7.1.

a) Compute the number of additions and multiplications in Algorithms 4.2 and 4.3 (forward and backward substitution).

b) Show that the product $\mathbf{L}_1\mathbf{L}_2$ of two lower triangular matrices $\mathbf{L}_1, \mathbf{L}_2$ is again a lower triangular matrix. Also show that the inverse of a (invertible) lower triangular matrix is lower triangular.

Solution.

a)

Algorithm 4.2 (solve $\mathbf{Lx} = \mathbf{b}$ using *forward substitution*)
Input: $\mathbf{L} \in \mathbb{R}^{n\times n}$ *lower triangular, invertible,* $\mathbf{b} \in \mathbb{R}^n$
Output: *solution* $\mathbf{x} \in \mathbb{R}^n$ *of* $\mathbf{Lx} = \mathbf{b}$

for $j = 1{:}n$ ***do***
$$x_j := \left(b_j - \sum_{k=1}^{j-1} l_{jk}x_k\right) \Big/ l_{jj}$$
[[*convention: empty sum = 0*]]
end for

Number of additions: n loop iterations of $\max(j-2, 0)$ additions, where j is the current loop index, for a total of $\sum_{j=2}^{n} j - 2 = \frac{1}{2}(n^2 - 3n + 2)$ additions.

Number of multiplications: n loop iterations of $j-1$ multiplications, where j is the current loop index, for a total of $\sum_{j=1}^{n} j - 1 = \frac{1}{2}(n-1)n$ multiplications.

Algorithm 4.3 (solution of $\mathbf{Ux} = \mathbf{b}$ using *back substitution*)
Input: $\mathbf{U} \in \mathbb{R}^{n\times n}$ *upper triangular, invertible,* $\mathbf{b} \in \mathbb{R}^n$
Output: *solution* $\mathbf{x} \in \mathbb{R}^n$ *of* $\mathbf{Ux} = \mathbf{b}$

for $j = n{:}{-1}{:}1$ ***do***
$$x_j := \left(b_j - \sum_{k=j+1}^{n} u_{jk}x_k\right) \Big/ u_{jj}$$
end for

Number of additions: n loop iterations of $\max(n-(j+1), 0)$ additions, where j is the current loop index, for a total of $\sum_{j=1}^{n-1} n - (j+1) = \frac{1}{2}(n-3)n$ additions.

Number of multiplications: n loop iterations of $n - j$ multiplications, where j is the current loop index, for a total of $\sum_{j=1}^{n} n - j = \frac{1}{2}(n-1)n$ multiplications.

b) We show product of lower triangular matrices is triangular:

$$\text{for } j > i : (\mathbf{L}_1\mathbf{L}_2)_{ij} = \sum_{k=1}^{n} \underbrace{\mathbf{L}_{1_{ik}}}_{0 \text{ for } k>i} \underbrace{\mathbf{L}_{2_{kk}}}_{0 \text{ for } j>k} = 0$$

Since $k \geq j \implies k > i$.

We show inverse of triangular is triangular:

Let $k \in \{1, ..., n\}$ arbitrary, we consider multiplication of $\mathbf{L}$ with the k-th column of the inverse, which has to result in the k-th unit vector:

$$\begin{pmatrix} \mathbf{L}_{11} & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & & \vdots \\ \mathbf{L}_{(k-1)1} & & & 0 & & 0 \\ \mathbf{L}_{k1} & & & \mathbf{L}_{kk} & \ddots & 0 \\ \vdots & & & & & \vdots \\ \mathbf{L}_{(n-1)1} & & & & \ddots & 0 \\ \mathbf{L}_{n1} & \cdots & \cdots & \mathbf{L}_{kk} & \cdots & \mathbf{L}_{nn} \end{pmatrix} \begin{pmatrix} \mathbf{L}^{-1}_{1k} \\ \vdots \\ \mathbf{L}^{-1}_{n(k-1)} \\ \mathbf{L}^{-1}_{nk} \\ \vdots \\ \mathbf{L}^{-1}_{(n-1)k} \\ \mathbf{L}^{-1}_{nk} \end{pmatrix} = \begin{pmatrix} 0 = \mathbf{L}_{11} \cdot \mathbf{L}^{-1}_{1k} & \implies \mathbf{L}^{-1}_{1k} = 0 \\ \vdots & \implies \mathbf{L}^{-1}_{2k}, ..., \mathbf{L}^{-1}_{(k-2)k} = 0 \\ 0 = \sum_{j=1}^{k-1} \mathbf{L}_{1j} \cdot \mathbf{L}^{-1}_{jk} & \implies \mathbf{L}^{-1}_{(k-1)k} = 0 \\ 1 & \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix}$$

Therefore in the k-th column, $\mathbf{L}^{-1}_{ik} = 0$ for $k > i$. Since k was chosen arbitrarily, this means that $\mathbf{L}^{-1}$ has to be lower triangular.

□

7.2

a) Explain Crout's algorithm from Chapter 4.3.1 in the lecture notes

b) Modify the algorithm to compute a Cholesky factorization

$$\mathbf{C}^T \mathbf{C} = \mathbf{A}$$

and realize your algorithm in Matlab/Python.

Solution.

a) We want to solve a system of n^2 equations for n^2 unknowns, an in order to do so traverse these equations in a specific order ("Crout ordering"). This involves alternating between row and column traversal: first we go through the first row, then the first column, then the second row,.. This is done to allow our equations to either consist of lots of zeros and/or elements already computed in previous steps.

b) See Python.

□

7.4.

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 10^{-6} & 1 \\ 1 & 0 \end{pmatrix}$$

a) Compute by hand the LU-factorization of $\mathbf{A}$. Calculate $\mathbf{A}^{-1}$, $\mathbf{L}^{-1}$, $\mathbf{U}^{-1}$ and compute the three condition numbers $\kappa_\infty(\mathbf{A}) = \|\mathbf{A}\|_\infty \|\mathbf{A}^{-1}\|_\infty$, $\kappa_\infty(\mathbf{L}) = \|\mathbf{L}\|_\infty \|\mathbf{L}^{-1}\|_\infty$, $\kappa_\infty(\mathbf{U}) = \|\mathbf{U}\|_\infty \|\mathbf{U}^{-1}\|_\infty$. Here, $\|.\|_\infty$ is the row-sum norm

b) Repeat the calculation of a) for the matrix $\tilde{\mathbf{A}}$ that is obtained from $\mathbf{A}$ by interchanging the two rows. What do you observe?

Solution.

a)

$$\mathbf{A} = \begin{pmatrix} 10^{-6} & 1 \\ 1 & 0 \end{pmatrix} = \mathbf{LU} \overset{Crout}{=} \begin{pmatrix} 1 & 0 \\ \mathbf{L}_{21} & 1 \end{pmatrix}\begin{pmatrix} 10^{-6} & 1 \\ 0 & \mathbf{U}_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/10^{-6} & 1 \end{pmatrix}\begin{pmatrix} 10^{-6} & 1 \\ 0 & -10^{6} \end{pmatrix}$$

Solve

$$\mathbf{AA}^{-1} = I, \begin{pmatrix} 10^{-6} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{A}_{11}^{-1} \\ \mathbf{A}_{21}^{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 10^{-6} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{A}_{12}^{-1} \\ \mathbf{A}_{22}^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{A}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -10^{-6} \end{pmatrix}$$

$$\mathbf{LL}^{-1} = I, \begin{pmatrix} 1 & 0 \\ 10^{6} & 1 \end{pmatrix}\begin{pmatrix} \mathbf{L}_{11}^{-1} \\ \mathbf{L}_{21}^{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 10^{6} & 1 \end{pmatrix}\begin{pmatrix} \mathbf{L}_{12}^{-1} \\ \mathbf{L}_{22}^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{L}^{-1} = \begin{pmatrix} 1 & 0 \\ -10^{6} & 1 \end{pmatrix}$$

$$\mathbf{UU}^{-1} = I, \begin{pmatrix} 10^{-6} & 1 \\ 0 & -10^{6} \end{pmatrix}\begin{pmatrix} \mathbf{U}_{11}^{-1} \\ \mathbf{U}_{21}^{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 10^{-6} & 1 \\ 0 & -10^{6} \end{pmatrix}\begin{pmatrix} \mathbf{U}_{12}^{-1} \\ \mathbf{U}_{22}^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\mathbf{U}^{-1} = \begin{pmatrix} 10^{6} & 1 \\ 0 & -10^{-6} \end{pmatrix}$$

$$\kappa_\infty(\mathbf{A}) = \|\mathbf{A}\|_\infty \left\|\mathbf{A}^{-1}\right\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \max_i \sum_{j=1}^{n} |a_{ij}^{-1}| = (10^{-6}+1)(1+10^{-6})$$

$$\kappa_\infty(\mathbf{L}) = \|\mathbf{L}\|_\infty \left\|\mathbf{L}^{-1}\right\|_\infty = \max_i \sum_{j=1}^{n} |l_{ij}| \max_i \sum_{j=1}^{n} |l_{ij}^{-1}| = (10^{6}+1)(10^{6}+1)$$

$$\kappa_\infty(\mathbf{U}) = \|\mathbf{U}\|_\infty \left\|\mathbf{U}^{-1}\right\|_\infty = \max_i \sum_{j=1}^{n} |u_{ij}| \max_i \sum_{j=1}^{n} |u_{ij}^{-1}| = (0+10^{6})(10^{6}+1)$$

b)

$$\tilde{\mathbf{A}} = \begin{pmatrix} 1 & 0 \\ 10^{-6} & 1 \end{pmatrix} = \tilde{\mathbf{L}}\tilde{\mathbf{U}} = \begin{pmatrix} 1 & 0 \\ 10^{-6} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Same as in a) we get

$$\tilde{\mathbf{A}}^{-1} = \tilde{\mathbf{A}}$$

$$\tilde{\mathbf{L}}^{-1} = \begin{pmatrix} 1 & 0 \\ -10^{-6} & 1 \end{pmatrix}$$

$$\tilde{\mathbf{U}}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\kappa_\infty(\tilde{\mathbf{A}}) = \left\|\tilde{\mathbf{A}}\right\|_\infty \left\|\tilde{\mathbf{A}}^{-1}\right\|_\infty = \max_i \sum_{j=1}^{n} |\tilde{a}_{ij}| \max_i \sum_{j=1}^{n} |\tilde{a}_{ij}^{-1}| = (10^{-6}+1)(10^{-6}+1) \approx 1$$

$$\kappa_\infty(\tilde{\mathbf{L}}) = \left\|\tilde{\mathbf{L}}\right\|_\infty \left\|\tilde{\mathbf{L}}^{-1}\right\|_\infty = \max_i \sum_{j=1}^{n} |\tilde{l}_{ij}| \max_i \sum_{j=1}^{n} |\tilde{l}_{ij}^{-1}| = (10^{-6}+1)(10^{-6}+1) \approx 1$$

$$\kappa_\infty(\tilde{\mathbf{U}}) = \left\|\tilde{\mathbf{U}}\right\|_\infty \left\|\tilde{\mathbf{U}}^{-1}\right\|_\infty = \max_i \sum_{j=1}^{n} |\tilde{u}_{ij}| \max_i \sum_{j=1}^{n} |\tilde{u}_{ij}^{-1}| = (0+1)(0+1) = 1$$

□

7.5.

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a tridiagonalmatrix of the form

$$\mathbf{A} = \begin{pmatrix} d_1 & e_1 & & & \\ c_2 & d_2 & e_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & e_{n-1} \\ & & & c_n & d_n \end{pmatrix}$$

Assume that $\mathbf{A}$ has an LU-factorization.

a) Show: the factors $\mathbf{L}$ and $\mathbf{U}$ have the form

$$\mathbf{L} = \begin{pmatrix} 1 & & & & \\ l_2 & 1 & & & \\ & l_3 & 1 & & \\ & & \ddots & \ddots & \\ & & & l_n & 1 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} u_1 & f_1 & & & \\ & u_2 & f_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & f_{n-1} \\ & & & & u_n \end{pmatrix}$$

Hint: do the induction step of Thm. 4.17 of the notes

b) Formulate an algorithm that computes the l_i and the u_i for $i = 2, ..., n$ and realize your algorithm in Matlab/Python. Input are the vectors $\mathbf{d}, \mathbf{e}, \mathbf{c}$ (i.e., the diagonals of $\mathbf{A}$), output are the vectors $\mathbf{l}, \mathbf{u}$, and $\mathbf{f}$ (i.e., the diagonals of $\mathbf{L}$ and $\mathbf{U}$).

Solution.

a) On paper

b) see Python

□