Hybrid Systems
Modeling, Analysis and Control

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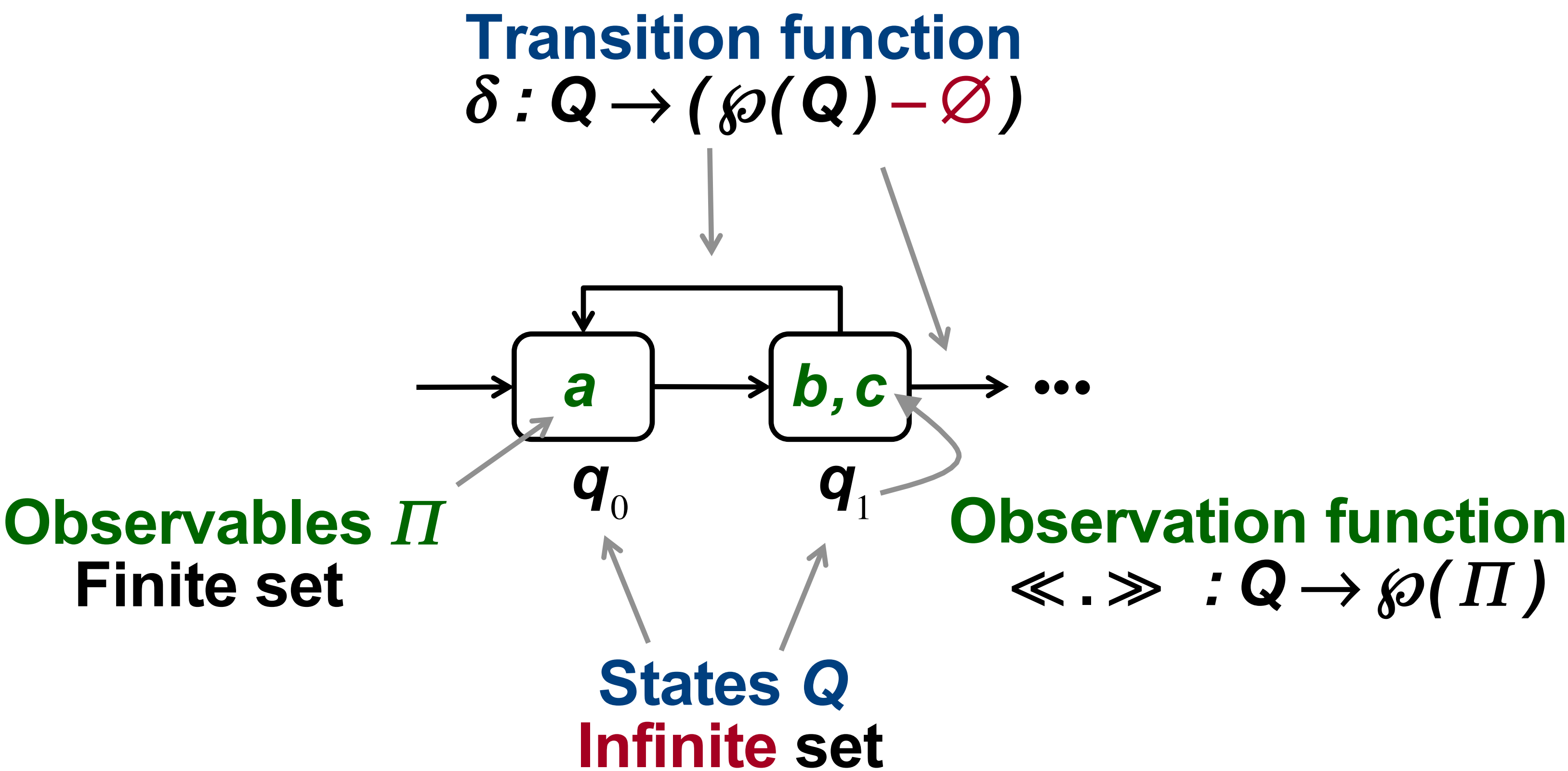
Lecture 7

(Infinite) Transition Structures



Transition Structures $\mathcal{K} = (Q, Q_0, \Pi, \ll . \gg, \delta)$

- **States** Q: A possibly infinite set
- **Initial states** Q_0: Nonempty subset of Q
- **Observables** Π: A finite set (alphabet) of propositions
- **Transition relation** δ: Binary relation over Q
- **Observation function** $\ll . \gg : Q \to \wp(\Pi)$

Observation function $\ll . \gg : Q \to \wp(\Pi)$ is also written as

Satisfaction relation $\models \; \subseteq \; Q \times \Pi$ s.t. $q \models \pi$ iff $\pi \in \ll q \gg$

∞ Transition Structure $\mathcal{K}_H = (Q, \Pi, \ll . \gg, \delta)$

States Q: Pairs (m,x) consisting of

Discrete modes: $m \in \Pi$

Continuous variables: $x \in \mathbb{R}^n$

Observables Π: $\ll m,x \gg \; = \; m$

Transition relation δ: $(m',x') \in \delta(m,x)$ iff

Discrete switches: $\exists e \in E.\ a \in L.$

$$e = (m, m') \ \wedge \ \text{label}(e) = a$$

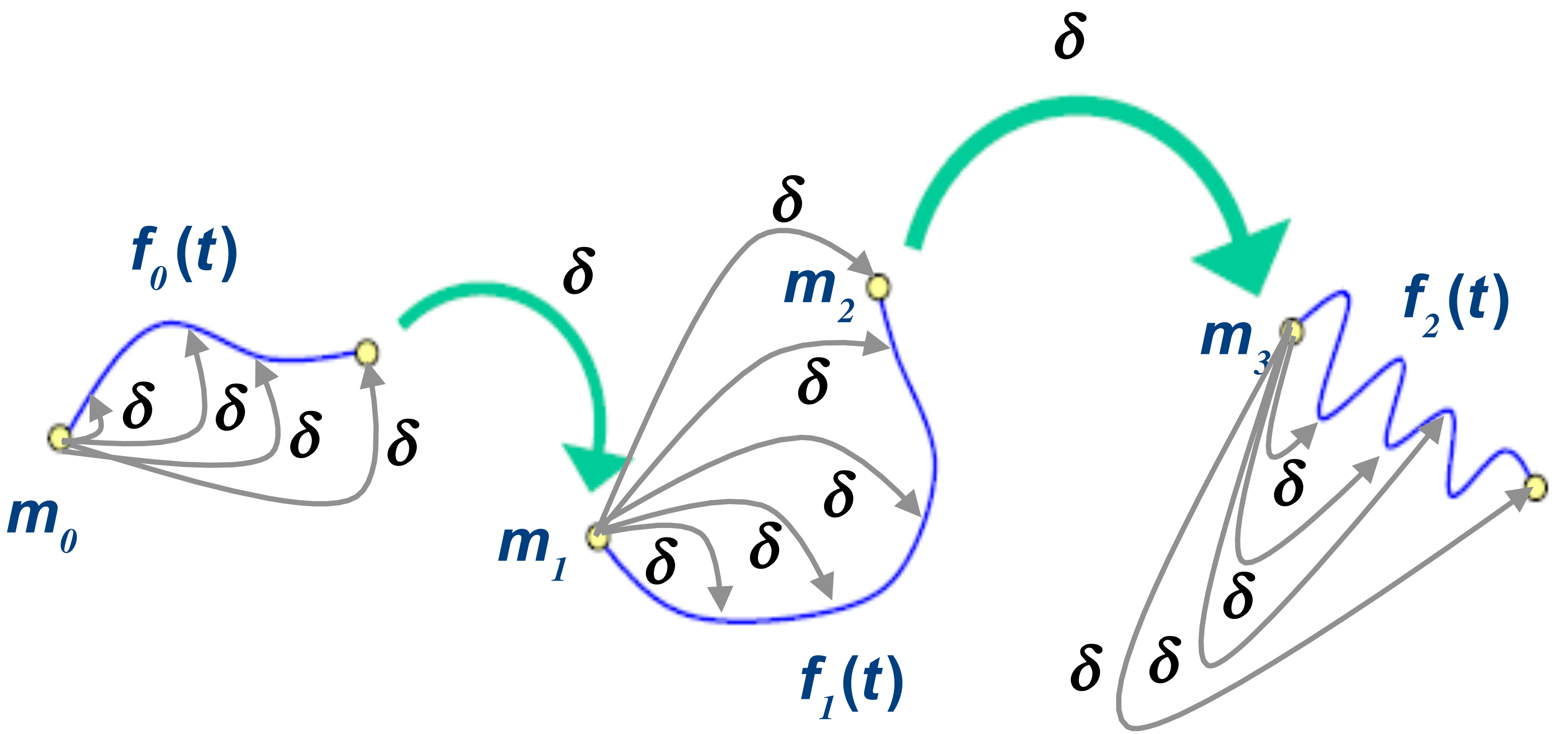
$$\text{guard}(e)(x) \geq 0 \ \wedge \ \text{action}(x, x')$$

Continuous flow: $\exists\, v \in V.\ f : [0, \mathrm{T}] \to \mathbb{R}.$

$$v = m = m' \ \wedge \ f(0) = x \ \wedge \ f(\mathrm{T}) = x'$$

$$\forall 0 \leq t \leq \mathrm{T}.\ \ f(t) \in \text{inv}(v) \ \wedge \ \dot{f}(t) \in \text{flow}(v)(f(t))$$

Intuition of the Construction



Time is abstracted away

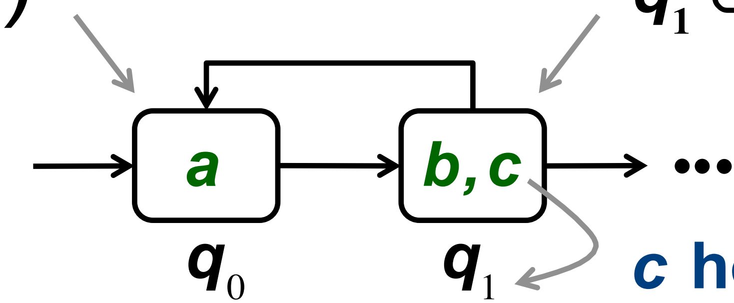
Semantics of Transition Structures

Predecessor of q_1

$q_0 \in \delta^{-1}(q_1)$

Successor of q_0

$q_1 \in \delta(q_0)$



c holds at q_1

$c \in \ll q_1 \gg$

q_0-run of $\mathcal{K}$: $r = q_0 q_1 \ldots \quad q_{i+1} \in \delta(q_i)$

Trace of r: $\ll r \gg \; = \; \ll q_0 \gg \ll q_1 \gg \ldots \; = \; \{a\}\{b,c\}\ldots$

L^q: All q-runs, $\quad \ll L^q \gg$: The associated traces

$L(\mathcal{K})$: Language of $\mathcal{K} = \bigcup_{q \in Q_0} \ll L^q \gg$

Forward Reachability Algorithm

$\textbf{Pre}(\textbf{P}) = \{q \in Q \mid \exists p \in P . \delta(q) = p\}$

$\textbf{Post}(\textbf{P}) = \{q \in Q \mid \exists p \in P . \delta(p) = q\}$

$\textbf{Pre*}(\textbf{P}) = \bigcup_{i \in \mathbb{N}} \textbf{Pre}^i(\textbf{P})$

$\textbf{Post*}(\textbf{P}) = \bigcup_{i \in \mathbb{N}} \textbf{Post}^i(\textbf{P})$

$\textbf{Reach(T)} = \textbf{Post*}(Q_0)$

Reachability problem: **Given a transition system $\mathcal{K} = (Q, Q_0, \Pi, \ll . \gg, \delta)$ and a proposition $\pi \in \Pi$, is Reach(T) $\cap [\![\pi]\!] \neq \varnothing$?**

initially $R := Q_0$;
while (true) {
  if $R \cap [\![\pi]\!]_{\mathcal{K}} \neq \varnothing$: **then return "unsafe" end if;**

  if $Post(R) \subseteq R$: **then return "safe" end if;**

  $R := R \cup Post(R)$
}

Backward Reachability Algorithm

Backward Reachability problem: Given a transition system $\mathcal{K}$ and a proposition $\pi \in \Pi$, is Pre*($[\![\pi]\!]_{\mathcal{K}}$) $\cap Q_0 \neq \emptyset$?

initially $R := [\![\pi]\!]_{\mathcal{K}}$;
while (true) {

if $R \cap Q_0 \neq \emptyset$: then return "unsafe" end if;

if Pre$(R) \subseteq R$: then return "safe" end if;

$R := R \cup$ Pre(R)

}

Quotient Structures

Atomic State-Equivalence: $p \cong^A q$ iff $\ll p \gg \; = \; \ll q \gg$

Atomic Regions A: The equivalence classes of $\cong^A$

Quotient $\mathcal{K}_{/\cong} = (Q_{/\cong}, \Pi, \ll . \gg_{/\cong}, \delta_{/\cong})$: If $\cong$ refines $\cong^A$

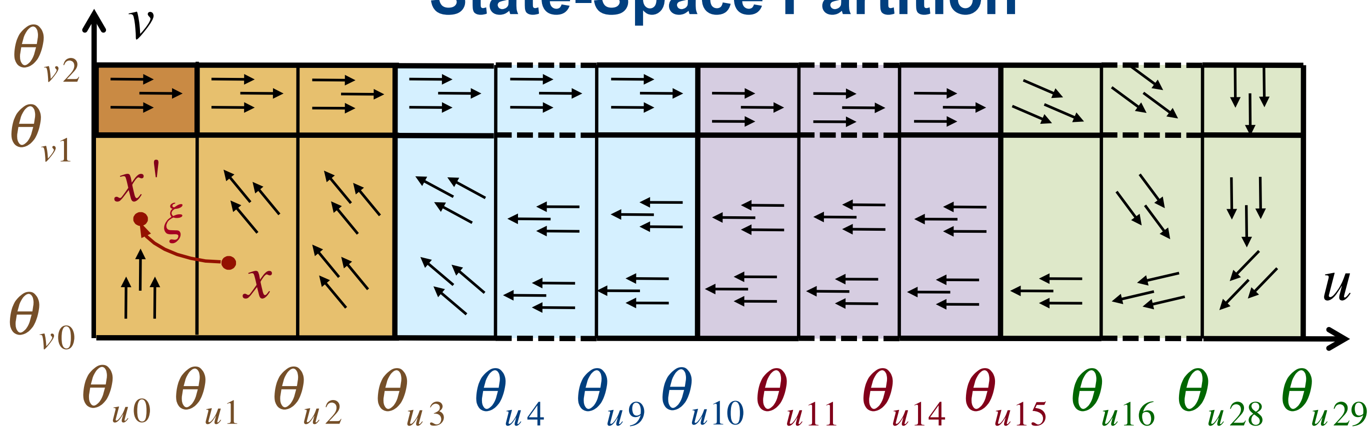
$Q_{/\cong}$ is the set of equivalence classes of $\cong$

$R \in \delta_{/\cong}(P)$ if $\exists p \in P . r \in R.\; r \in \delta(p)$

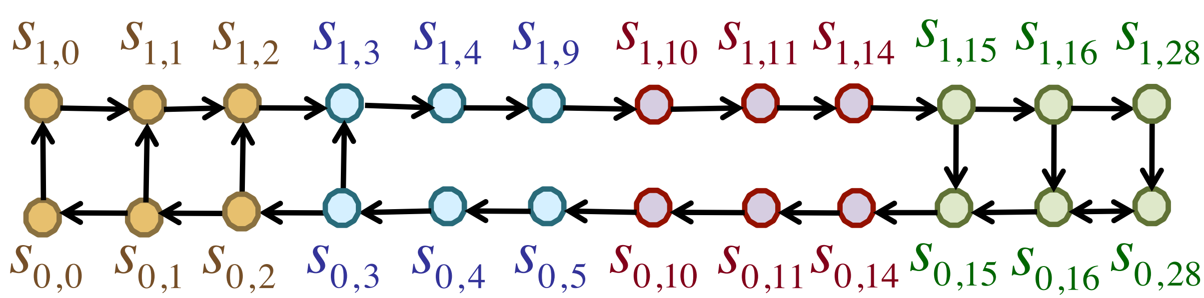
$\pi \in \; \ll R \gg_{/\cong}$ if $\exists r \in R.\; \pi \in \; \ll r \gg$

Example of Quotient Structures

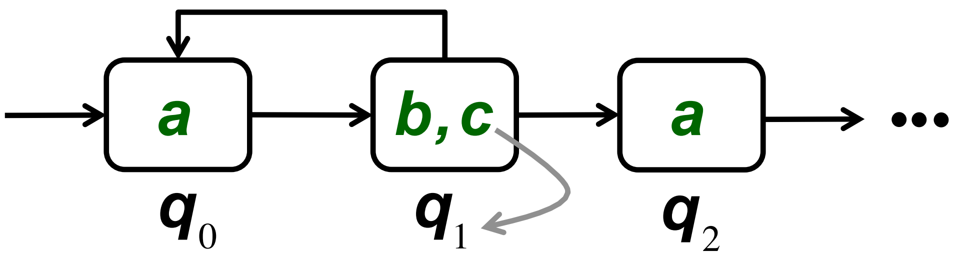
State-Space Partition



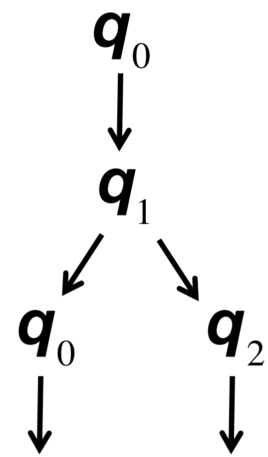
Finite Quotient Structure



Transition Structures: Semantics



$b, c \in \ll q_1 \gg$



q_0-run of $\mathcal{K}$: $r = q_0 q_1 \ldots \quad q_{i+1} \in \delta(q_i)$

Trace of r: $\ll r \gg \; = \; \ll q_0 \gg \ll q_1 \gg \ldots \; = \; \{a\}\{b, c\}\ldots$

State Logic

State logic $\mathcal{L}$: Formulas interpreted over states of $\mathcal{K}$

$[\![\varphi]\!]_{\mathcal{K}}$: **Set of states of $\mathcal{K}$ satisfying φ**

$\mathcal{L}$-MC problem: For $\varphi \in \mathcal{L} \wedge q \in Q_{\mathcal{K}}$ is $q \in [\![\varphi]\!]_{\mathcal{K}}$?

Fully Abstract Semantics

$\mathcal{L}$ - **equivalence:** $\boldsymbol{p} \cong^{\mathcal{L}} \boldsymbol{q}$ **if** $\forall \varphi \in \mathcal{L}.\ \boldsymbol{p} \in [\![\varphi]\!]_{\mathcal{K}}$ **iff** $\boldsymbol{q} \in [\![\varphi]\!]_{\mathcal{K}}$

$\cong^{\mathcal{L}}$ **is the fully abstract semantics for** $\mathcal{L}$

$\cong$ **has finite index** $\Rightarrow$ **finite-state MC**

Syntax of Computation-Tree Logic
(State Logic)

Propositions: Every proposition $\pi \in \Pi$ is a formula

Formulas: If φ_1 and φ_2 are formulas, then so are:

$\varphi_1 \vee \varphi_2$ $\quad \neg\varphi_1$ $\quad \mathbf{E}\mathcal{X}\varphi_1$ $\quad \mathbf{E}\mathcal{G}\varphi_1$ $\quad \varphi_1 \mathbf{E}\mathcal{U}\, \varphi_2$

Logic operators: $\wedge$, $\Rightarrow$, $\Leftrightarrow$ from $\vee$ and $\neg$ as usual

Temporal operators: $\mathbf{A}\mathcal{X}$, $\mathbf{A}\mathcal{G}$, $\mathbf{A}\mathcal{F}$ from $\mathbf{E}\mathcal{X}$, $\mathbf{E}\mathcal{G}$, $\mathbf{E}\mathcal{F}$:

Possibly eventually: $\mathbf{E}\mathcal{F}\varphi = \text{true}\ \mathbf{E}\mathcal{U}\, \varphi$

Inevitably next: $\mathbf{A}\mathcal{X}\varphi = \neg\mathbf{E}\mathcal{X}\neg\varphi$

Inevitably always: $\mathbf{A}\mathcal{G}\varphi = \neg\,\mathbf{E}\mathcal{F}\neg\varphi$

Inevitably eventually: $\mathbf{A}\mathcal{F}\varphi = \neg\,\mathbf{E}\mathcal{G}\neg\varphi$

Semantics of Computation-Tree Logic

Interpretation of φ: Over tree of q_0-runs on $\mathcal{K}$

Notation: $q \models_B \varphi$ where q the current node in $\mathcal{K}$

$q_0 \models_B \pi$ if $\pi \in \ll q_0 \gg$

$q_0 \models_B \varphi_1 \vee \varphi_2$ if either $q_0 \models_B \varphi_1$ or $q_0 \models_B \varphi_2$

$q_0 \models_B \neg\varphi$ if $q_0 \not\models_B \varphi$

$q_0 \models_B \mathsf{E}\mathcal{X}\varphi$ if $\exists q_0 q_1 \in \mathcal{K}.\; q_1 \models_B \varphi$

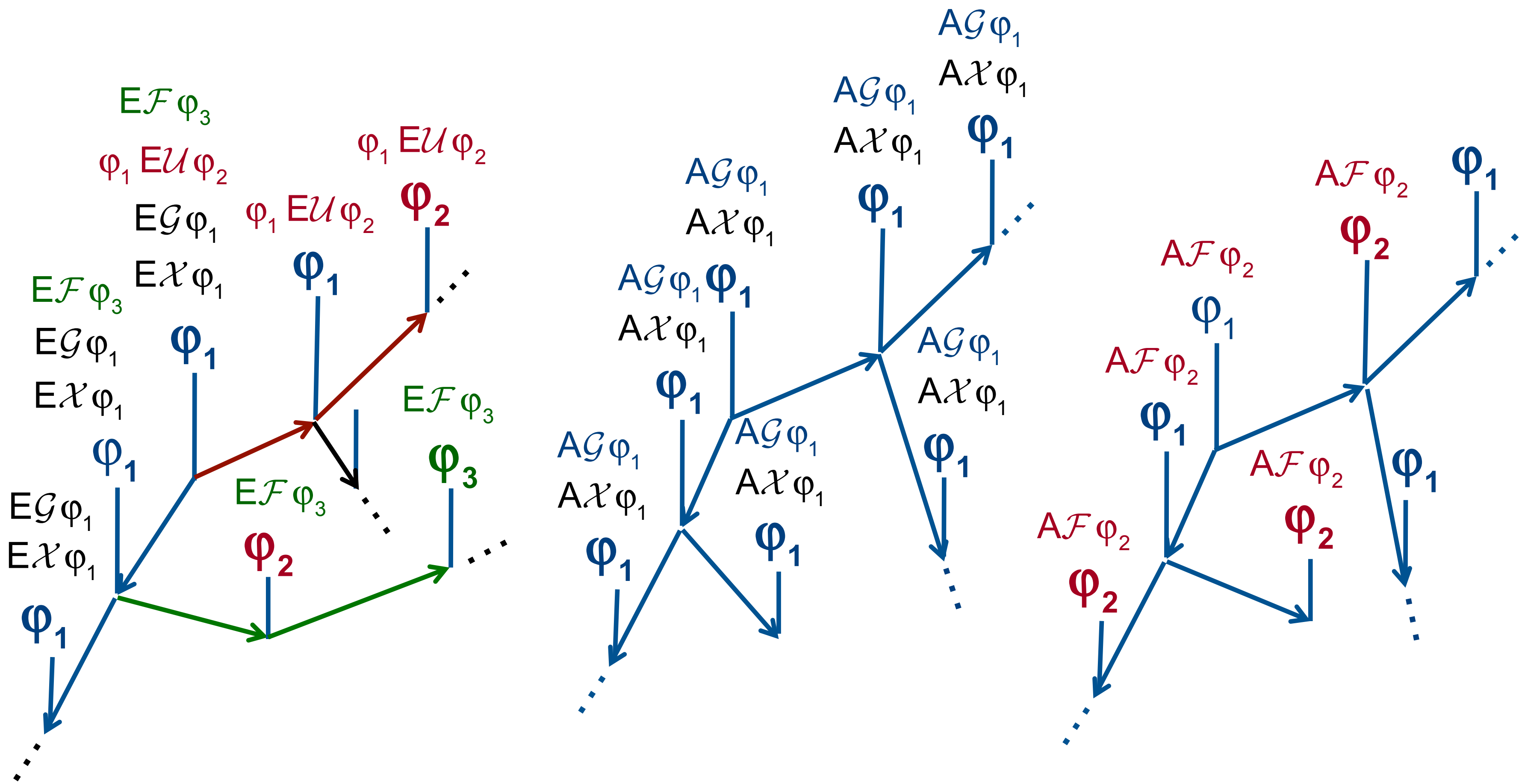
$q_0 \models_B \mathsf{E}\mathcal{G}\varphi$ if $\exists q_0 q_1 \ldots \in \mathcal{K}.\; \forall i \geq 0.\; q_i \models_B \varphi$

$q_0 \models_B \varphi_1 \mathsf{E}\mathcal{U}\, \varphi_2$ if $\exists q_0 q_1 \ldots \in \mathcal{K}, i \geq 0.\; q_i \models_B \varphi_2$ and $\forall 0 \leq j < i.\; q_j \models_B \varphi_1$

Satisfying structure: $\mathcal{K} \models_B \varphi$ if $\exists q_0 \in Q_{0,\mathcal{K}}.\; q_0 \models_L \varphi$

CTL MC: Given structure $\mathcal{K}$ and CTL formula φ does $\mathcal{K} \models_B \varphi$?

Power Series Interpretation



Bisimulation

Simulation: $p \prec_s q$ implies

(bc) $\ll p \gg \; = \; \ll q \gg$

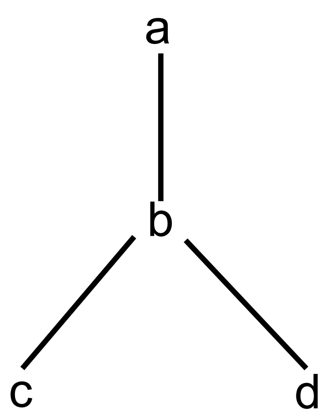
(is) $\forall p' \in \delta(p).\ \exists q' \in \delta(q).\ p' \prec_s q'$

Bisimulation $\cong^b$ **:** A symmetric simulation

Bisimilarity $\cong^B$ **:** $p \cong^B q$ if $\exists \cong^b .\ p \cong^b q$

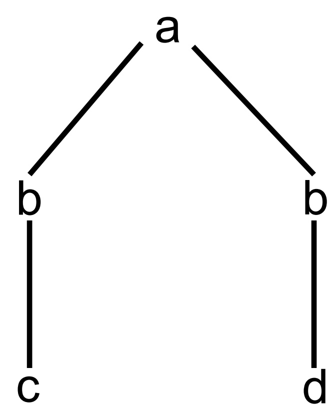
Thm: $\mathcal{K}$ satisfies an CTL formula iff $\mathcal{K}_{/\cong^B}$ satisfies it

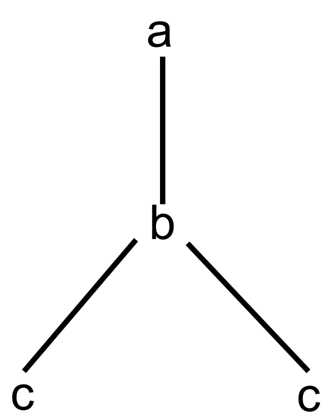
Trace Equivalence vs Bisimulation



$\cong^{\mathcal{L}}$

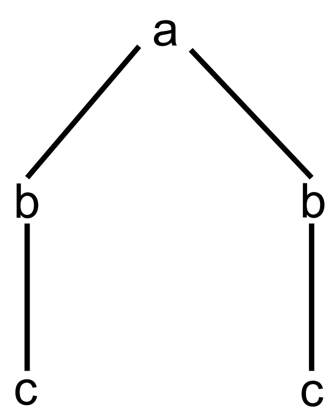
$\geq_{\boldsymbol{B}}$





$\cong^{\mathcal{L}}$

$\cong_{\boldsymbol{B}}$



Partition Refinement
Coarsest Bisimulation Algorithm

Coarsest bisimulation: **Bisimulation with fewest equivalence classes**

initially $Q_{/\cong^B} := \{ [\![\pi]\!] \mid \pi \in \Pi \}$

while $\exists P, P' \in Q_{/\cong^B}.\ (P \cap \mathrm{Pre}(P') \neq \varnothing) \wedge (P \cap \mathrm{Pre}(P') \subset P)$ **do**

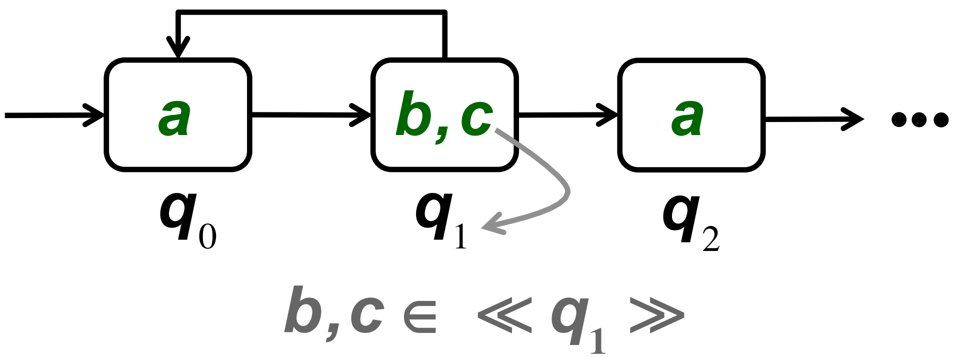
$P_1 := P \cap \mathrm{Pre}(P');\ \ P_2 := P \setminus \mathrm{Pre}(P')$

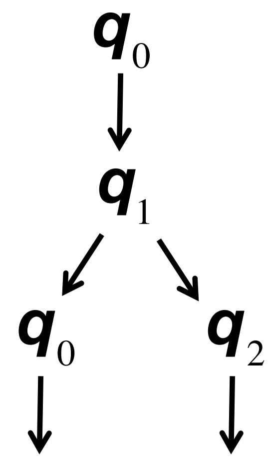
$Q_{/\cong^B} := (Q_{/\cong^B} \setminus P) \cup \{P_1, P_2\}$

end while

return $Q_{/\cong^B}$

Transition Structures: Semantics





q_0-run of $\mathcal{K}$: $r = q_0 q_1 \ldots \quad q_{i+1} \in \delta(q_i)$

Trace of r: $\ll r \gg \; = \; \ll q_0 \gg \ll q_1 \gg \ldots \; = \; \{a\}\{b, c\}\ldots$

Syntax of Linear Temporal Logic
(Trace Logic)

Propositions: Every proposition $\pi \in \Pi$ is a formula

Formulas: If φ_1 and φ_2 are formulas then so are:

$\varphi_1 \vee \varphi_2$ $\quad \neg\varphi_1$ $\quad \mathcal{X}\varphi_1$ (neXt) $\quad \varphi_1 \,\mathcal{U}\, \varphi_2$ (Until)

Logic operators: $\wedge$, $\Rightarrow$, $\Leftrightarrow$ from $\vee$ and $\neg$ as usual

Temporal operators: $\mathcal{G}$ and $\mathcal{F}$ from $\mathcal{X}$ and $\mathcal{U}$ as follows:

Finally (eventually): $\mathcal{F}\varphi = \text{true}\ \mathcal{U}\ \varphi$

Globally (always): $\mathcal{G}\varphi = \neg\ \mathcal{F}\neg\varphi$

Semantics of Linear Temporal Logic

Interpretation of φ: Over the infinite traces w in L($\mathcal{K}$)

Notation: $(w,i) \models_L \varphi$ where i the current index in w

$(w,i) \models_L \pi$ if $\pi \in w(i)$

$(w,i) \models_L \varphi_1 \vee \varphi_2$ if either $(w,i) \models_L \varphi_1$ or $(w,i) \models_L \varphi_2$

$(w,i) \models_L \neg\varphi$ if $(w,i) \not\models_L \varphi$

$(w,i) \models_L \mathcal{X}\varphi$ if $(w,i+1) \models_L \varphi$

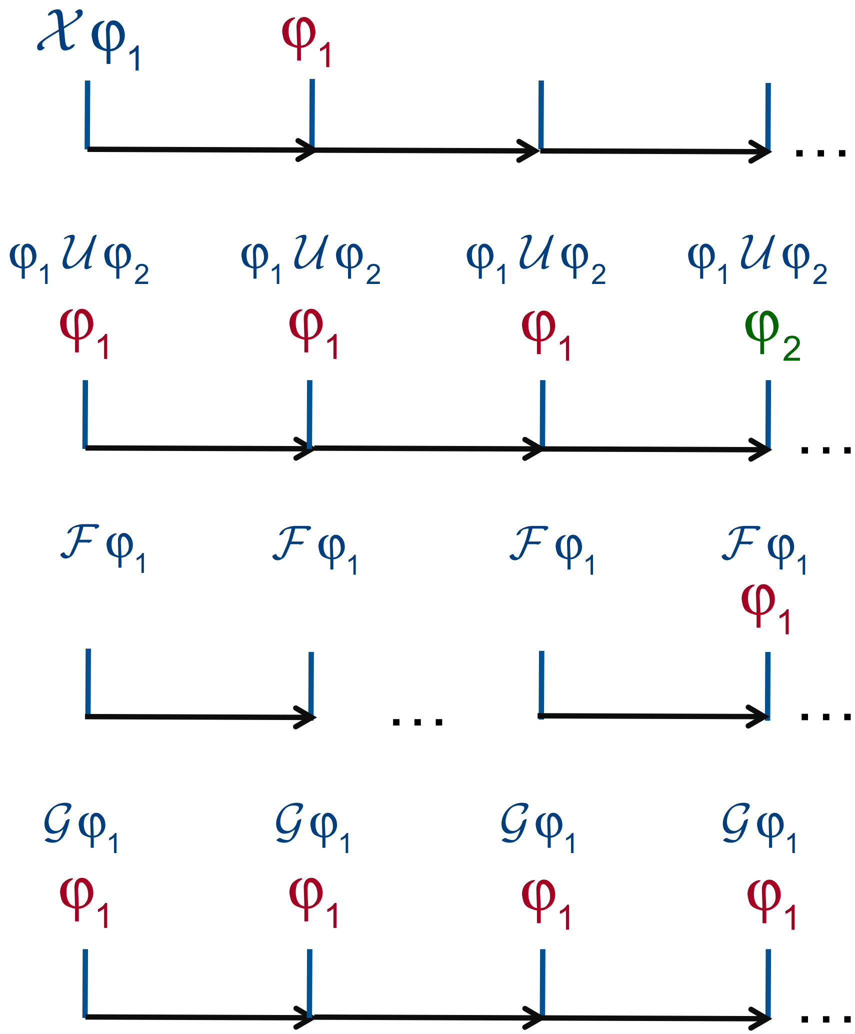
$(w,i) \models_L \varphi_1 \mathcal{U} \varphi_2$ if $\exists j \geq i.\ (w,j) \models_L \varphi_2$ and $\forall \mathrm{i} \leq \mathrm{k} < j.\ (w,k) \models_L \varphi_1$

Satisfying trace: $w \models_L \varphi$ if $(w,0) \models_L \varphi$

Satisfying structure: $\mathcal{K} \models_L \varphi$ if $\exists w \in L(\mathcal{K}).\ w \models_L \varphi$

LTL MC: Given structure $\mathcal{K}$ and LTL formula φ does $\mathcal{K} \models_L \varphi$?

Power Series Interpretation



Important LTL Formulas

$\mathcal{F}\mathcal{G}\varphi$: **Stabilization**

$\mathcal{G}\mathcal{F}\varphi$: **Oscillation**

$\mathcal{G}(\varphi \Rightarrow \mathcal{F}\psi)$: **Response**

Language (Trace) Equivalence

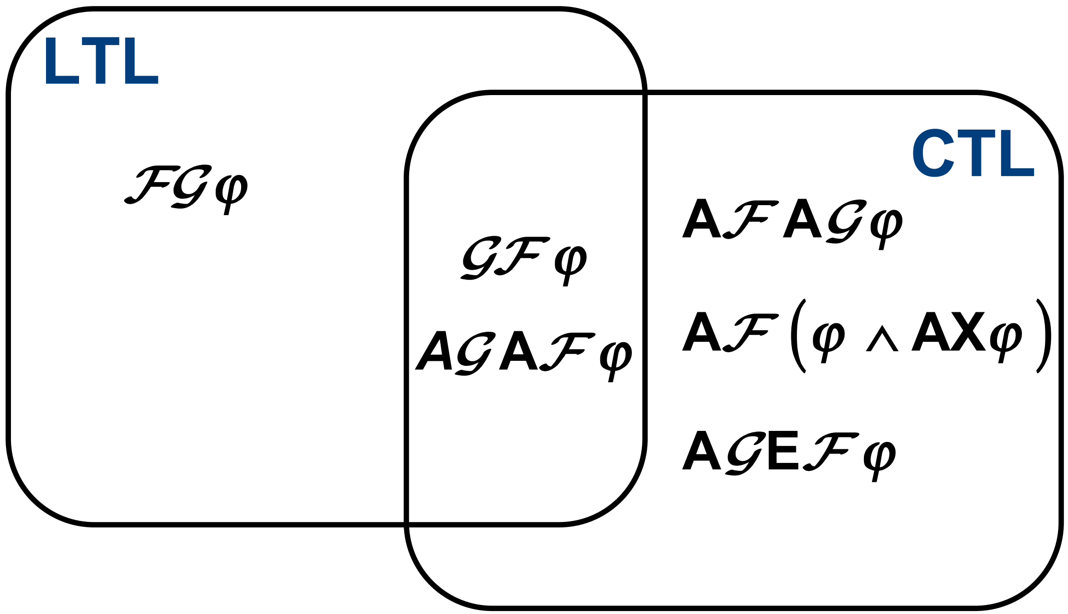
Language equivalence: Given $\mathcal{K}$, an equivalence relation $\cong_L$ on $Q_{\mathcal{K}}$ is a language equivalence of $\mathcal{K}$ if:

$$\forall p, q \in Q.\ \text{ if } \ p \cong_L q \ \text{ then } \ \ll L^p \gg \ = \ \ll L^q \gg$$

Language equivalence quotient of $\mathcal{K}$: Quotient $\mathcal{K}_{/\cong^L}$

Thm: $\mathcal{K}$ satisfies an LTL formula iff $\mathcal{K}_{/\cong^L}$ satisfies it

Expressiveness of CTL vs LTL



CTL versus LTL