Problem Set 2

Problem 2.1 Let $\mathsf{x} \sim \mathcal{U}(a,b)$ be a random variable uniformly distributed in the interval $[a,b]$. Consider the transformation $\mathsf{y} = \mathsf{x}^2$.

a) Find the pdf of y for the case $b > a > 0$.

b) Repeat a) for the case $a < 0 \wedge b > 0$.

Problem 2.2 Consider a random variable x, distributed according to the pdf

$$f_{\mathsf{x}}(x) = \begin{cases} c\left(1 - \frac{|x|}{a}\right), & -a \le x \le a \\ 0, & \text{otherwise} \end{cases},$$

where $a > 0$.

a) Specify the constant c.

b) Calculate the cdf $F_{\mathsf{x}}(x)$. Sketch the pdf and the cdf of x.

c) Find the moments $m_{\mathsf{x}}^{(k)}$ and the central moments $m_{\mathsf{x}-\mu_{\mathsf{x}}}^{(k)}$ for $k = 1, 2, 3$.

Problem 2.3 Let $\mathsf{y} = \tanh(\mathsf{x}/2)$, where $\mathsf{x} \sim \mathcal{U}(a,b)$.

a) Calculate and sketch $f_{\mathsf{y}}(y)$.

b) Calculate the mean μ_{y}.

Problem 2.4 Let $\mathsf{x} \sim \mathcal{N}(\mu_{\mathsf{x}}, \sigma_{\mathsf{x}}^2)$ be a Gaussian random variable with mean $\mu_{\mathsf{x}} = 4$ and variance $\sigma_{\mathsf{x}}^2 = 5$.

a) Calculate the probability that x is in the interval $[-2, 3]$.

b) A random variable y is obtained from x via the clipping operation $g(\cdot)$ as follows:

$$\mathsf{y} = g(\mathsf{x}) = \begin{cases} a, & \mathsf{x} \le 2 \\ \mathsf{x}, & 2 < \mathsf{x} < 6 \\ b, & \mathsf{x} \ge 6 \end{cases}.$$

Find expressions for a and b such that the mean square error $\mathrm{E}\{(\mathsf{x}-\mathsf{y})^2\}$ is minimized and evaluate these expressions numerically. Find and sketch the pdf of y.

c) The random variable y is now quantized (binned) with Q levels (bins) yielding a new random variable z, i.e.,

$$\mathsf{z} = q(\mathsf{y}) = k \qquad \text{if} \quad \mathsf{y} \in [g_{k-1}, g_k] \subset \mathbb{R},\ k = 1, 2, \ldots, Q,$$

where g_i denotes the ith quantization boundary and $g_0 = -\infty$, $g_Q = \infty$. For $Q = 3$, find the remaining g_i such that $p_{\mathsf{z}}(z) = 1/Q$.