.5\times 10^{-6} \cdot (90.8)^{2.5} \cdot \pi^{1.25}} \cdot \frac{(0.187\times 10^{-3})^{-1/4} - (0.01\times 10^{-3})^{-1/4}}{-1/4}$$

$$\approx 45$$

$\therefore$ The subject can lift 45 times before bone factures.

This is likely to be an underestimate of the actual number of lifts, because we ignore the microstructure of the bone and the osteons and collagen fibres. They all limit crack propagation.



The spring constant and damping coefficients in the model in Figure 2-35 are replaced with the corresponding material constants, i.e. Young's moduli E_0, E_1 in place of the spring constant k_0, k_1, and viscosity μ in place of the damping coefficient η_0.

Let σ = stress applied ; σ_0 = stress in the upper leg ; σ_1 = stress in the lower leg

ε = strain of the entire model

ε_0 = strain of the elastic element with Young's modulus E_0

ε_1 = strain of the elastic element with Young's modulus E_1

ε_μ = strain of the viscostic element.

$$\therefore \quad \varepsilon = \varepsilon_1 \quad \text{and} \quad \varepsilon = \varepsilon_0 + \varepsilon_\mu \Rightarrow \dot{\varepsilon} = \dot{\varepsilon}_0 + \dot{\varepsilon}_\mu$$

We know $\sigma_0 = E_0 \varepsilon_0 \Rightarrow \dot{\sigma}_0 = E_0 \dot{\varepsilon}_0$; also $\sigma_0 = \mu \dot{\varepsilon}_\mu$

$$\therefore \dot{\varepsilon} = \frac{\dot{\sigma}_0}{E_0} + \frac{\sigma_0}{\mu}$$

But $\sigma_0 = \sigma - \sigma_1 = \sigma - E_1\varepsilon_1 = \sigma - E_1 \varepsilon$

$$\therefore \dot{\varepsilon} = \frac{1}{E_0}(\dot{\sigma} - E_1 \dot{\varepsilon}) + \frac{1}{\mu}(\sigma - E_1 \varepsilon)$$

After rearranging,

$$\sigma + \frac{\mu}{E_0}\dot{\sigma} = E_1 \varepsilon + \mu\left(1 + \frac{E_1}{E_0}\right)\dot{\varepsilon}$$

The internal friction will be an extreme when the first derivative of $\tan\delta$ with respect to ω is equal to zero. The expression for $\tan\delta$ is

$$\tan\delta = \frac{\omega(\tau_\sigma - \tau_\varepsilon)}{1+\omega^2\tau_\sigma\tau_\varepsilon}$$

Recalling the quotient rule for differentiation, we get

$$\frac{d}{d\omega}(\tan\delta) = \frac{(\tau_\sigma-\tau_\varepsilon)(1+\omega^2\tau_\sigma\tau_\varepsilon) - \omega(\tau_\sigma-\tau_\varepsilon)(2\omega\tau_\sigma\tau_\varepsilon)}{(1+\omega^2\tau_\sigma\tau_\varepsilon)^2}$$

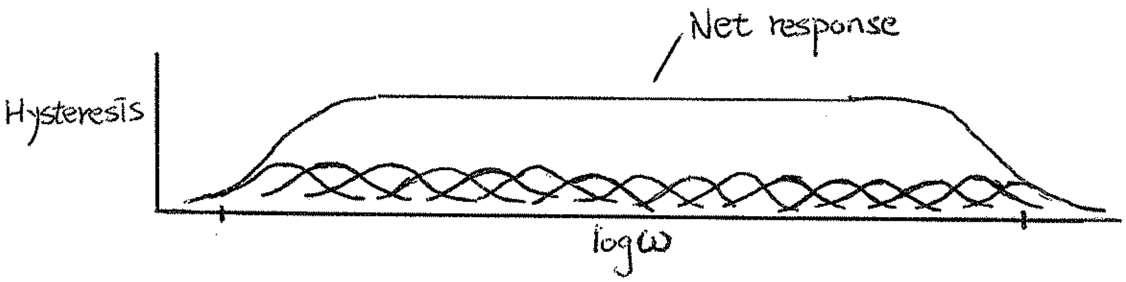
$$= \frac{(\tau_\sigma-\tau_\varepsilon) + \omega^2(\tau_\sigma-\tau_\varepsilon)(\tau_\sigma\tau_\varepsilon) - 2\omega^2(\tau_\sigma-\tau_\varepsilon)(\tau_\sigma\tau_\varepsilon)}{(1+\omega^2\tau_\sigma\tau_\varepsilon)^2}$$

$$= \frac{(\tau_\sigma-\tau_\varepsilon) - \omega^2(\tau_\sigma-\tau_\varepsilon)(\tau_\sigma\tau_\varepsilon)}{(1+\omega^2\tau_\sigma\tau_\varepsilon)^2}$$

$$= \frac{(\tau_\sigma-\tau_\varepsilon)(1-\omega^2\tau_\sigma\tau_\varepsilon)}{(1+\omega^2\tau_\sigma\tau_\varepsilon)^2}$$

Setting $\frac{d}{d\omega}(\tan\delta) = 0$, we get that the internal friction is a maximum when $\omega = (\tau_\sigma\tau_\varepsilon)^{-1/2}$

The standard linear model exhibits internal friction (and therefore, hysteresis) over a rather narrow range of frequencies, which is inconsistent with the experimental evidence. To improve the model to account for insensitivity of internal damping to frequency, we would need to add additional elements to model, or alternatively add exponential terms to the governing equation (i.e., relaxation or creep response) where the time constants for each exponential term differ. In doing so, the improved model would have an internal damping peak that was "spread out" (see figure below) and not as sensitive to frequency.



a) From the text, $V_{pushoff}^2 = 2gc\left[\frac{F_{equiv}}{W} - 1\right]$, measured with respect to the platform. The total velocity is just $V_p + V_{pushoff}$. The elevation of the centre h is

$$h = \frac{(V_p + V_{pushoff})^2}{2g} = \frac{\left[V_p + \sqrt{2gc\,(F_{equiv}/W - 1)}\right]^2}{2g}$$

Alternative Solution:

Developing the equation as in the text, with $V_T = V_p + V_{pushoff}$

$$V_T(t) = gt\left[(F_{equiv}/W) - 1\right] + V_p \quad -① $$

$$\Rightarrow t = \frac{V_T(t) - V_p}{g\left[F_{equiv}/W - 1\right]}$$

Integrate ① gives:

$$z(t) - z_0 = \frac{1}{2}gt^2\left[F_{equiv}/W - 1\right] + V_p t \quad -②$$

at end of pushoff $t = \tau$ and $z(\tau) - z_0 = c + V_p\tau$

sub it in ②:

$$c + V_p\tau = \frac{1}{2}g\tau^2\left[F_{equiv}/W - 1\right] + V_p\tau$$

but $\tau = \dfrac{V_T(\tau) - V_p}{g\left[F_{equiv}/W - 1\right]}$

$$\therefore \quad 2c = \left(\frac{V_T(\tau) - V_p}{g\left[F_{equiv}/W - 1\right]}\right)^2 g\left[F_{equiv}/W - 1\right]$$

$$\therefore \quad V_T(\tau) = \sqrt{2gc\left[F_{equiv}/W - 1\right]} + V_p$$

and $h = \dfrac{V_T^2(\tau)}{2g} = \dfrac{\left[\sqrt{2gc\,(F_{equiv}/W - 1)} + V_p\right]^2}{2g}$

b) Given $F_{equiv}/W = 2$, $C = 1.5\ ft$, $V_p = 5\ ft/s$, $g = 32.2\ ft/s^2$

$$h = \frac{\left[5\ ft/s + \sqrt{2(32.2\ ft/s^2)(1.5\ ft)(2-1)}\right]^2}{2(32.2\ ft/s^2)} = 3.41\ ft$$

$$F(t) - W = \frac{W}{g}\frac{dV}{dt} \quad \Rightarrow \quad \frac{dV}{dt} = \frac{g}{W}F - g$$

$$V(t) = \frac{g}{W}\int F\,dt \; - gt \; + \cancel{\text{const.}} \rightarrow 0 \text{ , since } V=0 \text{ at } t=0$$

$$V(\tau) = \frac{g}{W}\int_0^{\tau}\left[\beta W \sin\left(\frac{\pi t}{\tau}\right) + W\left(1 - \frac{t}{\tau}\right)\right]dt \; - g\tau \quad ; W = 160 \text{ lbm}$$

$$= g\tau\int_0^1\left[3\sin(\pi\beta) + (1-\beta)\right]d\beta \; - g\tau \quad ; \beta = t/\tau$$

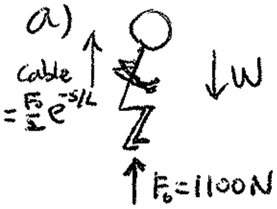
$$= g\tau\left(-\frac{3}{\pi}\cos\pi\beta\Big]_0^1 + 1 - \frac{1}{2} - 1\right)$$

$$= g\tau\left(\frac{6}{\pi} - \frac{1}{2}\right) = (32.2 \text{ ft/s}^2)(180\times10^{-3}\text{ s})\left(\frac{6}{\pi} - \frac{1}{2}\right)$$

$$= 8.17 \text{ ft/s}$$

$$h = \frac{V(\tau)^2}{2g} = \frac{(8.17 \text{ ft/s})^2}{2(32.2 \text{ ft/s}^2)} = 1.04 \text{ ft.}$$

a)



$$+\uparrow \sum F = ma$$

$$\therefore F_0 + \frac{F_0}{2} e^{-s/L} - W = mv \cdot \frac{dv}{ds}$$

where s is measured from the beginning of the pushoff.

$$\therefore F_0\left(1 + \frac{1}{2} e^{-s/L} - \frac{W}{F_0}\right) = \frac{1}{2} m \frac{d(v^2)}{ds}$$

$$\frac{1}{2} m \int_0^{v_p} d(v^2) = F_0 \int_0^c \left(1 + \frac{1}{2} e^{-s/L} - \frac{W}{F_0}\right) ds$$

where v_p is the pushoff speed and c is the crouch depth

$$\therefore \frac{1}{2} m v_p^2 = F_0 \left[s - \frac{L}{2} e^{-s/L} - \frac{W}{F_0} s \right]_{s=0}^{c}$$

$$\frac{1}{2} m v_p^2 = F_0 c \left[1 - \frac{W}{F_0} + \frac{L}{2c}\left(1 - e^{-c/L}\right) \right]$$

Given: $\frac{W}{F_0} = \frac{(75 \times 9.81)N}{1100\ N} = 0.6689$; $\frac{L}{c} = \frac{0.4m}{0.4m} = 1$

$$\therefore \frac{1}{2} m v_p^2 = F_0 c \left[1 - 0.6689 + \frac{1}{2}(1 - e^{-1}) \right] = 0.6472 F_0 c$$

Now, apply conservation of energy in airborne phase:

$$mgh = \frac{1}{2} m v_p^2 = 0.6472 F_0 c$$

$$\therefore h = \frac{0.6472 F_0 c}{mg} = 0.6472 \frac{F_0}{W} c = \frac{0.6472}{0.6689} c$$

$$\therefore h = 0.387 m$$

b) When the catch fails to disengage, we can apply the same analysis to the airborne phase. Now the start velocity is V_p, the final velocity is zero (at top of jump), and distances are measured from beginning of pushoff phase. Note that F_0 does not act.

$$+\uparrow \sum F = ma$$

$$\therefore \frac{F_0}{2} e^{-s/L} - W = \frac{1}{2} m \frac{d(v^2)}{ds}$$

$$F_0 \int_c^{c+h} \left(\frac{1}{2} e^{-s/L} - \frac{W}{F_0} \right) ds = \frac{1}{2} m \int_{V_p^2}^{0} d(v^2) = -\frac{1}{2} m V_p^2 = -0.6472 F_0 C$$

$$\therefore \left[-\frac{L}{2} e^{-s/L} - \frac{W}{F_0} s \right]_{s=c}^{c+h} = -0.6472 C$$

$$\therefore \frac{L}{2} \left(e^{-(c+h)/L} - e^{-c/L} \right) + \frac{W}{F_0} h = 0.6472 C$$

$$\frac{1}{2} \left(e^{-c/L} \right) \left(e^{-h/L} - 1 \right) + \frac{W}{F_0} \frac{h}{L} = 0.6472 \frac{C}{L}$$

We know $W/F_0 = 0.6689$ and $c/L = 1$

$$\therefore \frac{1}{2} e^{-1} \left(e^{-h/L} - 1 \right) + 0.6689 \frac{h}{L} = 0.6472$$

$$\Rightarrow e^{-h/L} + 3.637 \frac{h}{L} = 4.519$$

Numerical solution is $\frac{h}{L} = 1.156$

$$\Rightarrow h = 0.462 \text{ m.}$$

a) During pushoff: $+\uparrow \sum F = m\frac{dV}{dt} = \frac{W}{g}\frac{dV}{dt} = F - \frac{W}{g}g_{moon}$

Replace F by F_{equiv}:

$$g\left(\frac{F_{equiv}}{W} - \frac{g_{moon}}{g}\right) = \frac{dV}{dt}$$

$$\Rightarrow V = gt\left(\frac{F_{equiv}}{W} - \frac{g_{moon}}{g}\right) = \frac{dz}{dt}$$

$$\Rightarrow z = \frac{1}{2}gt^2\left[\frac{F_{equiv}}{W} - \frac{g_{moon}}{g}\right]$$

eliminate t

$$\rightarrow \frac{V_{pushoff}^2}{2g} = c\left[\frac{F_{equiv}}{W} - \frac{g_{moon}}{g}\right]$$

During the airborne phase:

$$\frac{1}{2}mV_{pushoff}^2 = mg_m h$$

$$\therefore h = \frac{V_{pushoff}^2}{2g_{moon}} = c \cdot \frac{g}{g_{moon}}\left[\frac{F_{equiv}}{W} - \frac{g_{moon}}{g}\right]$$

$$\Rightarrow h = c\left[\frac{F_{equiv}}{W}\frac{g}{g_{moon}} - 1\right]$$

b) From the text: $F_{equiv}/W = 2.7$, $c = 20''$

$$\therefore h = 20''\left[2.7 \times 6 - 1\right] = 304''$$

If the centre of gravity is $36''$ above ground, bar height would be

$$304'' + 36'' = 340'' = 28'4''$$

above ground.

From energy balance,

$$E_{potential} = E_{work} - E_{crouch}$$

(i.e. some energy in jumping is used in going from the crouch to an erect position)

$$mgh = Fc - Wc \Rightarrow Wh = (F - W)c$$

$$\therefore h = c\left(\frac{F}{W} - 1\right) \Rightarrow \frac{h}{c} = \frac{F}{W} - 1$$

$$\therefore F = W\left(\frac{h}{c} + 1\right) \Rightarrow F = 150 \text{ lbs}\left(\frac{22''}{15''} + 1\right) = 370 \text{ lbf}$$

Take moments about ankle since weight of body and compression force due to acceleration acts through there.

$$\sum M_{ankle} = T(2.5) - 370 \cdot 6 = 0$$

$$\therefore T = 888 \text{ lbf}$$

a) Assumes:
- total conservation of forward kinetic energy to elevation
- neglects mass of pole
- no pushing on pole in air
- no addition of forward kinetic energy added by pushoff.

Forward kinetic energy (KE) $= \frac{W}{2g} V_a^2$

KE associated with pushoff can be calculated by

$$KE_{vert} = \frac{W}{g} \cdot g\, h_{vert} = WC\left[\frac{F_o}{W} - 1\right]$$

$$\therefore \text{Total KE} = W\left[\frac{V_a^2}{2g} + C\left(\frac{F_o}{W} - 1\right)\right]$$

This gives a total height H of

$$\frac{W}{g} gH = W\left[\frac{V_a^2}{2g} + C\left(\frac{F_o}{W} - 1\right)\right]$$

by conservation of energy

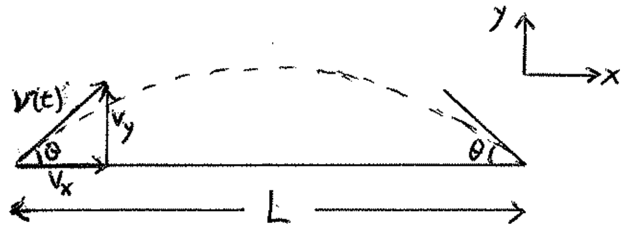
$$\therefore H = \frac{V_a^2}{2g} + C\left(\frac{F_o}{W} - 1\right)$$

b) Given: $V_a = 9\ m/s$, $F_o/W = 2$, $C = 0.25\,m$

Then $H = \frac{(9\ m/s)^2}{2(9.81\ m/s^2)} + (0.25m)(2-1) = 4.38m$

If centre of gravity starts 30 cm from the floor, the total height of her centre of gravity is

$$4.38m + 0.3m = 4.68m$$



$V_x = V(t)\cos\theta = V\cos\theta = \text{const.}$

$V_y = V(t)\sin\theta$

$$\frac{dV_y}{dt} = -g \Rightarrow V_y = -gt + c_1$$

At $t=0$, $V_y(0) = V(0)\sin\theta = V\sin\theta$

$\therefore V_y(t) = -gt + V\sin\theta$

At maximum height, $V_y(t) = 0$

$\Rightarrow 0 = -gt + V\sin\theta \quad \Rightarrow t = V/g \sin\theta$

$\therefore$ Total time in air is

$$T = 2 \cdot \frac{V}{g}\sin\theta$$

$\therefore$ Total distance travelled is

$$L = (V\cos\theta)T = \frac{2V^2}{g}\cos\theta\sin\theta = \frac{V^2}{g}\sin 2\theta$$

For maximum L, set $\frac{dL}{d\theta} = 0$

$$\Rightarrow \frac{dL}{d\theta} = \frac{2V^2}{g}\cos 2\theta = 0 \quad \Rightarrow \theta = 45^\circ$$

$$\therefore L_{V=10\,m/s} = \frac{V^2}{g}\sin 90^\circ = \frac{(10\,m/s)^2}{(9.81\,m/s^2)} = 10.2\,m$$

In air:

For 3 full rotations, $\Delta\theta = 6\pi$, $\Delta t = 0.95\,s$

$$\therefore \omega_{air} = \frac{\Delta\theta}{\Delta t} = const \quad , \text{ since } \alpha = 0 \text{ rad/s}^2 \text{ in air}$$

$$= \frac{6\pi}{0.95\,s} = 19.84 \text{ rad/s}$$

For set-up:

Starting : $\omega = 0$ rad/s

ending : $\omega = 19.84$ rad/s

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{19.84 \text{ rad/s}}{0.35\,s} = 56.69 \text{ rad/s}^2$$

$$I_G = m k_G^2 = (70\,kg)(18\times10^{-2}\,m)^2 = 2.268\ kg\,m^2$$

$$\therefore M = I_G \alpha = (2.268\ kg\cdot m^2)(56.69 \text{ rad/s}^2) = 128.6\ N\cdot m$$

a) $\overset{+}{\rightarrow} \sum F_x = m a_x = m \frac{dV_x}{dt}$

$$\Rightarrow mV_{x_2} - mV_{x_1} = \int_{t_1}^{t_2} F_x dt = -\frac{1}{2}(200N)(0.4s) = -40 N \cdot s$$

$$\therefore V_{x_2} = V_{x_1} - \frac{40 N \cdot S}{m} = 2 m/s - \frac{40 N \cdot S}{65 kg}$$

$$\therefore V_{x_2} = 1.385 m/s$$

b) We will neglect kinetic energy associated with motion of legs relative to the centre of gravity. Then there is an interchange between potential and kinetic energy.

$$\therefore \frac{1}{2} m V_{x_1}^2 + mgh_1 = \frac{1}{2} m V_{x_2}^2 + mgh_2$$

Note: y-component velocity is neglected for kinetic energy.

$$\therefore h_2 - h_1 = \frac{V_{x_1}^2 - V_{x_2}^2}{2g} = 10.6 cm$$

From Table 10-2, mass of total leg is

$$m = (0.161)(60\,kg) = 9.66\,kg$$

From Figure 10-31, F_x and F_y can be approximated:

$$F_x = -\frac{2.5}{13} \times 200\,N = -38.5\,N$$

$$F_y = \frac{43}{62} \times (60\,kg)(9.81\,m/s^2) = 408\,N$$

$$\overset{+}{\rightarrow} \sum F_x = m a_x$$

$$-38.5\,N + R_x = (9.66\,kg)(-0.25\,m/s^2)$$

$$\therefore R_x = 36.1\,N$$

$$+\uparrow \sum F_y = m a_y$$

$$408\,N + R_y - (9.66\,kg)(9.81\,m/s^2) = (9.66\,kg)(-0.75\,m/s^2)$$

$$\therefore R_y = -320.5\,N$$

From Figure 10-21, length of the total arm is

$$L = (0.818 - 0.377)H$$
$$= 0.441 \times 1.8\,m = 0.794\,m$$

From Table 10-2:

mass of the total arm, $m = 0.05 \times 70\,kg = 3.5\,kg$

radius of gyration, $k_A = 0.645 \times 0.794\,m = 0.512\,m$ (about proximal end)

$$\therefore\ I_A = m k_A^2 = (3.5\,kg)(0.512\,m)^2 = 0.918\,kg\,m^2$$
$$= 0.918\,N \cdot m \cdot s^2$$

$$\Sigma M_A = I_A \alpha$$

$$\therefore\ 10\,N \cdot m = 0.918\,N \cdot m \cdot s^2 \cdot \alpha$$
$$\Rightarrow\ \alpha = 10.89\,rad/s^2$$

Now, $\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$

$$= 2(10.89\,rad/s^2)\left(80^\circ \cdot \frac{\pi}{180^\circ}\,rad\right)$$
$$= 30.41\,rad^2/s^2$$

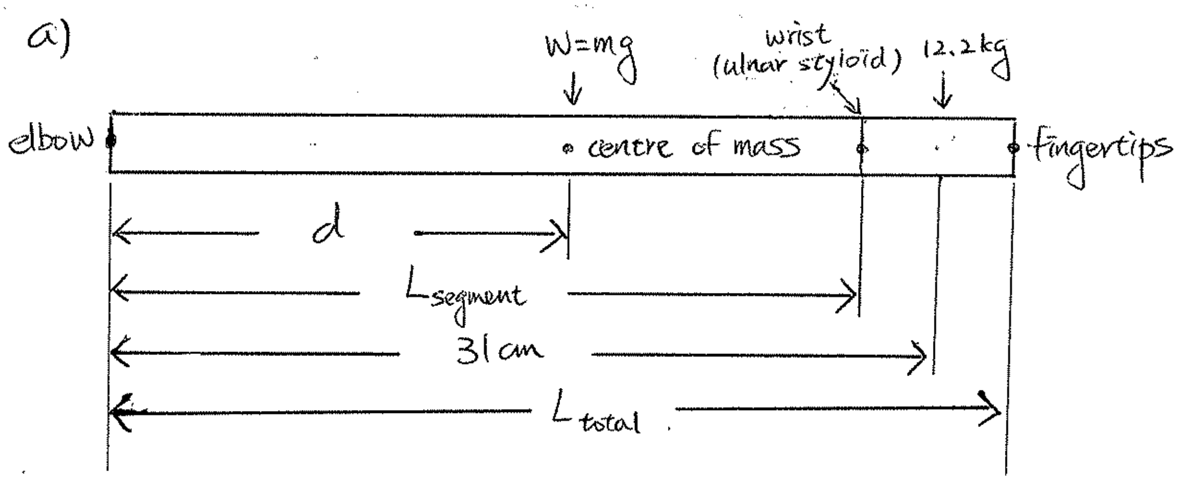
Given $\omega_0 = 0$ $\Rightarrow$ $\omega = 5.51\,rad/s$

Then the speed of the hand is

$$V = L\omega = (0.794\,m)(5.51\,rad/s) = 4.37\,m/s$$

Note that this solution gives the speed of the fingertips, since the anthropometric data for arm length (Figure 10-21) include the length of the entire hand. The linear speed of the the palm will be slightly less than that of the fingertips.

a)



From Figure 10-21:

$$L_{total} = (0.630 - 0.377)(183\text{ cm}) = 46.3\text{ cm}$$

$$L_{segment} = (0.630 - 0.485)(183\text{ cm}) = 26.5\text{ cm}$$ (elbow axis to ulnar styloid)

From Table 10-2 for forearm and hand:

$$d = 0.682\,L_{segment} = 18.1\text{ cm from elbow}$$

$$m = 0.022\,M = (0.022)(80\text{ kg}) = 1.76\text{ kg}$$

$$k = 0.827\,L_{segment} = 21.9\text{ cm about elbow}$$

$$\therefore\ I = mk^2 = (1.76\text{ kg})(21.9\text{ cm})^2 = 844\text{ kg}\cdot\text{cm}^2$$

$$+\circlearrowleft\ \Sigma M_{elbow} = I\alpha$$

$$\therefore (1.76\text{ kg})(981\text{ cm/s}^2)(18.1\text{ cm}) + (12.2\text{ kg})(981\text{ cm/s}^2)(31\text{ cm})$$

$$= (844\text{ kg cm}^2)\,\alpha$$

$$\Rightarrow \alpha = 476.6\text{ rad/s}^2$$

b) without forearm weight:

$$\alpha = \frac{(12.2\text{ kg})(981\text{ cm/s}^2)(31\text{ cm})}{844\text{ kg}\cdot\text{cm}^2} = 439.6\text{ rad/s}^2$$

$$\therefore \text{the difference is } \frac{439.6 - 476.6}{476.6} = -7.8\%$$

a) Using the scale bar, the vertical displacement of centre of mass from image 1 to image 6 is about 0.96 m

Assuming no energy loss, the time between image 1 to image 6 is

$$t = \left(2 \cdot 0.96\,m \cdot \frac{1}{9.8\,m/s^2}\right)^{1/2} = 0.443\,s = 443\,msec$$

And there are 5 time intervals,

$\therefore$ the flash interval is $\frac{443\,msec}{5} \approx 90\,msec.$

b) let V_{x_1} = initial horizontal velocity = 0 m/s ($t=0$)

V_{x_2} = horizontal velocity at $t = 400$ msec

V_x = horizontal velocity at $t \geq 400$ msec

Assuming no energy loss, V_x should be constant and equal to V_{x_2} throughout the motion.

From the image, the horizontal displacement of centre of mass between image 1 to image 6 is about 0.68 m.

$\therefore 0.68\,m = V_x \cdot \Delta t = V_x \cdot 0.443\,s$

$\therefore V_x = V_{x_2} = 1.53\,m/s$

Then, acceleration, a_x, during pushoff is

$$a_x = \frac{V_{x_2} - V_{x_1}}{400\,msec} = \frac{1.53\,m/s}{0.4\,s} = 3.83\,m/s^2$$

$\therefore$ The constant horizontal force is

$$F_x = m \cdot a_x = 82.2\,kg \cdot 3.83\,m/s^2 = 314.83\,N$$

c) let ω_1 = initial angular velocity at $t=0$ $= 0$ rad/s

ω_2 = angular velocity at $t = 400$ msec

ω = angular velocity at $t \geq 400$ msec

Assuming no energy loss, ω should be constant and equal to ω_2 throughout the motion.

$$\therefore \omega = \left(340^\circ \cdot \frac{\pi}{180^\circ}\right) \times \frac{1}{13 \times 0.09\,s} = 5.07 \text{ rad/s}$$

(since there are 13 intervals and 90 msec per interval)

$\therefore$ the angular acceleration, α, during pushoff is

$$\alpha = \frac{\omega_2 - \omega_1}{\Delta t} = \frac{5.07 \text{ rad/s}}{0.4\,s} = 12.68 \text{ rad/s}^2$$

Now, we know the average moment during pushoff is 100 Nm, and $M = I\alpha$

$$\therefore \quad I = \frac{M}{\alpha} = \frac{100 \text{ Nm}}{12.68 \text{ rad/s}^2} = 7.89 \text{ Nms}^2$$

Also, $I = mk^2$,

$\therefore$ the average radius of gyration is

$$k = \left(\frac{I}{m}\right)^{1/2} = \left(\frac{7.89 \text{ N.m.s}^2}{82.2 \text{ kg}}\right)^{1/2} = 0.31 \text{ m}$$

Using equation 10.24, the angular acceleration at frame 5 is

$$\alpha_5 = \frac{\theta_4 - 2\theta_5 + \theta_8}{\Delta t^2} = \frac{(-0.349) - 2(-0.070) + (0.122)}{(0.0417s)^2}$$

$$= -50 \quad rad/s^2$$

Note: - θ's are in radians
- Δt = time increment between frames = 1/24 = 0.0417s

Also, the mass moment of inertia is

$$I = mk^2$$
$$= (12.08\,kg)(0.53\,m)^2$$
$$= 3.39 \quad kg \cdot m^2$$

Note: According to Table 10-2 and Figure 10-21, for total leg
$m = 75\,kg \times 0.161 = 12.08\,kg$
$k = (0.530\,H)(0.560) = (0.530 \times 1.8\,m)(0.560) = 0.53\,m$

∴ the moment extered on the swing leg at the hip is

$$M = I\alpha$$
$$= 3.39\,kg \cdot m^2 \cdot (-50 \quad rad/s^2)$$
$$= -169.5\ Nm$$