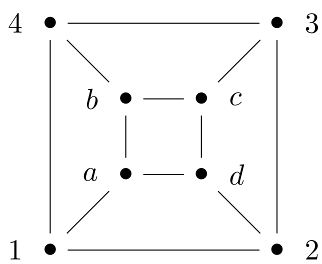
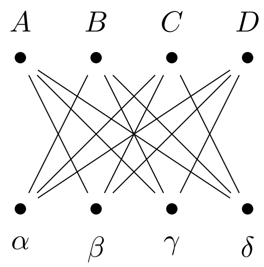
1. ÜBUNG
104.283 Diskrete Mathematik für Informatik

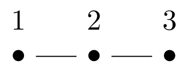
(1) A simple undirected graph is called cubic if each of its vertices has degree 3.
  (a) Find a cubic graph with 6 vertices!
  (b) Is there a cubic graph with an odd number of vertices?
  (c) Prove that for all $n \geq 2$ there exists a cubic graph with $2n$ vertices!

(2) Reformulate the following problems using a suitable graph theoretical model, and find a solution:
  (a) Show that in every city at least two of its inhabitants have the same number of neighbours! Hint: use the pigeonhole principle.
  (b) Some friends go (separately) on holidays. Each of them sends three postcards. Is it possible that everyone receives postcards from precisely those to whom he sent postcards?

(3) Are the following two graphs isomorphic?





(4) (a) Compute the number of walks of length ℓ from i to j in the graph 

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  (b) How could you use the adjacency matrix to compute the number of triangles (i.e., cycles of length three) in a (loopless) graph? Perform the computation for two graphs of your choice on four vertices.

(5) Let $G = (V, E)$ be a simple graph with at least five vertices. The complement $\bar{G}$ of G is the graph with the same vertex set as G, where two vertices are adjacent if and only if they are not adjacent in G.
  Show that at least one of G and $\bar{G}$ contains a cycle. Furthermore, characterise all trees T such that $\bar{T}$ is also a tree.

(6) Prove that the following statements are all equivalent.
   (a) G is a tree, i.e., G is connected and has no cycles.
   (b) Every two vertices of G are connected by a unique one path.
   (c) G is connected and $|V| = |E| + 1$.
   (d) G is a minimally connected graph, i.e., every edge is a bridge.
   (e) T is a maximally acyclic graph, i.e., adding any edge yields a cycle.

(7) Show that every graph with at least as many edges as vertices contains a cycle.

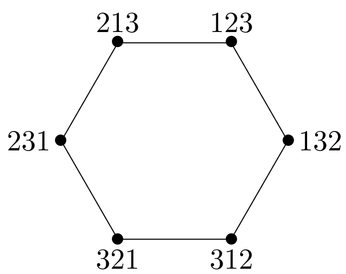
(8) Conclude from the statement

   a connected graph has an Eulerian circuit if and only if all its vertices have even degree

   that a graph contains an Eulerian trail from u to v if and only if u and v are the only vertices of odd degree of the graph.

(9) Let G be a graph with at least two vertices. Prove or disprove:
   (a) Deleting a vertex of maximal degree Δ cannot increase the average degree.
   (b) Deleting a vertex of minimal degree δ cannot decrease the average degree.

(10) Let G_n be the graph whose vertices are the permutations of $\{1, 2, \ldots, n\}$. Two vertices in G_n are adjacent if they differ by swapping two numbers next to each other:



Show that G_n is connected.