Project and Enterprise Financing

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Lectures 6, 7, and 8: Equity Financing: Overview

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1. Equity Instruments

2. Return Calculation

3. Risk Quantification

4. Basics of Modern Portfolio Theory

5. Market Model

6. CAPM

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8. Market Efficiency and Market Anomalies

1. Equity Instruments

1.1 Equity financing of a joint stock company (AG, Aktiengesellschaft)

- Limitation of liability to the nominal (share) capital (Min: €70,000)
- Nominal (share) capital (Grundkapital): Stocks (=Shares)
  - ➪ easy to sell (usually by agreement and transfer)
  - ➪ easier to raise capital: unlimited number of shareholders
  - ➪ Stress ratio is solved (especially for traded stock company)

    Large long term capital requirements ⇔ Liquidity considerations of small investors
- Shareholders do not have a direct influence on the management

1.1.1 Types of shares

- **Distinction based on the level of rights:**

  Common shares (Stammaktien):

  - Equal voting rights in the annual general meeting (AGM)
  - Equal rights to receive dividends
  - Equal share of liquidation proceeds
  - Statutory subscription right (gesetzliches Bezugsrecht)
  - Most common type of shares

Preference shares (Vorzugsaktien):

- Do not have voting rights
- Take priority for the purpose of profit distribution
- Are entitled to higher dividends
- Reason: equity ↑, upkeeping of voting right structure (see e.g. Google)

- **Distinction based on transfer conditions:**

Bearer shares (Inhaberaktien): Shares made out to the bearer

- The bearer owns all rights associated with the shares
- Transfer: by agreement and transfer
- Advantage: easy to transfer
- Most common type

Registered shares (Namensaktien):

- Registered under the shareholder's name
- Registration in the share register (Aktienbuch) of the joint stock company
- Transfer: Documentation on the back of the security (signature of the the transferor, name of the new owner, endorsement (Indossament), adjustments in the share register)

1.1.2 Increase in share capital (Kapitalerhöhung)

❍ **Types to increase the share capital:**

- **Ordinary capital increase** (ordentliche Kapitalerhöhung):
  Issue of new shares, general case

- **Authorized capital increase** (genehmigte Kapitalerhöhung):
  Special case of the ordinary capital increase, the annual general meeting approves in advance the capital increase; the actual execution only takes place at a later date (e.g. when a higher agio can be achieved).

- **Contingent capital increase** (bedingte Kapitalerhöhung):
  Execution upon the occurrence of certain conditions (e.g. convertible bond, warrant bond).

- **Capital increase from company funds** (Kapitalerhöhung aus Gesellschafts-mitteln):

  No liquidity inflow.

  Increase of share capital by converting reserves into share capital.

❍ **Seasoned Equity Offering** (Ordentliche Kapitalerhöhung)

Reasons for a capital increase:

(i) Capacity expansions,

(ii) Large restructurings

(iii) Interest in other firms

(iv) Capital structure improvements

Placement with public investors: Typically a third-party issue (a banking syndicate underwrites the issue)

Advantage:

- consulting capacity
- distribution network
- Risk ↓

2. Return calculation

2.1 Discrete Returns

A Return is defined as the relative price change of an asset between two points in time, i.e. between *t-1* and *t.* The period between *t-1* and *t* is very often one day, but it can also be any other period, e.g. a week, a month, a year, or periods smaller than one day, e.g. 10 minutes. The length of the period depends on the purpose. Smaller periods have the advantage that the number of return periods is larger and that certain statistical statements are more reliable. The return for one period (from *t-1* until *t*) is defined as:

$$R_{t-1,t} = \frac{P_t - P_{t-1}}{P_{t-1}}$$

P_t: Price at time *t* (e.g. the closing price on day *t*)

P_{t-1}: Price at time *t-1* (e.g. the closing price on day *t-1*)

Example

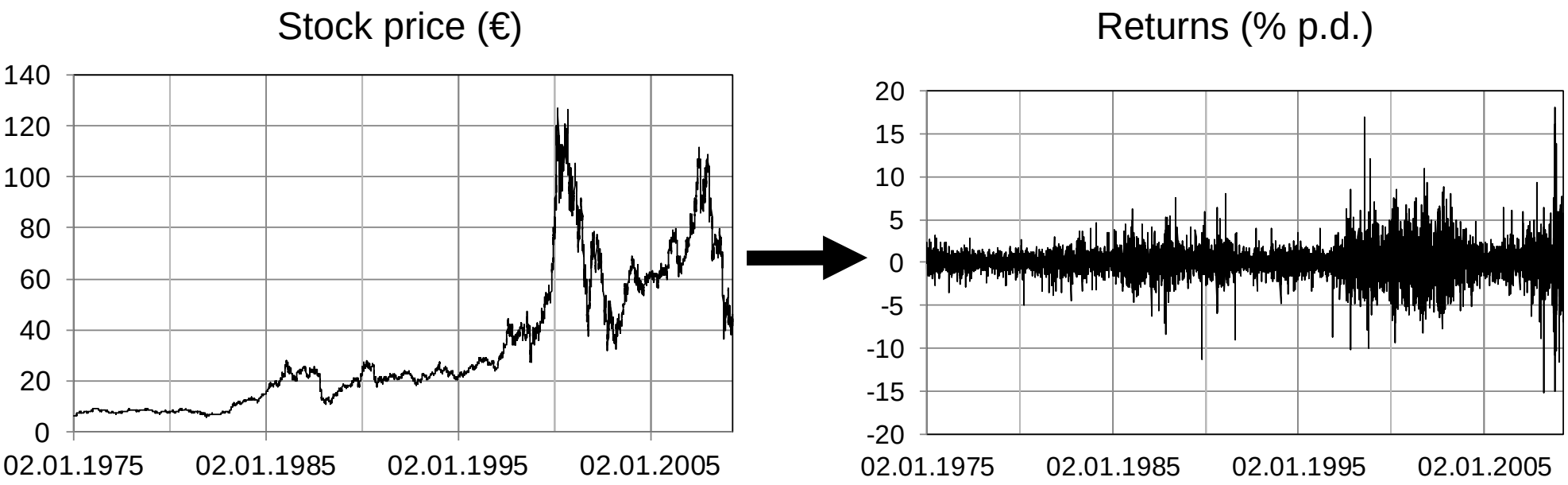
Return calculation for Siemens equity prices:

Trading days	Closing price	Return (%) discrete
13.01.2014	98.52	
14.01.2014	98.73	0.21315
15.01.2014	101.35	2.65370
16.01.2014	100.00	-1.33202
17.01.2014	100.55	0.55000

E.g., the return for the 15.1.2014 can be calculated as:

$$R = \frac{101.35 - 98.73}{98.73} = 2.6537\%$$

If the return calculation is performed for many intervals, a so-called return time series appears. The following figure reveals the time series of daily returns for Siemens using a 34 years long period (2.1.1975 bis 31.3.2009, 8934 observations).



❍ Consideration of dividend payments

Dividends are part of the performance for the equity owner (the investor) and, thus, have to be considered in the return calculation:

$$R_{t-1,t} = \frac{P_t - P_{t-1} + D_{t-1,t}}{P_{t-1}}$$

$D_{t-1,t}$: Dividend payments (or other cash payments) between *t-1* and *t* (Assumption: reinvestment at *t*)

Example

Return calculation for Siemens equity prices. On 29.1.2014 (= ex-day) a dividend of €3 for the last business year is payed (*Dividendenabschlag*).

Trading days	Closing price	Dividend	Return (%) discret
27.01.2014	97.41		
28.01.2014	98.94		1.571
29.01.2014	94.87	3.00	-1.081
30.01.2014	94.14		-0.769
31.01.2014	93.96		-0.191

Example: Reinvestment of Dividends

Day 1: P_1 = €39
Day 2: P_2 = €40, €2.9 Dividend (ex-Day) $R_{1,2}$ = 10% ((40 – 39 + 2.90)/39)
Day 3: P_3 = €41 $R_{2,3}$ = 2.5% ((41 – 40)/40)

Implicitly: Reinvestment of the dividend at €40.

Investor with 10,000 Shares on Day 1:
Day 1: 10,000 Shares
Day 2: 10,725 Shares = 10,000 + (2.9/40)·10,000
Day 3: 10,725 Shares

Wealth development:
Day 1: 10,000 Shares x €39 = €390,000
Day 2: 10,725 Shares x €40 = €429,000 = €390,000 · (1 + **0.1**)
Day 3: 10,725 Shares x €41 = €439,725 = €390,000 · (1 + 0.1) · (1 + **0.025**)

❍ Correction for artificial price changes

The market value of equity is defined as share price times the number of shares outstanding. The stock price is changing over time due to supply and demand.

But the stock price can also change due to artificial effects. Typical examples are:

Stock split, seasoned equity offering (*Kapitalerhöhung*), capital reduction (*Kapital-erhabsetzung*)

In these cases, the number of shares outstanding is increasing or decreasing without changes in the market value of equity.

E.g., at a 10:1 stock split the number of shares outstanding increases tenfold and correspondingly reduces the share price to a tenth compared to the share price immediately before the split. This price change of -90% (e.g. from €100 to €10) is an artificial process, as investors wealth is not affected.

Thus, artificial price changes have to be considered when calculating returns! This can be done via a correction factor *f*:

$$R_{t-1,t} = \frac{P_t - f_t \cdot P_{t-1} + D_{t-1,t}}{f_t \cdot P_{t-1}}$$

f_t: Correction factor for time *t*

f = 1: Value of subscription rights = 0

f < 1: Value of subscription rights > 0

$$f_t = \frac{P_{t-1} - SR_t}{P_{t-1}}$$

SR_t: Value of subscription rights for time *t*

❍ **Subscription right** (*Bezugsrecht*)

Issue of new shares: Affects the old shareholders

- Percentage fraction ↓, lower influence, capital dilution effect (Kapitalverwässerungseffekt)
- Share price ↓, as: issue price of the new shares **<** current share price (market)

  But: share capital (number of shares outstanding) ↑

Subscription right: Protection of the old shareholders against this disadvantage

- Offert to buy new shares proportional to the # of existing (old) shares
- Ex-Day: Subscription right discount (*Bezugsrechtsabschlag*), thereafter: trading of subscription rights takes place
- Subscription right discount = computed subscription right value (SR)

Computing of the subscription right value

$$SR_t = P_{t-1} - P_{t-1,ex} = P_{t-1} - \frac{P_{t-1} \cdot old + IP \cdot new}{old + new}$$

$$= \frac{P_{t-1} \cdot new - IP \cdot new}{old + new} = \frac{P_{t-1} - IP}{\frac{old}{new} + 1}$$

old: Number of shares an investor has to have to buy "new" new shares.

P_t: Closing price on day t

P_{t-1}: Closing price on day t-1

IP: Issue price of the new shares

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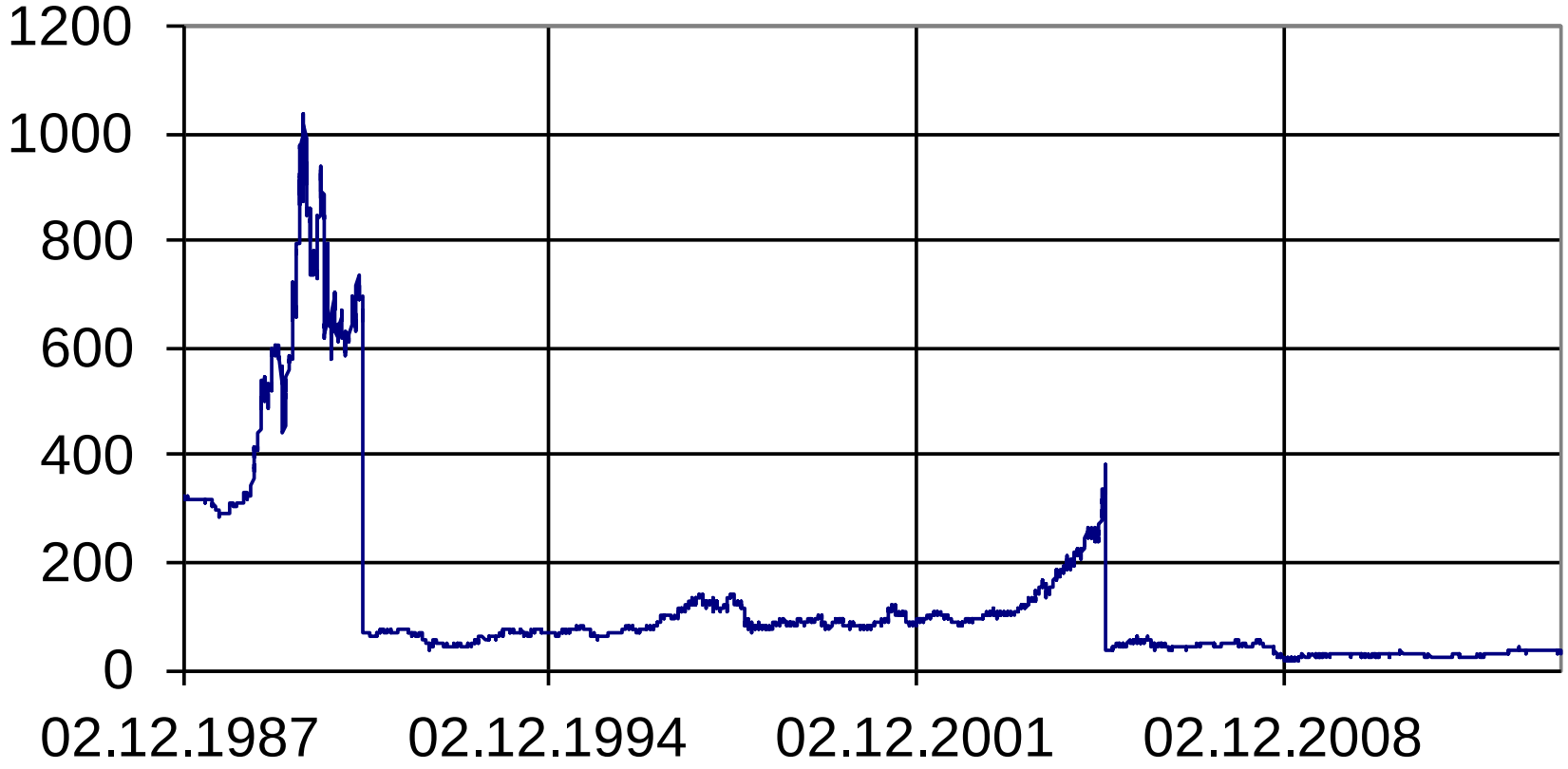
Example: OMV – Historical capital changes (2.12.1987 – 28.2.2014)

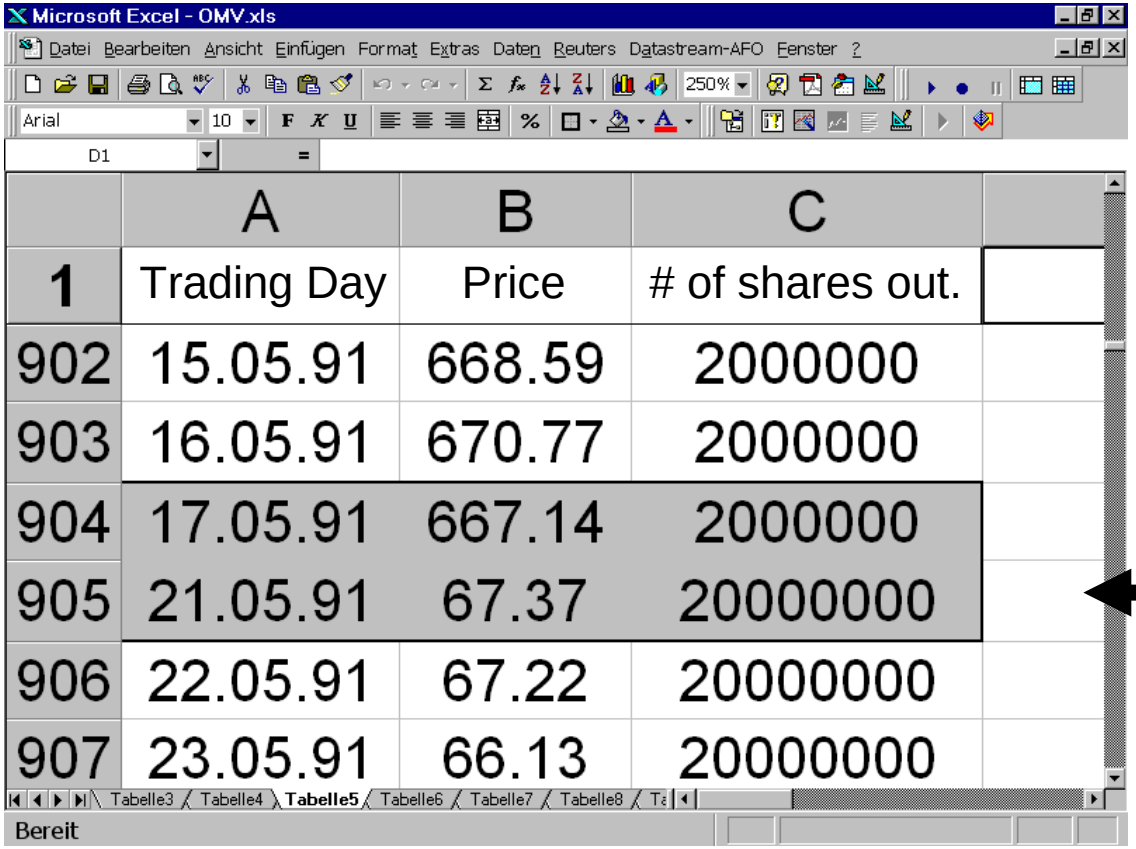
Ex-Day		# of shares Old	# of shares New	Issues price (IP)	# of shares outstanding before	# of shares outstanding after
21.04.1991	10:1 Split	1	9	0	2 Mio	20 Mio
21.05.1991	SEO	5	1	€65.4056	20 Mio	24 Mio
09.06.1994	SEO	8	1	€50.8710	24 Mio	27 Mio
02.12.2004	SEO	9	1	€219	27 Mio	30 Mio
11.07.2007	10:1 Split	1	9	0	30 Mio	300 Mio
19.05.2011	SEO	11	1	€27.50	300 Mio	327,272,727

SEO: Seasoned equity offering

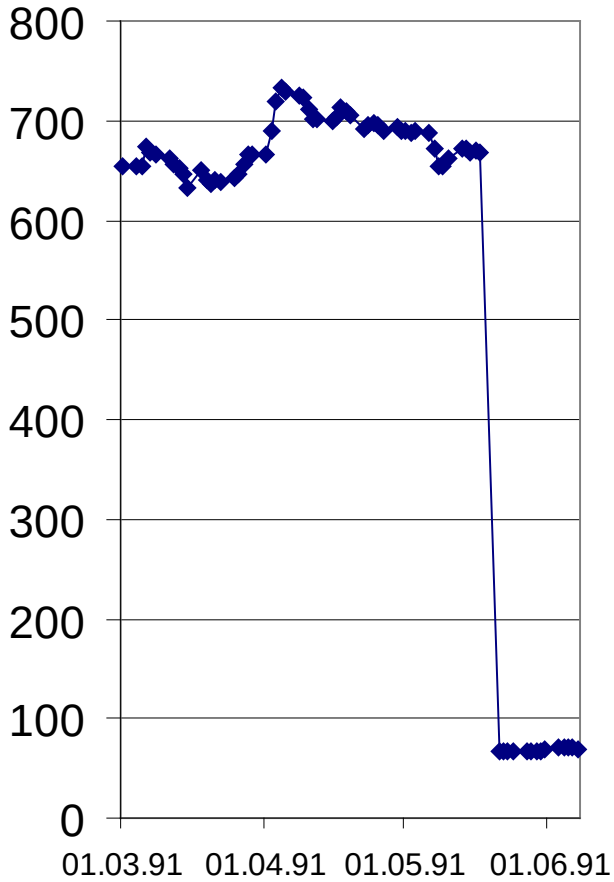
OMV – uncorrected share price (€) (2.12.1987 – 28.2.2014, 6516 observations)



(a) OMV: 10:1 Split (21.4.1991)



	A	B	C
1	Trading Day	Price	# of shares out.
902	15.05.91	668.59	2000000
903	16.05.91	670.77	2000000
904	17.05.91	667.14	2000000
905	21.05.91	67.37	20000000
906	22.05.91	67.22	20000000
907	23.05.91	66.13	20000000



$$SR_t = 667.14 - \frac{667.14 \cdot 1 + 0 \cdot 9}{9 + 1} = 667.14 - 66.71 = 600.43$$

$$f_t = \frac{667.14 - 600.43}{667.14} = 0.1$$

Microsoft Excel - OMV.xls

A900 = 15.05.1991

	A	B	C	D	E
1	Trading Day	Price	SR	f	Return
900	15.05.91	668.59	0	1	-0.4334
901	16.05.91	670.77	0	1	0.3261
902	17.05.91	667.14	0	1	-0.5412
903	21.05.91	67.37	600.4	0.1	0.9833
904	22.05.91	67.22	0	1	-0.2227
905	23.05.91	66.13	0	1	-1.6215
906	24.05.91	66.28	0	1	0.2268
907	27.05.91	66.5	0	1	0.3319

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(b) OMV: SEO 2 (9.6.1994)

8:1:€50.8710 = old:new:IP ➔ 8 subscription rights for one new share

Value of **one** subscription right (SR_t):

$$SR_t = 67.73 - \frac{67.73 \cdot 8 + (50.8710) \cdot 1}{1+8} = 67.73 - 65.86 = 1.8732$$

Correction factor:

$$f_t = \frac{67.73 - 1.8732}{67.73} = 0.9723$$

Microsoft Excel - OMV.xls

	A	B	C	D	E
1	Trading Day	Price	SR	f	Return
1695	06.06.94	64.97	0	1	-1.9765
1696	07.06.94	65.41	0	1	0.6772
1697	08.06.94	67.73	0	1	3.5469
1698	09.06.94	67.3	1.8732	0.9723	2.1915
1699	10.06.94	68.31	0	1	1.5007
1700	13.06.94	69.84	0	1	2.2398
1701	14.06.94	68.53	0	1	-1.8757
1702	15.06.94	69.11	0	1	0.8463

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❍ Dividend disadvantage

If the new shares are not (100%) entitled to dividends for the current fiscal year, the SR_t calculation has to be corrected:

$$SR_t = P_{t-1} - \frac{P_{t-1} \cdot old + new \cdot (IP + x \cdot Div)}{old + new} = \frac{P_{t-1} \cdot new - IP \cdot new - x \cdot Div \cdot new}{old + new}$$

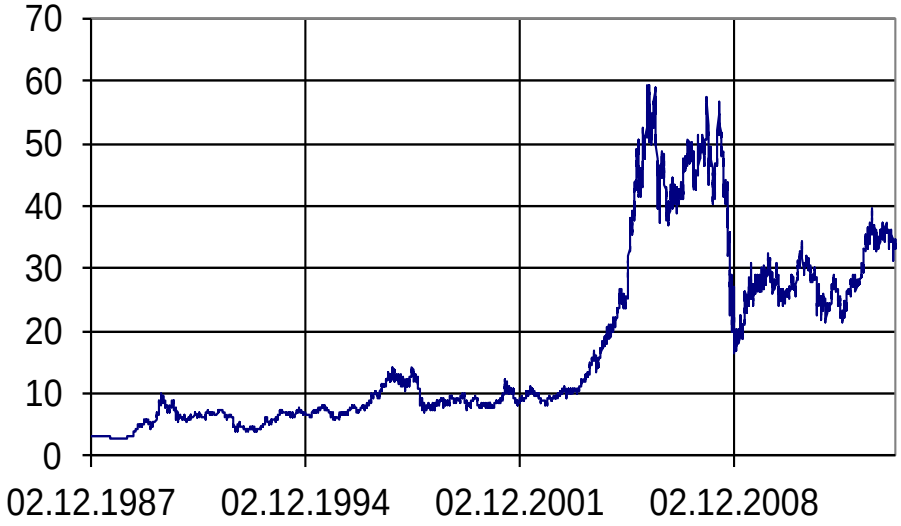
$$= \frac{P_{t-1} - IP - x \cdot Div}{\frac{old}{new} + 1}$$

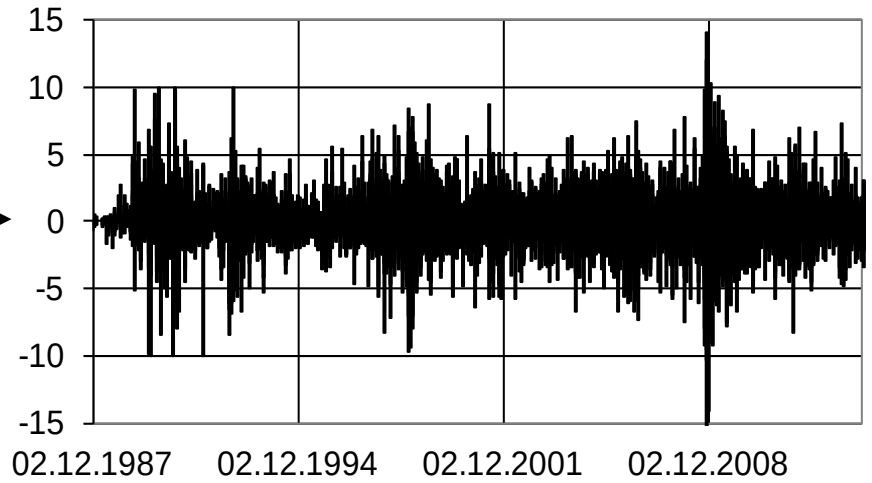
x: Dividend disadvantage = 1 - dividend entitlement of the new shares
(between 0 and 1)

Div: Dividend for the current fiscal year

IP: Issue price

OMV – corrected share price (€) and daily returns
(2.12.1987 – 28.2.2014, 6516 observations)





2.12.1987: IPO, issue price = €323.76 (ATS 4,455) ➔ 2x 10:1 Split: €3.2376

Cumulative correction factor for 4 SEOs: 0.956217

Issue date (1.12.1987):
Adjusted share price = €3.09582 (= 323.76·0.1·0.1·0.956217)

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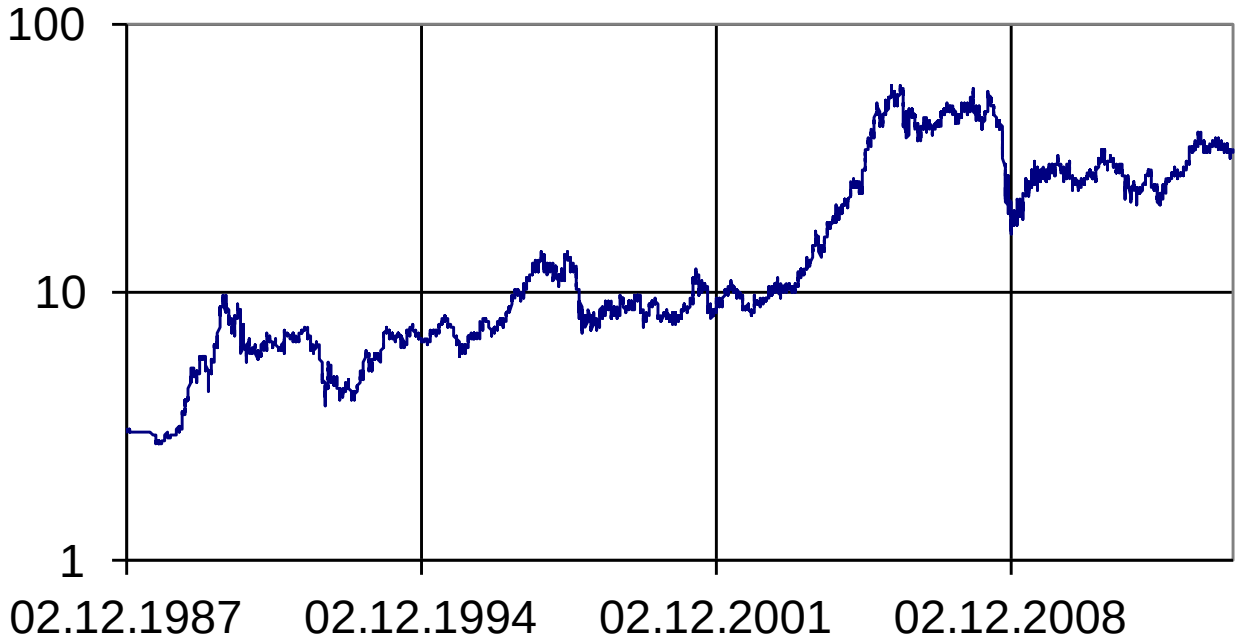
28.2.2014: Share price = €32.97 (i.e., x10.6498)

Wealth increase (26.25 years):
1.12.1987: €1,000
28.2.2014: €10,649.80
+964.98% or +9.43% p.a. (without dividends of 2-3% p.a.)

OMV – corrected share price (€) in logarithmic scalling:



2.2. Continuous Returns

- Discrete returns are not exact symmetric
- But this asymmetry is only important for really large returns

Example: Relationship between discrete and continuous returns

Lets assume we invest €100 over a period of 12 months and earn Z = €10 <u>after this 12 month</u>. Thus, the discrete return equals R = Z/100 = 10%.

If we would get instead two interest payments of €5 after 6 and 12 months the return equals:

$$R = \left(1 + \frac{\frac{Z}{2}}{100}\right)^2 - 1 = \left(1 + \frac{5}{100}\right)^2 - 1 = 10.25\% \text{ p.a.}$$

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Generally for n sub-periods or n interest payments:

$$R = \left(1 + \frac{\frac{Z}{n}}{100}\right)^n - 1$$

If the number of sub-periods becomes larger and larger (and, thus, its length smaller and smaller), then if $n \to \infty$:

$$R = \lim_{n \to \infty} \left(1 + \frac{\frac{Z}{n}}{100}\right)^n - 1 = e^{\frac{Z}{100}} - 1$$

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This implies for the example above:

$$R = \lim_{n\to\infty}\left(1+\frac{\frac{10}{n}}{100}\right)^{n} - 1 = e^{\frac{10}{100}} - 1 = e^{0.1} - 1 = 10.5171\%$$

Thus, if the payment interval becomes infinitesimal, the realized return increases from 10% to 10.5171%. This illustrates the relationship between realized (discrete) and continuously compounded return:

$$1 + R = e^{0.1}$$

$$\ln(1+R) = \ln(1+0.105171) \quad = \quad \ln\left(e^{0.1}\right) = 0.1 \text{ or } 10\% \; = \; R_{continuous}$$

A continuously compounded return of 10% p.a. corresponds to a discrete return of 10.5171% p.a.

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- Return calculation:

$$R_{t-1,t,\text{continuous}} = \ln\left(1 + R_{t-1,t}\right)$$

ln = natural logarithm

Example

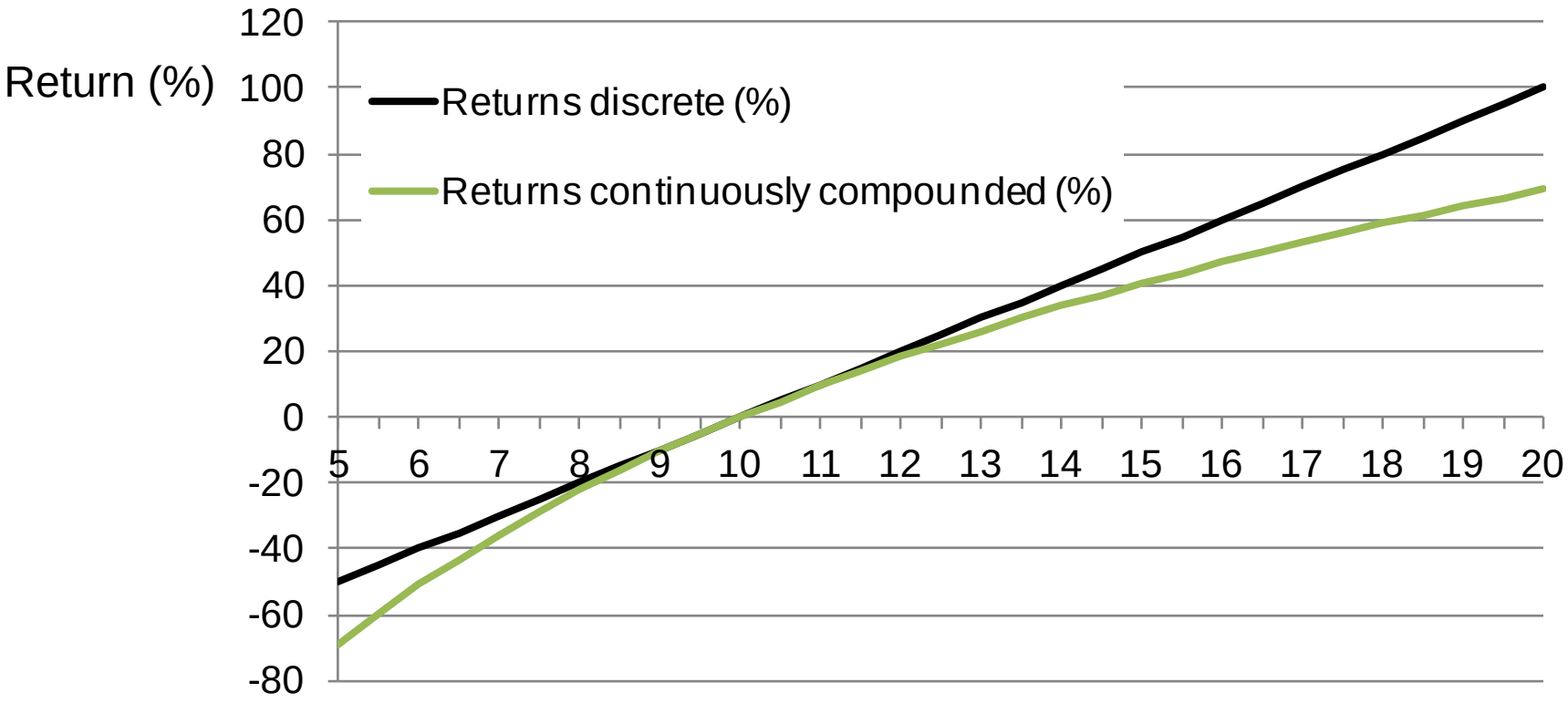
Calculation of discrete and continuous returns for Siemens:

Trading Day	Closing price (€)	Return discrete (%)	Return conti. (%)	discrete/conti.
13.01.2014	98.52			
14.01.2014	98.73	0.21315	0.21293	1.0011
15.01.2014	101.35	2.65370	2.61910	1.0132
16.01.2014	100.00	-1.33202	-1.34097	0.9933
17.01.2014	100.55	0.55000	0.54849	1.0027

E.g., continously compounded return for 15.1.2014:

$$R_{continuous} = \ln(1 + 0.0265370) = \ln\left(\frac{101.35}{98.73}\right) = 2.61910\%$$

Example: The following figure reveals that the differences between discrete and continuously compounded returns increase with the size of price fluctuations. The base price in this example is always €10.



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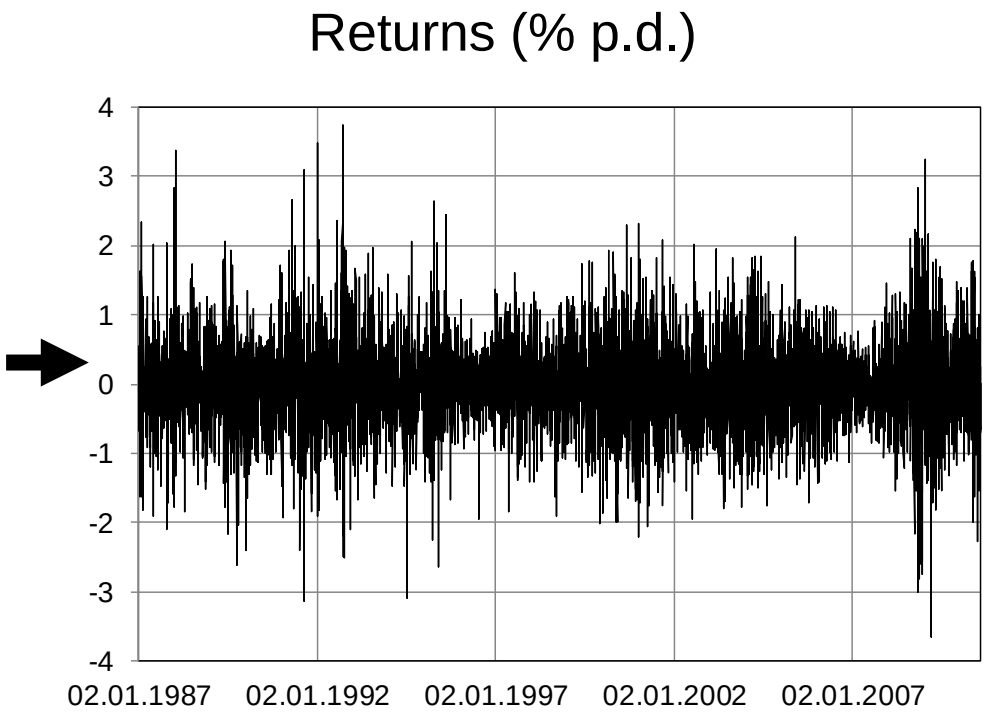
2.3 Time series of returns

- **FX Rates – USD/EUR (2.1.1987 bis 30.7.2010, 6072 observations)**



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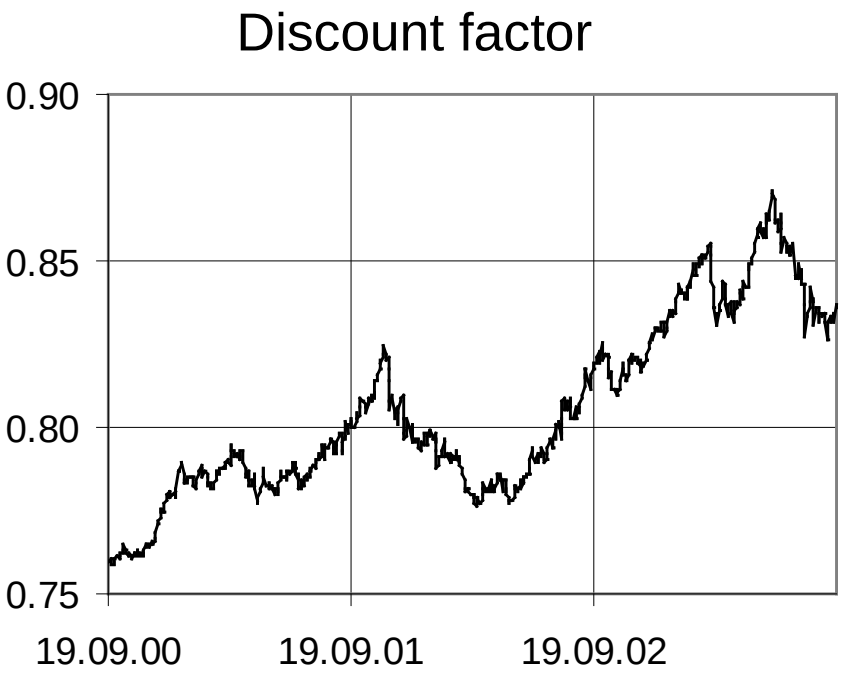
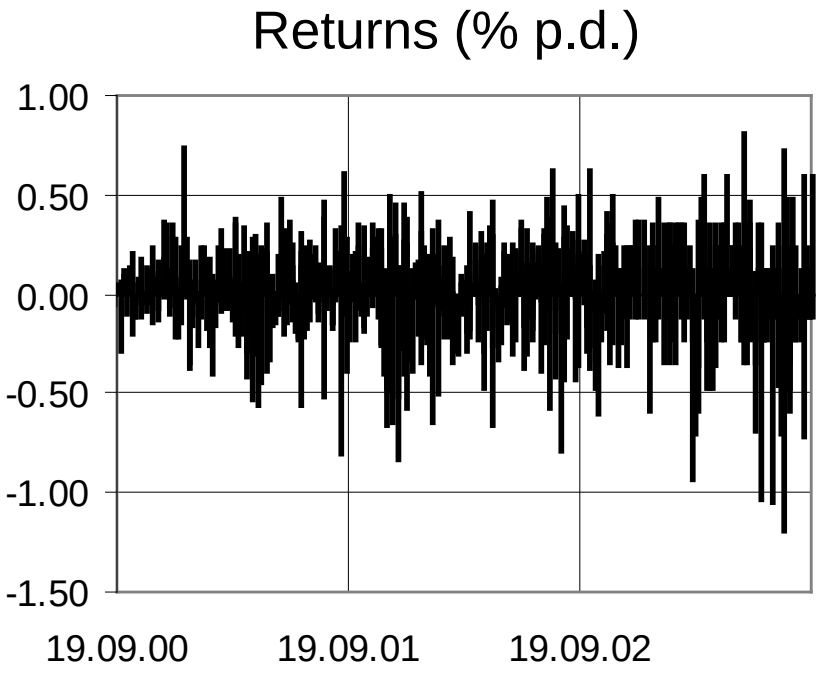
❍ Discount factor - Euro 5Y (19.9.2000 - 16.9.2003)





2.4. Buy-and-Hold Returns

A common method to calculate the performance of an asset based on a return time series over a longer period are Buy-and-Hold returns (BHRs). BHRs have the advantage that they are based on a realistic and ex-ante implementable trading strategy. The BHR for company *i* until *T* is defined as:

$$BHR_{i,T} = (1+R_{i,1})\cdot(1+R_{i,2})\cdot\ldots\ldots\cdot(1+R_{i,T}) - 1 = \left[\prod_{t=1}^{T}\left(1+R_{i,t}\right)\right] - 1$$

$R_{i,t}$ Return of comany *i* in (sub-) period *t* (e.g. trading day *t*)

If no dividends or artificial price changes take place, BHRs can also be calculated based on share prices. Let us consider the BHR calculation for 2 periods:

$$BHR_{i,0,2} = \left[\prod_{t=1}^{2}\left(1+R_{i,t}\right)\right]-1 \;= \left(1+R_{i,1}\right)\cdot\left(1+R_{i,2}\right)$$

$$BHR_{i,0,2} = \left(1+\frac{P_{i,1}-P_{i,0}}{P_{i,0}}\right)\cdot\left(1+\frac{P_{i,2}-P_{i,1}}{P_{i,1}}\right)-1$$

$$BHR_{i,0,2} = \frac{P_{i,0}+P_{i,1}-P_{i,0}}{P_{i,0}}\cdot\frac{P_{i,1}+P_{i,2}-P_{i,1}}{P_{i,1}}-1$$

$$BHR_{i,0,2} = \frac{P_{i,2}}{P_{i,0}}-1=\frac{P_{i,2}-P_{i,0}}{P_{i,0}}$$

2.4.1. Geometric Mean

The geometric mean corresponds to the constant, period based growth rate $GM_{\tau 1,\tau 2}$ that leads over the same period (i.e. between t = τ_1 and t = τ_2 over *n* (sub-) periods) to the same final value as the $BHR_{\tau 1,\tau 2}$. It can be calculated using the corresponding return time series or the corresponding BHR:

$$BHR_{\tau 1,\tau 2} = (1+R_{\tau 1+1})\cdot(1+R_{\tau 1+2})\cdot.....\cdot(1+R_{\tau 2}) - 1$$
$$= (1+GM_{\tau 1,\tau 2})^n - 1$$

$$GM_{\tau 1,\tau 2} = \left[(1+R_{\tau 1+1})\cdot(1+R_{\tau 1+2})\cdot.....\cdot(1+R_{\tau 2})\right]^{(1/n)} - 1 = \left[\prod_{t=\tau 1+1}^{\tau 2}(1+R_t)\right]^{(1/n)}$$

$$GM_{\tau 1,\tau 2} = \left[1+BHR_{\tau 1,\tau 2}\right]^{(1/n)} - 1$$

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If no dividends are payed between points in time τ_1 und τ_2 and/or no artificial pe-rice changes occur (or the price time series has been corrected correspondingly), the geometric mean can also be directly calculated by comparing the prices or index values at t = τ_1 and t = τ_2 ($P_{\tau 1}$ and $P_{\tau 2}$):

$$GM_{\tau 1,\tau 2} = \left[\frac{P_{\tau 2}}{P_{\tau 1}}\right]^{(1/n)} - 1$$

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Example: Annual returns of various US asset classes

Year	Large Company Stocks	Small Company Stocks	Long-Term Corporate Bonds	Long-Term Government Bonds	Intermediate-Term Government Bonds	U.S. Treasury Bills
1926	11.62	0.28	7.37	7.77	5.38	3.27
1927	37.49	22.10	7.44	8.93	4.52	3.12
1928	43.61	39.69	2.84	0.10	0.92	3.56
1929	-8.42	-51.36	3.27	3.42	6.01	4.75
1930	-24.90	-38.15	7.98	4.66	6.72	2.41
1931	-43.34	-49.75	-1.85	-5.31	-2.32	1.07
1932	-8.19	-5.39	10.82	16.84	8.81	0.96
1933	53.99	142.87	10.38	-0.07	1.83	0.30
1934	-1.44	24.22	13.84	10.03	9.00	0.16
1935	47.67	40.19	9.61	4.98	7.01	0.17
1936	33.92	64.80	6.74	7.52	3.06	0.18
1937	-35.03	-58.01	2.75	0.23	1.56	0.31
1938	31.12	32.80	6.13	5.53	6.23	-0.02
1939	-0.41	0.35	3.97	5.94	4.52	0.02
1940	-9.78	-5.16	3.39	6.09	2.96	0.00

Quelle: Ibbotson Associates.

Example:

BHR for Large Company Stocks, 1926 until 1928:

$$BHR_{1926,1928} = 1.1162 \cdot 1.3749 \cdot 1.4361 = +120.393\%$$

Yearly (constant) growth rate:

$$GM_{1926,1928} = [1 + 1.20393]^{(1/3)} - 1 = +30.137\% \text{ p.a.}$$

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Example: Index values of various US asset classes

Year	Large Stocks Total Returns	Large Stocks Capital Apprec	Small Stocks Total Returns	Long-Term Corp Bonds Total Returns	Long-Term Government Bonds Total Returns	Long-Term Government Bonds Capital Apprec	Intermediate-Term Government Bonds Total Returns	Intermediate-Term Government Bonds Capital Apprec	U.S. T-Bills Total Returns	Inflation
1925	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1926	1.116	1.057	1.003	1.074	1.078	1.039	1.054	1.015	1.033	0.985
1927	1.535	1.384	1.224	1.154	1.174	1.095	1.101	1.025	1.065	0.965
1928	2.204	1.908	1.710	1.186	1.175	1.061	1.112	0.997	1.103	0.955
1929	2.018	1.681	0.832	1.225	1.215	1.059	1.178	1.014	1.155	0.957
1930	1.516	1.202	0.515	1.323	1.272	1.072	1.258	1.048	1.183	0.899
1931	0.859	0.636	0.259	1.299	1.204	0.982	1.228	0.991	1.196	0.814
1932	0.789	0.540	0.245	1.439	1.407	1.109	1.337	1.041	1.207	0.730
1933	1.214	0.792	0.594	1.588	1.406	1.074	1.361	1.031	1.211	0.734
1934	1.197	0.745	0.738	1.808	1.547	1.146	1.483	1.092	1.213	0.749
1935	1.767	1.053	1.035	1.982	1.624	1.171	1.587	1.146	1.215	0.771
1936	2.367	1.346	1.705	2.116	1.746	1.225	1.636	1.165	1.217	0.780
1937	1.538	0.827	0.716	2.174	1.750	1.195	1.661	1.165	1.221	0.804
1938	2.016	1.035	0.951	2.307	1.847	1.229	1.765	1.216	1.221	0.782
1939	2.008	0.979	0.954	2.399	1.957	1.272	1.845	1.255	1.221	0.778
1940	1.812	0.829	0.905	2.480	2.076	1.319	1.899	1.280	1.221	0.786

Quelle: Ibbotson Associates.

BHR for Large Company Stocks, End 1928 until End 1931:

$$BHR_{1928,1931} = \left[\frac{0.859}{2.204}\right] - 1 = -61.025\%$$

Constant growth rate p.a.:

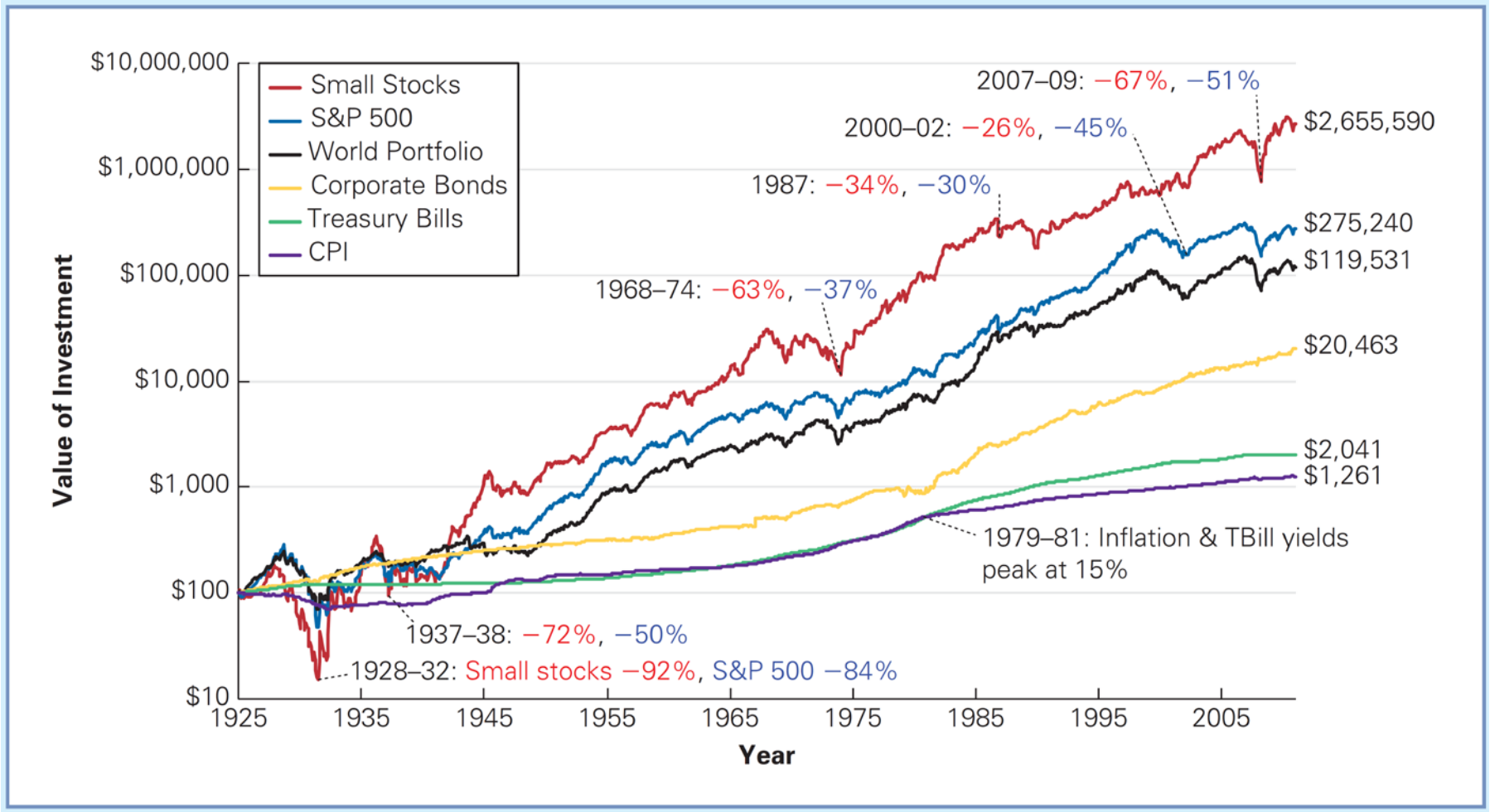
$$GM_{1928,1931} = \left[\frac{0.859}{2.204}\right]^{(1/3)} - 1 = -26.95\% \text{ p.a.}$$

Increased consequence for Small Company Stocks due to the crash in 1929:

$$BHR_{1928,1931} = \left[\frac{0.259}{1.710}\right] - 1 = -84.85\%$$

$$GM_{1928,1931} = \left[\frac{0.259}{1.710}\right]^{(1/3)} - 1 = -46.70\% \text{ p.a.}$$

Wealth development of $100 invested at the end of 1925 until the end of 2011[1]



[1] Source: Berk and DeMarzo (2014), Corporate Finance, 3rd edition, p. 314.

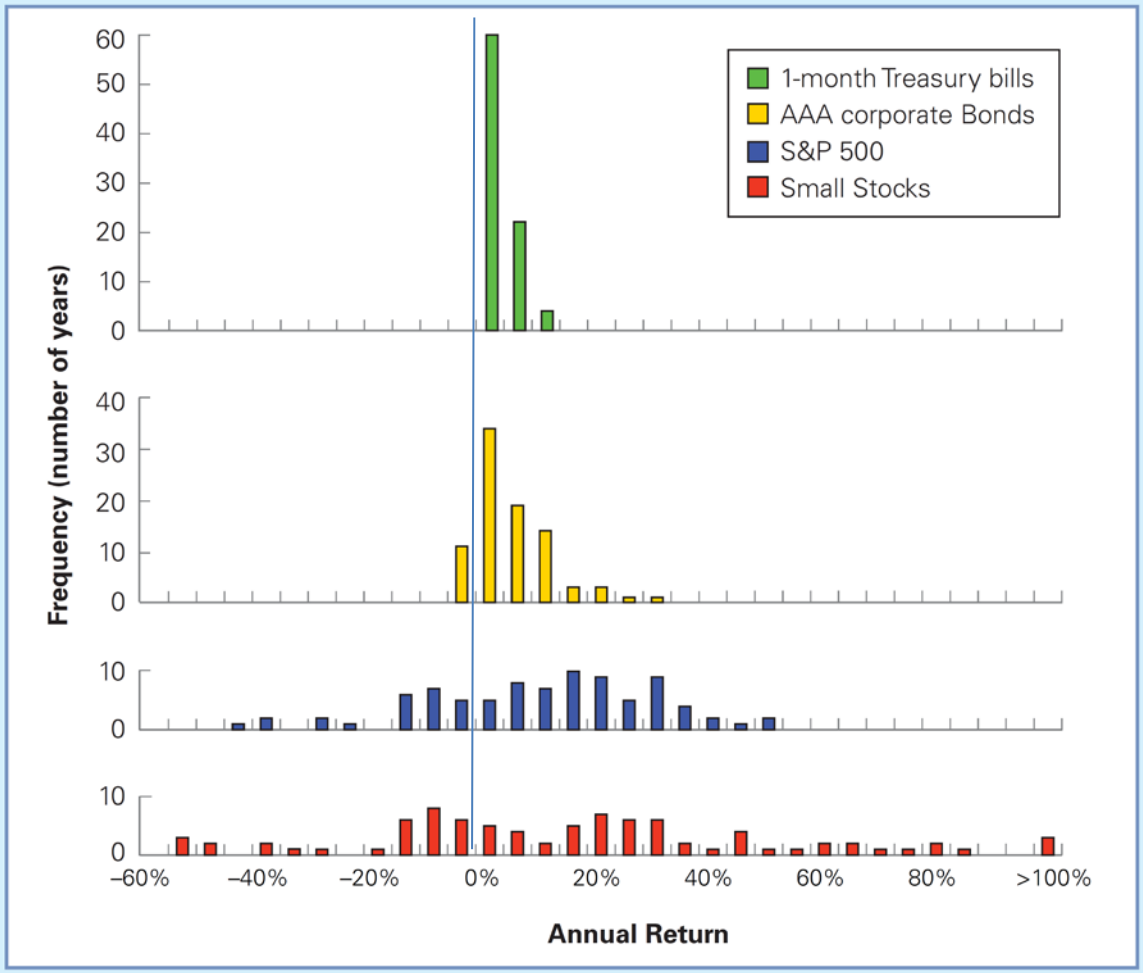
E.g., an investment of $100 in large US stocks (firms included in the Standard & Poor‘s 500 Index) increased during the 86 year period to $275,240. This impressive growth corresponds to an annual constant growth rate (geometric mean *GM*) of 9.6%:

$$GM_{\text{End 1925, End 2010}} = \left[\frac{275{,}240}{100}\right]^{(1/86)} - 1 = 9.6\%$$

An important fact from the graph:

A **higher performance** is associated with higher price fluctuations (**higher risk**).

Empirical distribution of annual returns for various asset classes 1926-2011[2]

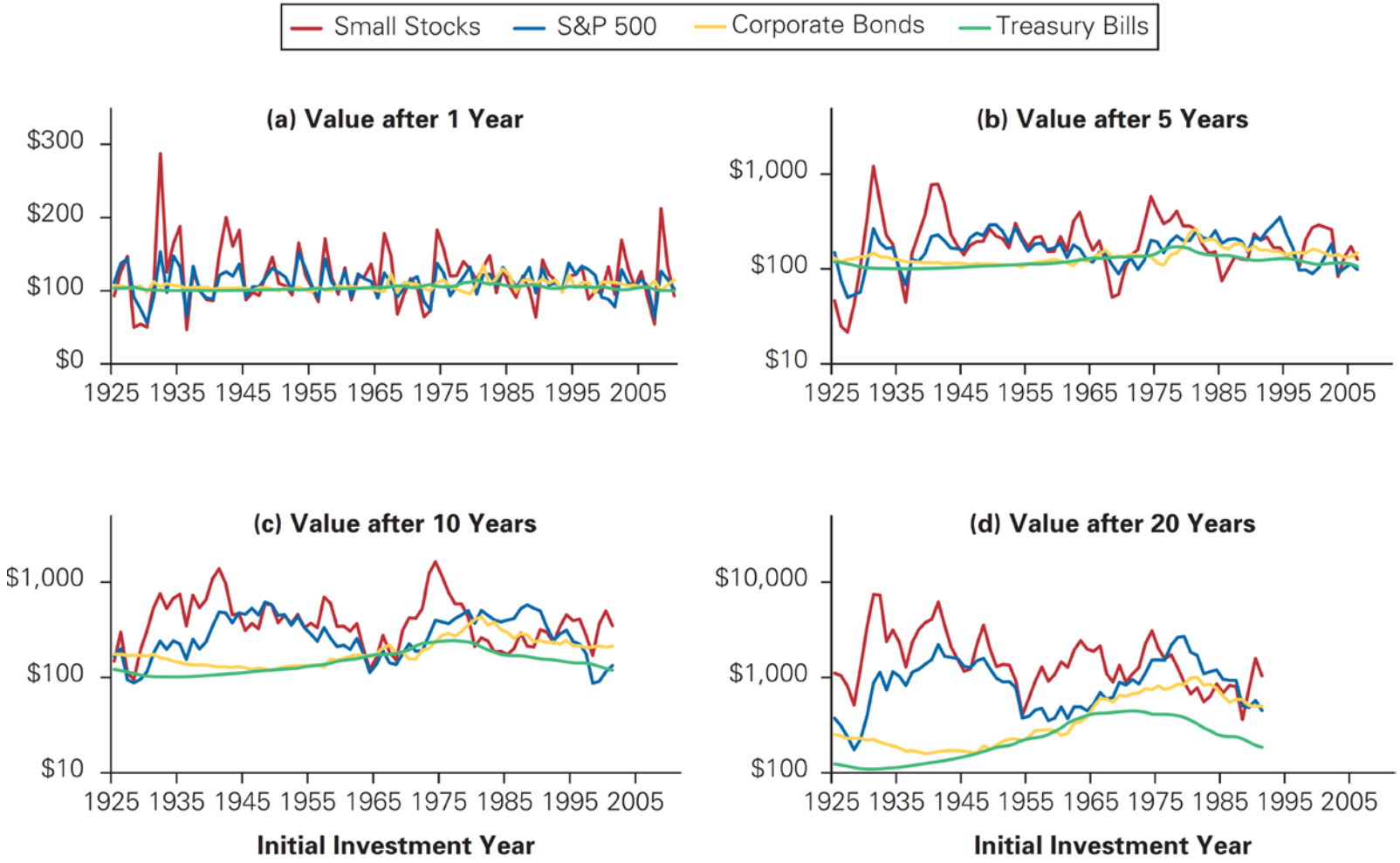


[2] Source: Berk and DeMarzo (2014), Corporate Finance, 3rd edition, p. 322.

Average annual return and return volatility (standard deviation)

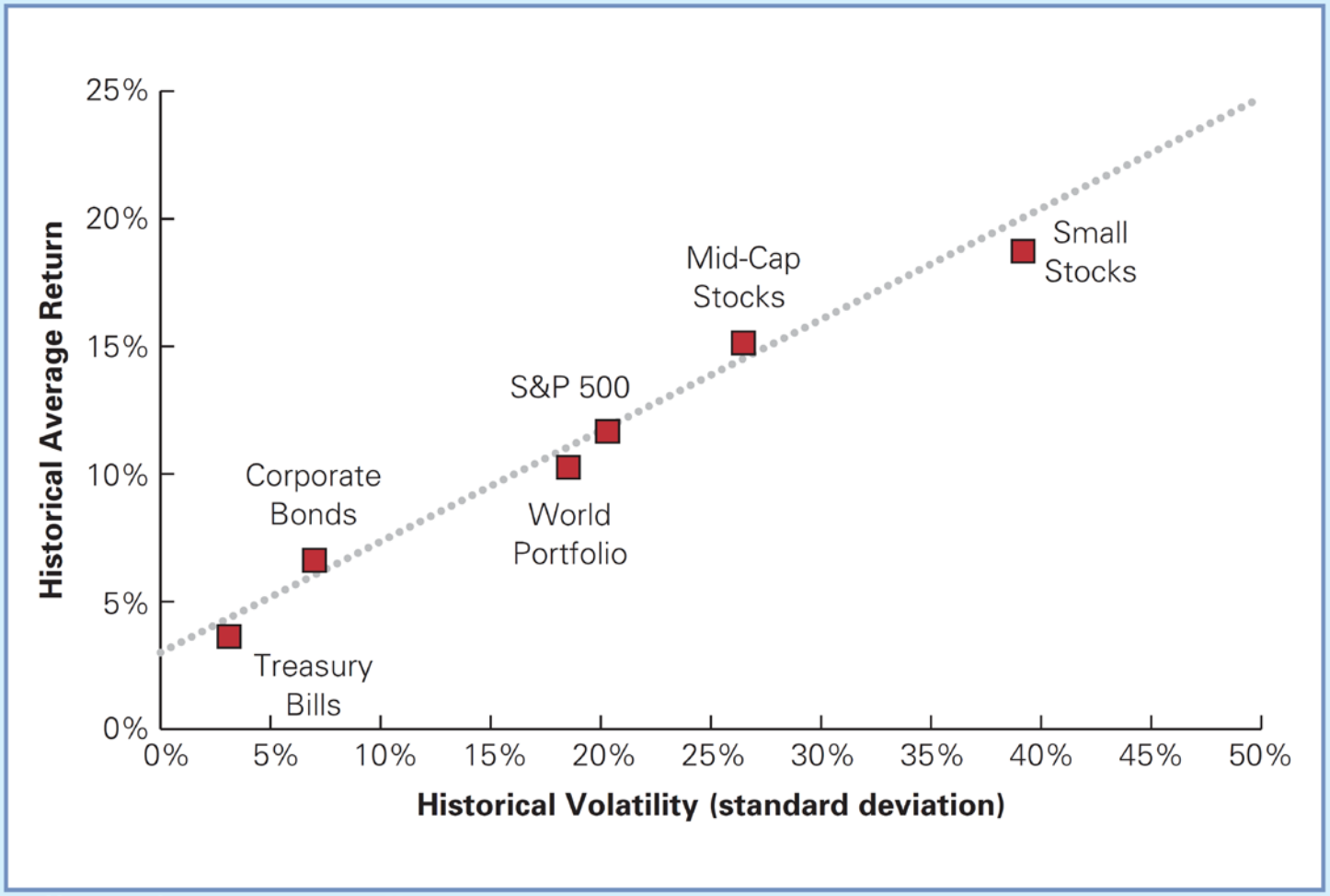
	Artihmetic mean	Geometric mean	Standard deviation
Small stocks	18.7%	12.6%	39.2%
S&P 500	11.7%	9.6%	20.3%
Corporate bonds	6.6%	6.4%	7.0%
Treasury bills	3.6%	3.6%	3.1%

The horizon matters: Value of $100 invested for different horizons[3]



[3] Source: Berk and DeMarzo (2014), Corporate Finance, 3rd edition, p. 315.

Historic trade-off between risk and return: Large portfolios[4]



[4] Source: Berk and DeMarzo (2014), Corporate Finance, 3rd edition, p. 327.

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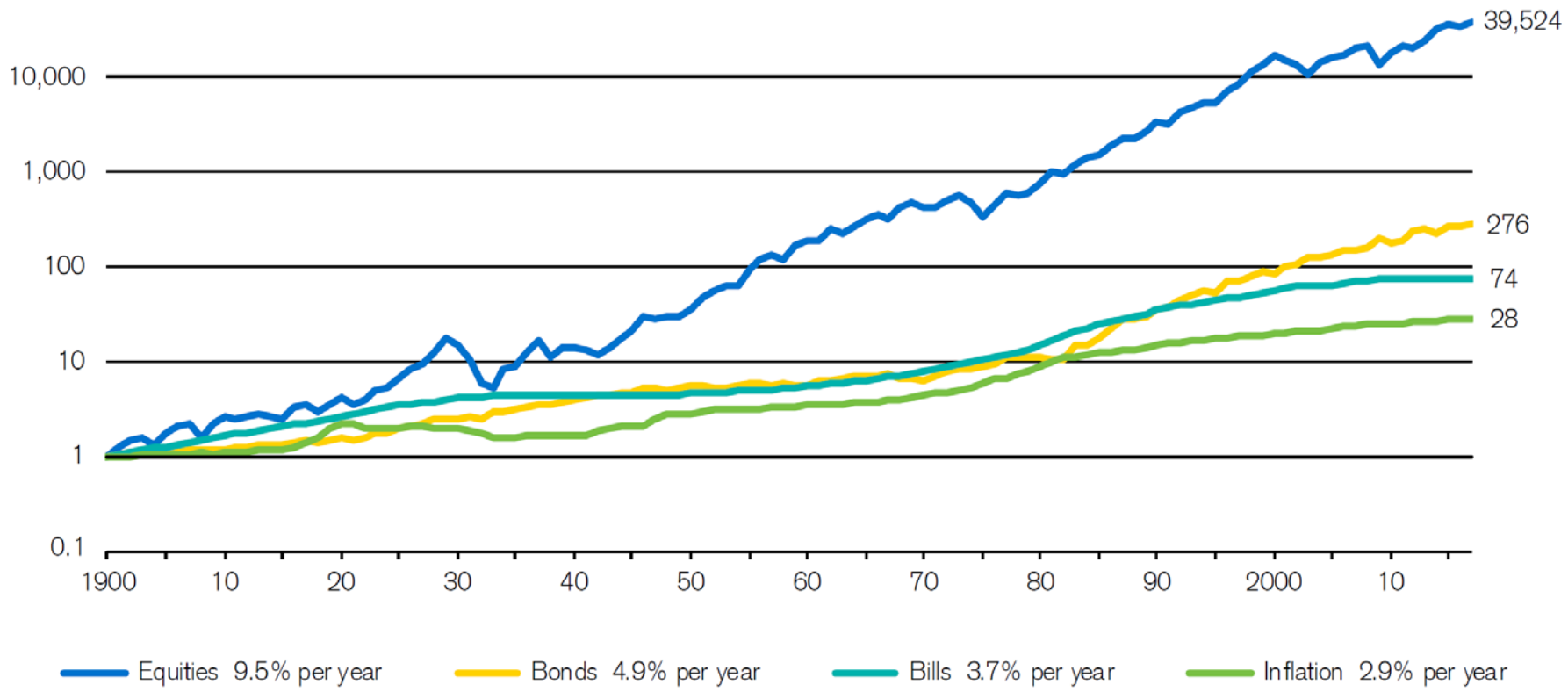
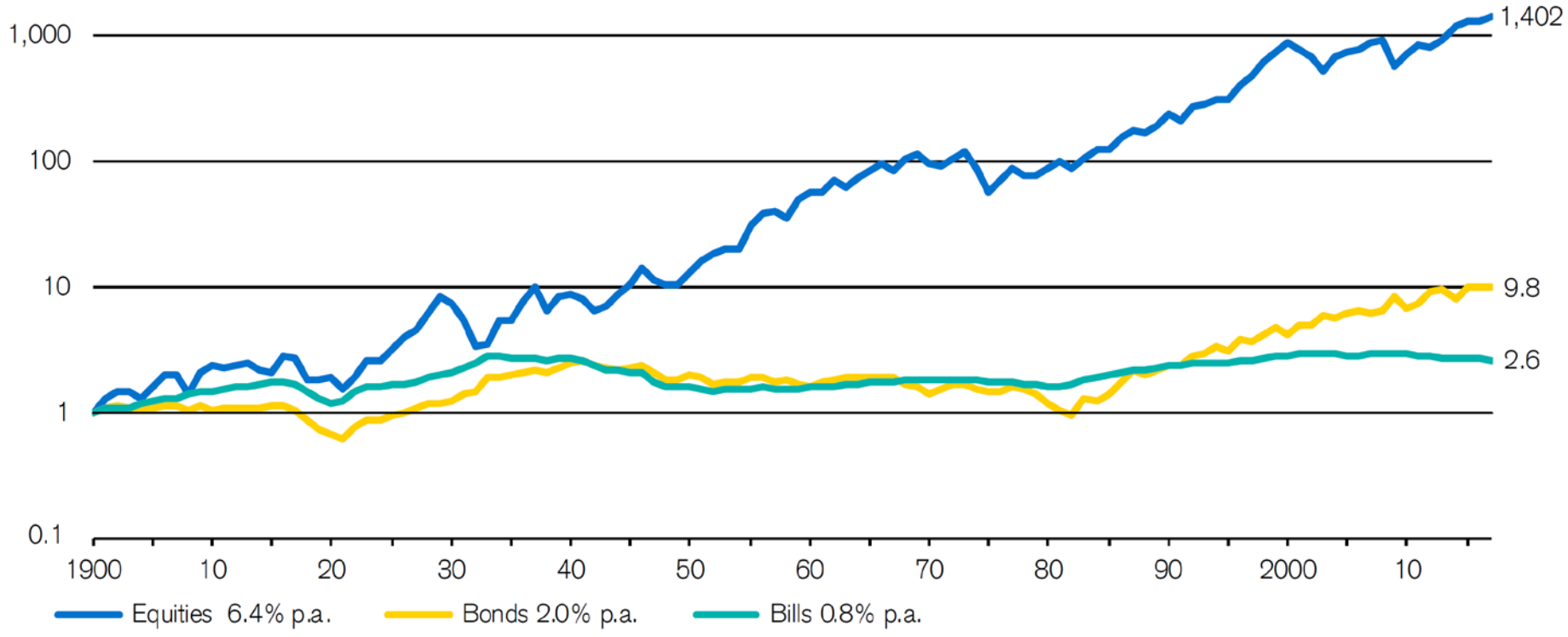
Performance: Nominal vs real

(a) Nominal Performance of US Asset Classes, 1900-2016[5]



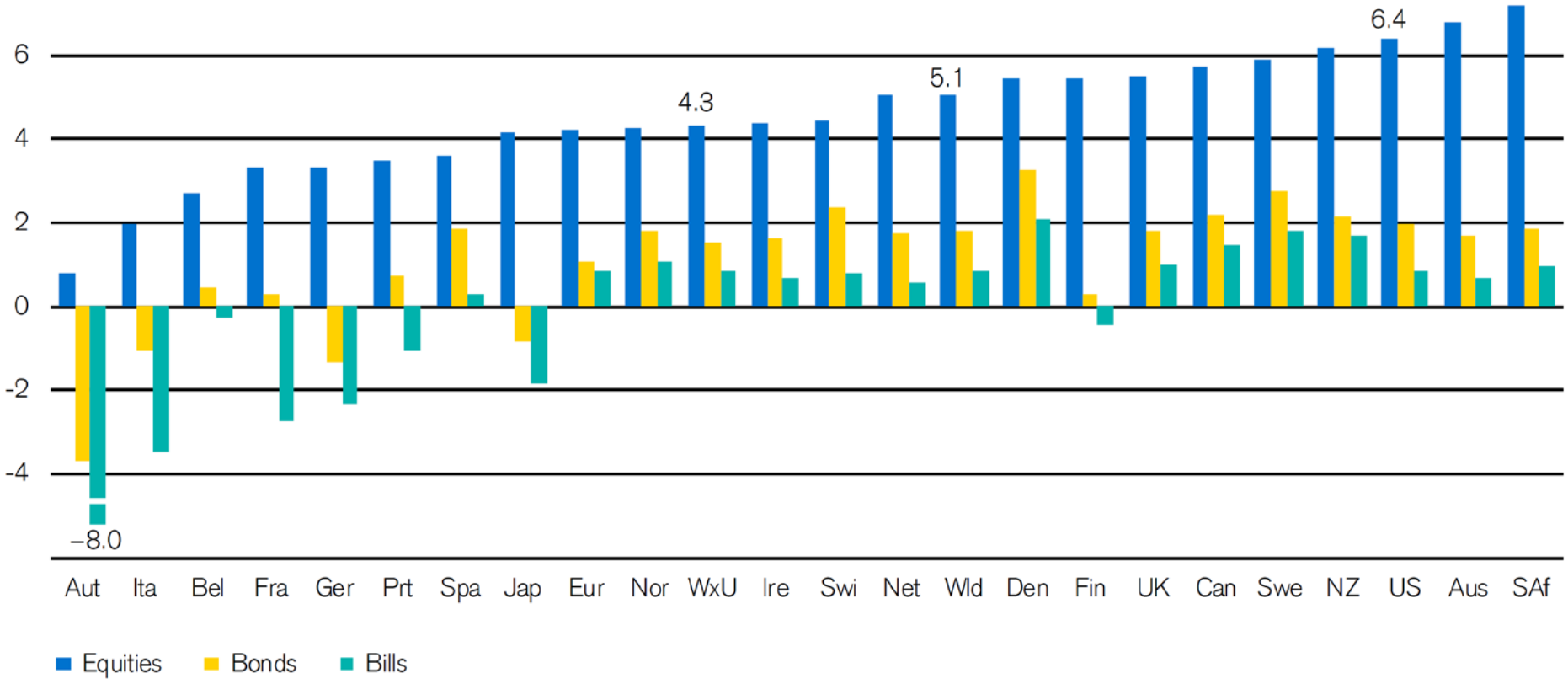
[5] Quelle: Dimson, Marsh, and Staunton, Credit Suisse Global Investment Returns Yearbook 2017 – Summary Edition, p. 11.

(b) Real Performance of US Asset Classes, 1900-2016[6]



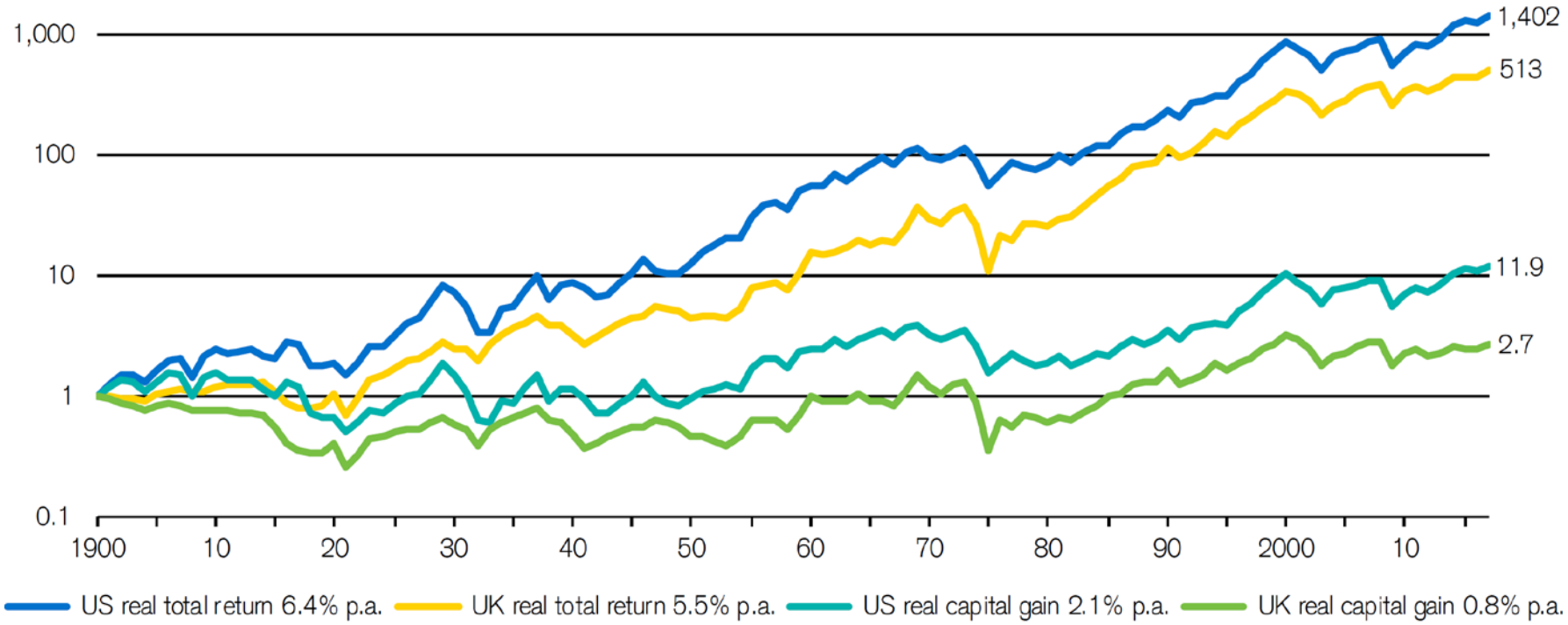
[6] Quelle: Dimson, Marsh, and Staunton, Credit Suisse Global Investment Returns Yearbook 2017 – Summary Edition, p. 12.

(c) Real performance of stock, bonds and bills in different countries, 1900-2016[7]



[7] Quelle: Dimson, Marsh, and Staunton, Credit Suisse Global Investment Returns Yearbook 2017 – Summary Edition, p. 13.

The implication of dividend payments and their re-investment (Large Stocks)[8]:



[8] Quelle: Dimson, Marsh, and Staunton, Credit Suisse Global Investment Returns Yearbook 2017 – Summary Edition, p. 13.

2.4.2. Arithmetic Mean

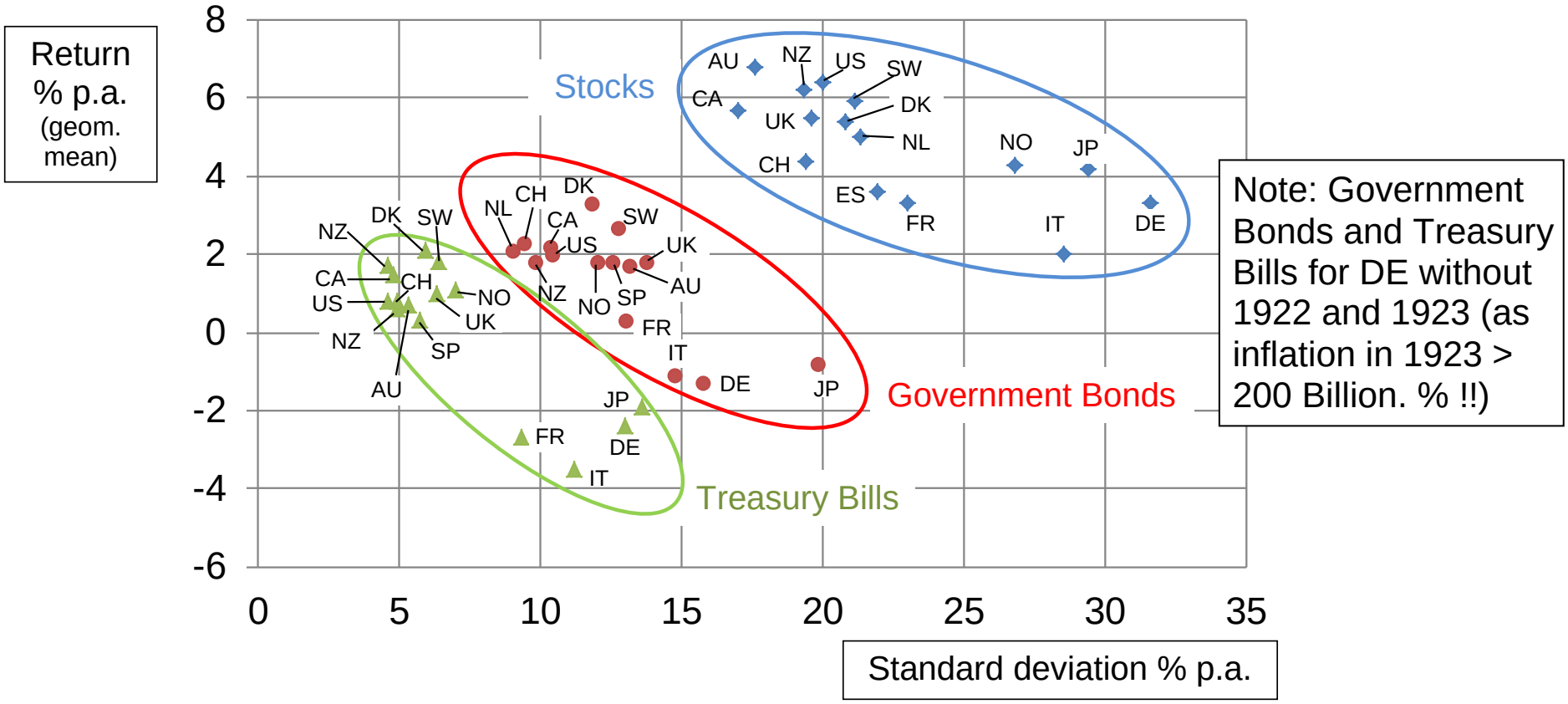
The arithmetic mean *AM* between t = τ_1 and t = τ_2 is defined as the simple average of period returns:

$$AM_{\tau 1,\tau 2} = \frac{1}{n} \cdot \sum_{t=\tau 1+1}^{\tau 2} R_t$$

Due to the compound interest effect the geometric mean is smaller than the arithmetic mean if the long run performance is positive.

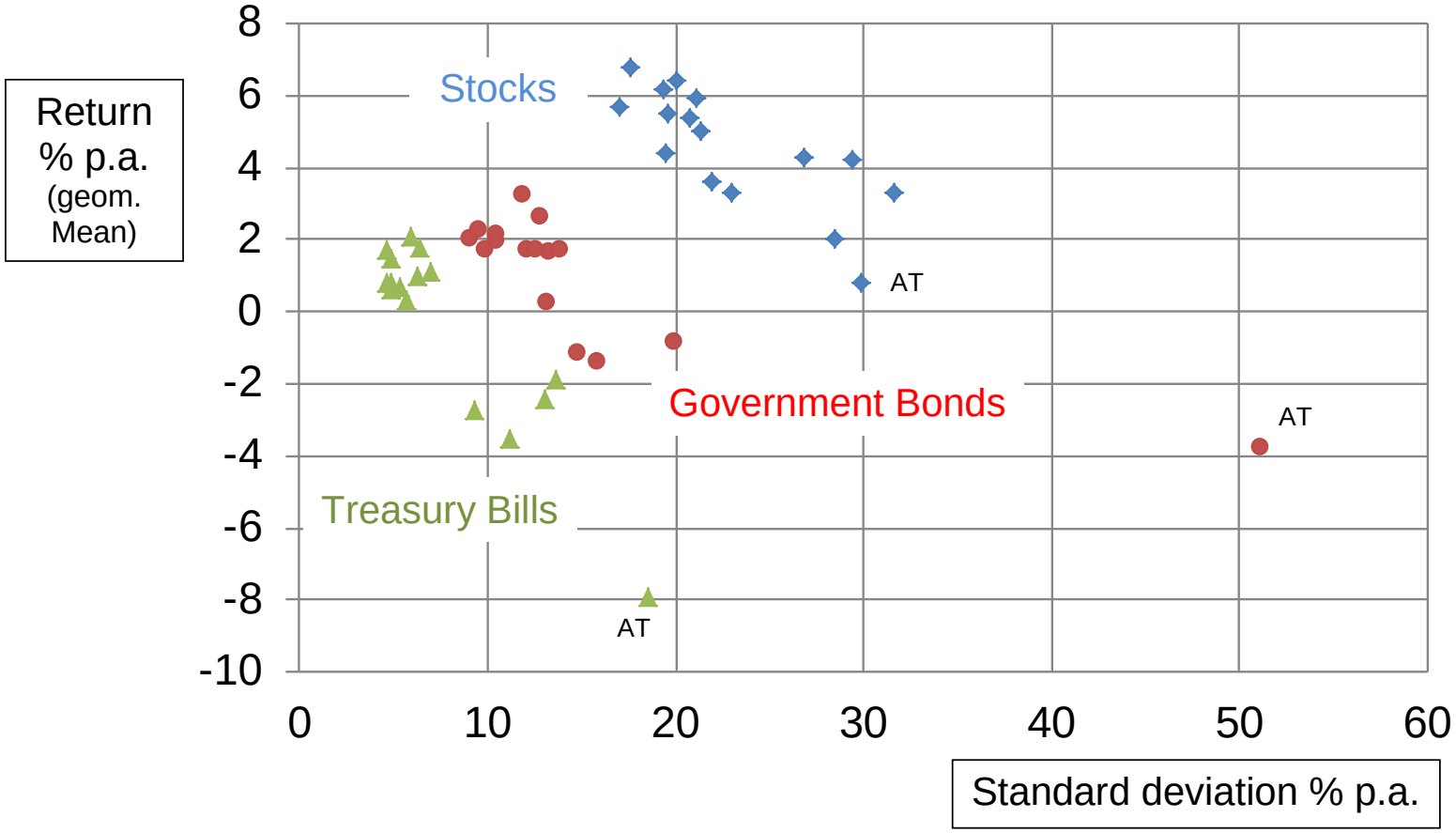
2.4.3. Risk-Return trade-off – International Evidence

Inflation adjusted performance in 15 countries, 1900-2016:[9]



[9] Data source: Dimson, Marsh, and Staunton, Credit Suisse Global Investment Returns Yearbook 2017 – Summary Edition, p. 14 and p. 16-17.

With Austria (AT):[10]



[10] Datenquelle: Dimson, Marsh, and Staunton, Credit Suisse Global Investment Returns Yearbook 2017 – Summary Edition, p. 14 and p. 16-17.

3. Risk Quantification

Ex-post returns are the basis for historical performance and variance calculations.

For investment decisions, potential future (ex-ante) returns are of interest.

Therefore, ex-ante returns are described via probability distributions.

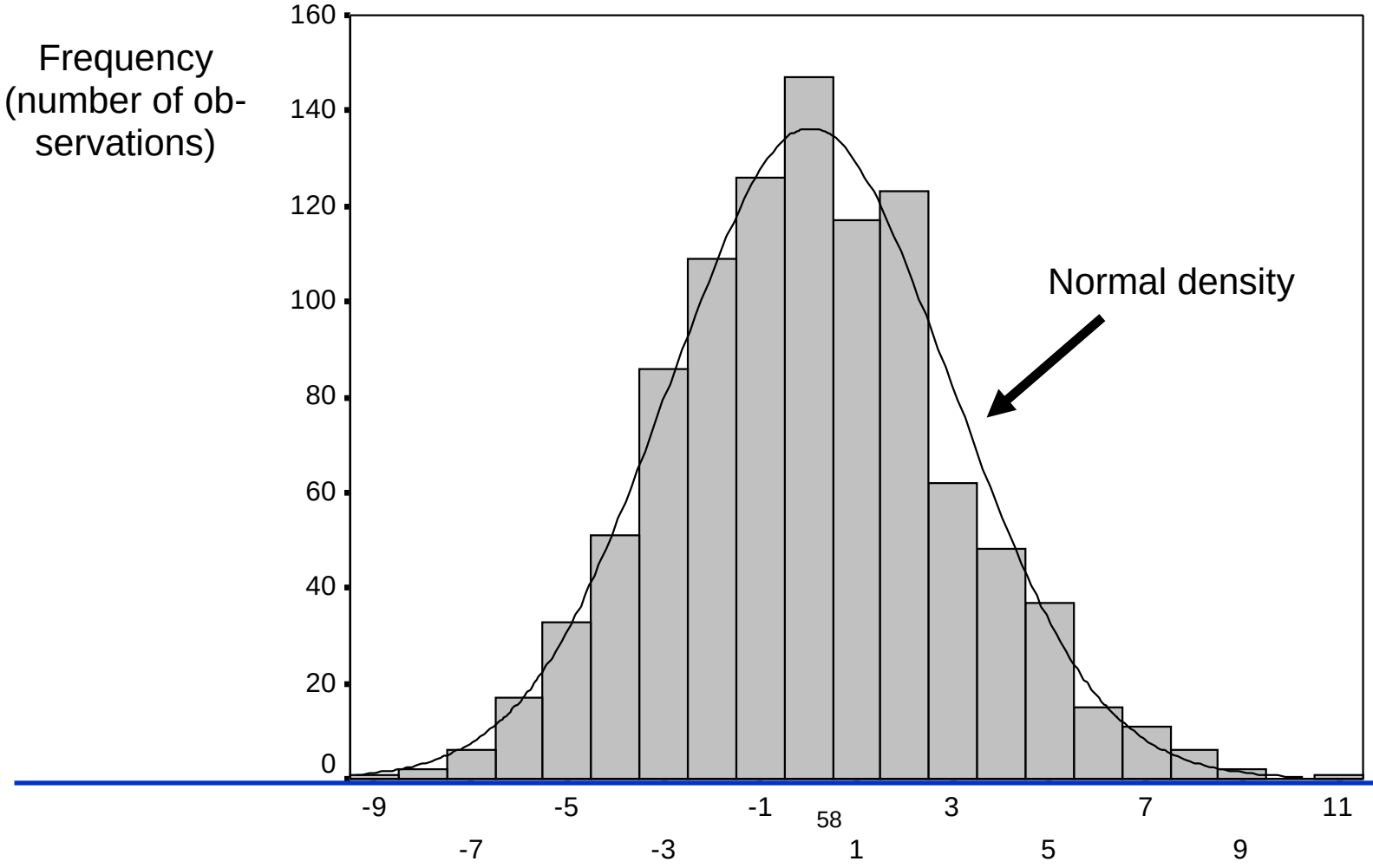
Especially the first two moments of the (ex-ante) return distribution are of interest.

As ex-ante returns are unknown, often historical observations are used as estimate for the (ex-ante) return distribution.

Ex-ante returns are typically modeled as random variables.

3.1 Empirical Distribution

Example: Empirical return distribution and the corresponding **normal density** for 1000 returns of **Siemens** (26.10.1999 until 16.9.2003).



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3.2 Normal Distribution

In finance returns are often assumed to be normal distributed.

A normal distribution is entirely characterized by the first two moments.

To characterize empirical distributions, sometimes also median and mode (*Modalwert*) as well as the third and fourth moment (skewness and kurtosis) are of interest.

Density function of the normal distribution:

$$f(x) = \frac{1}{\sqrt{2 \cdot \pi} \cdot \sigma} \cdot e^{\frac{(x-\mu)^2}{2 \cdot \sigma}}$$

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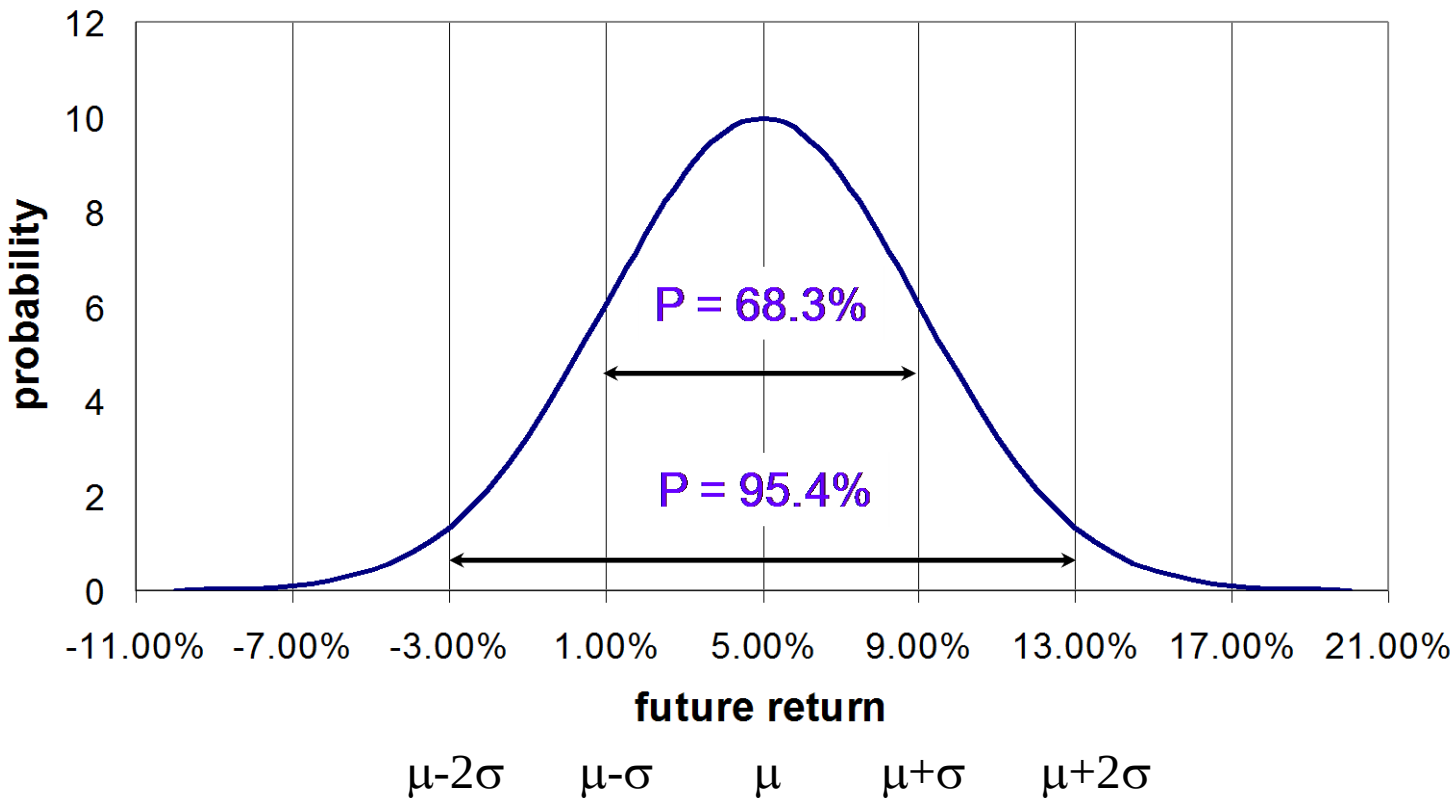
Example: Normal distribution with mean μ = 5% und standard deviation σ = 4%:



If returns are normally distributed, investors can **predict** that **future returns** will fall **within one standard deviation** above or below the mean about 68.3% of the time; and **within 2 standard deviations** about 95.4% of the time.

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1. Moment: Mean

See Arithmetic mean in 2.4.2

MS-Excel: Average(starting cell:ending cell)

- **Median** (important measure of location)
  - 50% of all observations (returns) are **smaller** than the median
  - 50% of all observations (returns) are **larger** than the median

MS-Excel: Median(starting cell:ending cell)

Median: Return that the center of a seris of observations (returns) sorted according to size ($R_1 \leq R_2 \leq ... \leq R_T$)

- Median for uneven T: $\text{Median} = R_{(T+1)/2}$

Example

Return time series: 4%, 8%, 5%, 5.5%, 5%

$\text{Median} = R_{(5+1)/2} = R_3 = 5\%$

- Median for even T: $\text{Median} = \frac{1}{2} \cdot \left(R_{(T/2)} + R_{(T/2)+1}\right)$

Example

Return time series: 4%, 8%, 5%, 5.5%, 5%, 7%

$$\text{Median} = \frac{1}{2} \cdot \left(R_{(6/2)} + R_{(6/2)+1}\right) = \frac{1}{2} \cdot \left(R_3 + R_4\right) = 5.25\%$$

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2. Moment: Variance

$$\sigma^2 = \frac{1}{N-1}\sum_{t=1}^{N}\left(R_t - \overline{R}\right)^2 \quad \Rightarrow \quad \sigma = \sqrt{\sigma^2}$$

- The standard deviation (**volatility**) often used as an estimate to represent **risk** associated with the expected return (mean).

MS-Excel: STDEV(starting cell:ending cell)

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- The **larger** the **standard deviation** is, the **more uncertain** the **outcomes** are. It is a measure of the goodness or confidence placed in the mean value.

- Being able to **measure** and determine the **past volatility of realized returns** provides some **insight** into the **riskiness** of that security as an investment.

- **Increased volatility**, or price fluctuations, is associated with **increased risk**.

3. Moment: Skewness

Reveals the degree of asymmetry for a return distribution. A normal distribution has a skewness of zero.

$$s = \frac{\frac{1}{N}\sum_{t=1}^{N}(R_t - \bar{R})^3}{\sqrt{\left(\frac{1}{N}\sum_{t=1}^{N}(R_t - \bar{R})^2\right)^3}} = \frac{\frac{1}{N}\sum_{t=1}^{N}(R_t - \bar{R})^3}{\sigma^3}$$

$s = 0 \quad \Rightarrow \quad$ Distribution = symmetric

$s < 0 \quad \Rightarrow \quad$ Distribution = skewed to the left

$s > 0 \quad \Rightarrow \quad$ Distribution = skewed to the right

MS-Excel: SKEWNESS(starting cell:ending cell)

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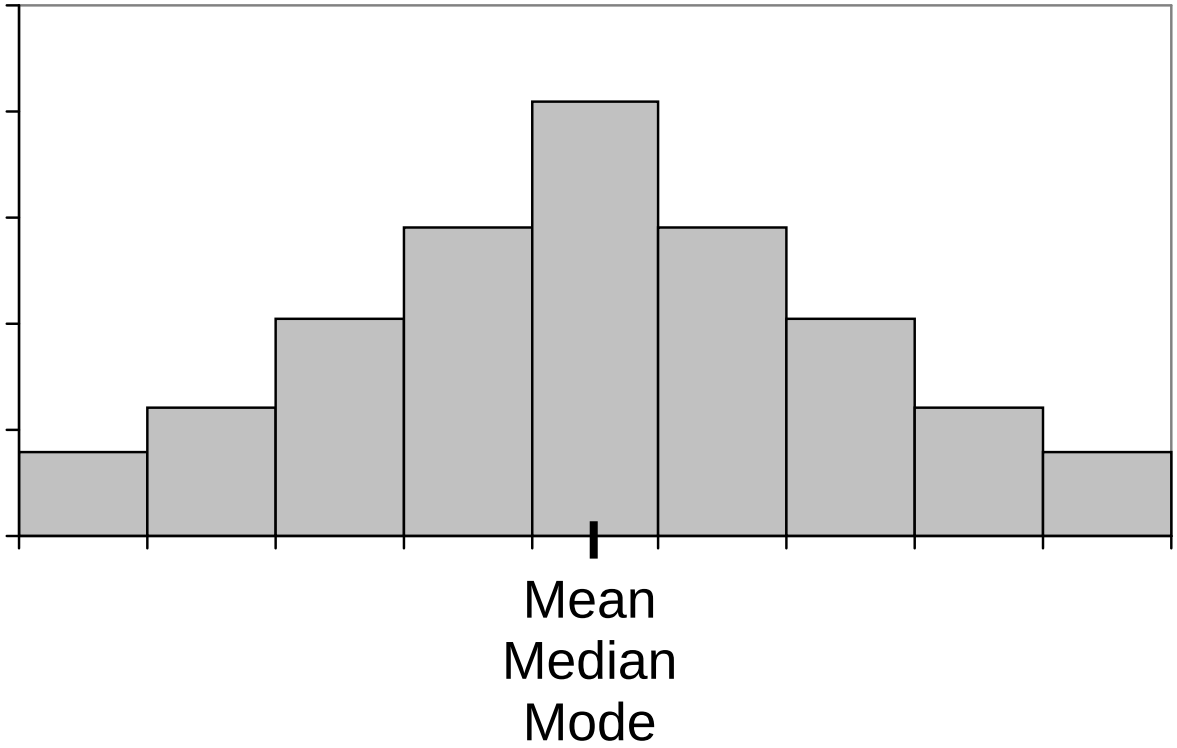
- **Symmetric distribution (s = 0)**

  $\Rightarrow$ Mean = Median = Mode



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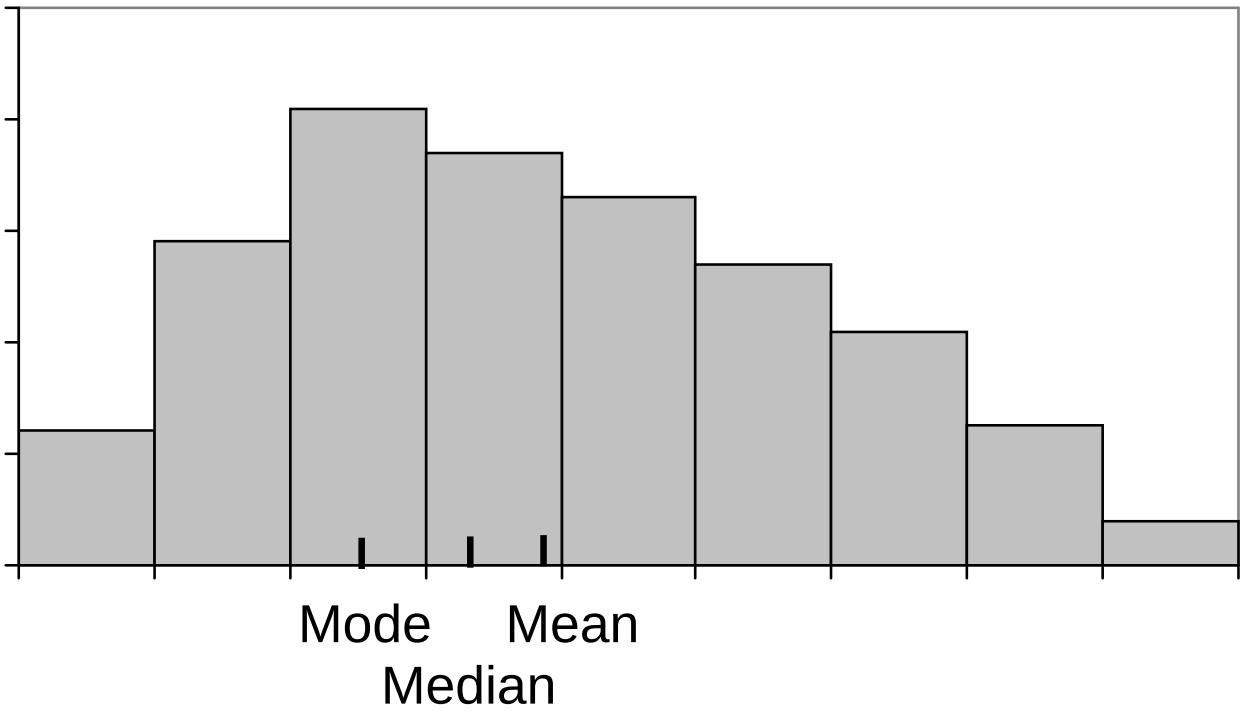
- **Distribution is skewed to the right (s > 0)**

  $\Rightarrow$ Mean > Median > Mode



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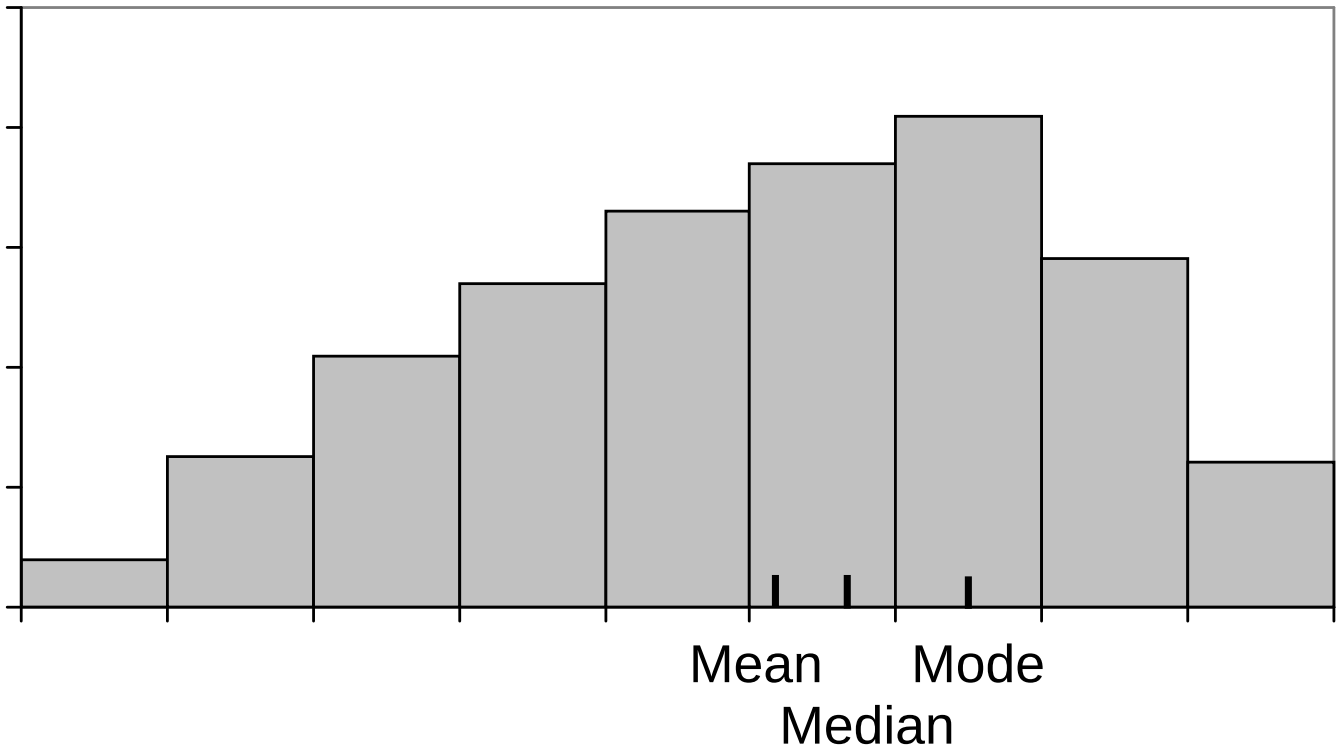
- **Distribution is skewed to the left (s < 0)**

  $\Rightarrow$ Mean < Median < Mode



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4. Moment: Kurtosis

The kurtosis shows, whether the maximum of a frequency distribution is higher or lower compared to a normal distribution. A normal distribution has a kurtosis of zero:

$$k = \frac{\frac{1}{n}\sum_{t=1}^{n}\left(R_t - \overline{R}\right)^4}{\left[\frac{1}{n}\sum_{t=1}^{n}(R_t - \overline{R})^2\right]^2} - 3$$

$k > 0 \quad \Rightarrow \quad$ leptokurtic distribution

$k < 0 \quad \Rightarrow \quad$ platykurtic distribution

MS-Excel: KURT(starting cell:ending cell)

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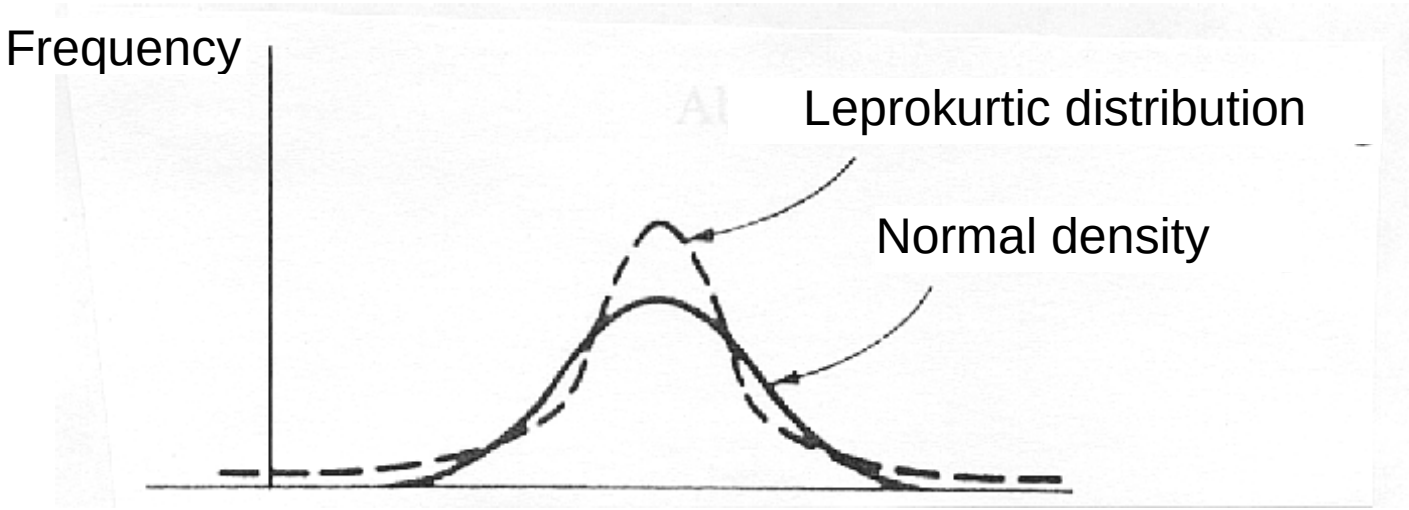
Leptocurtic versus normal distribution



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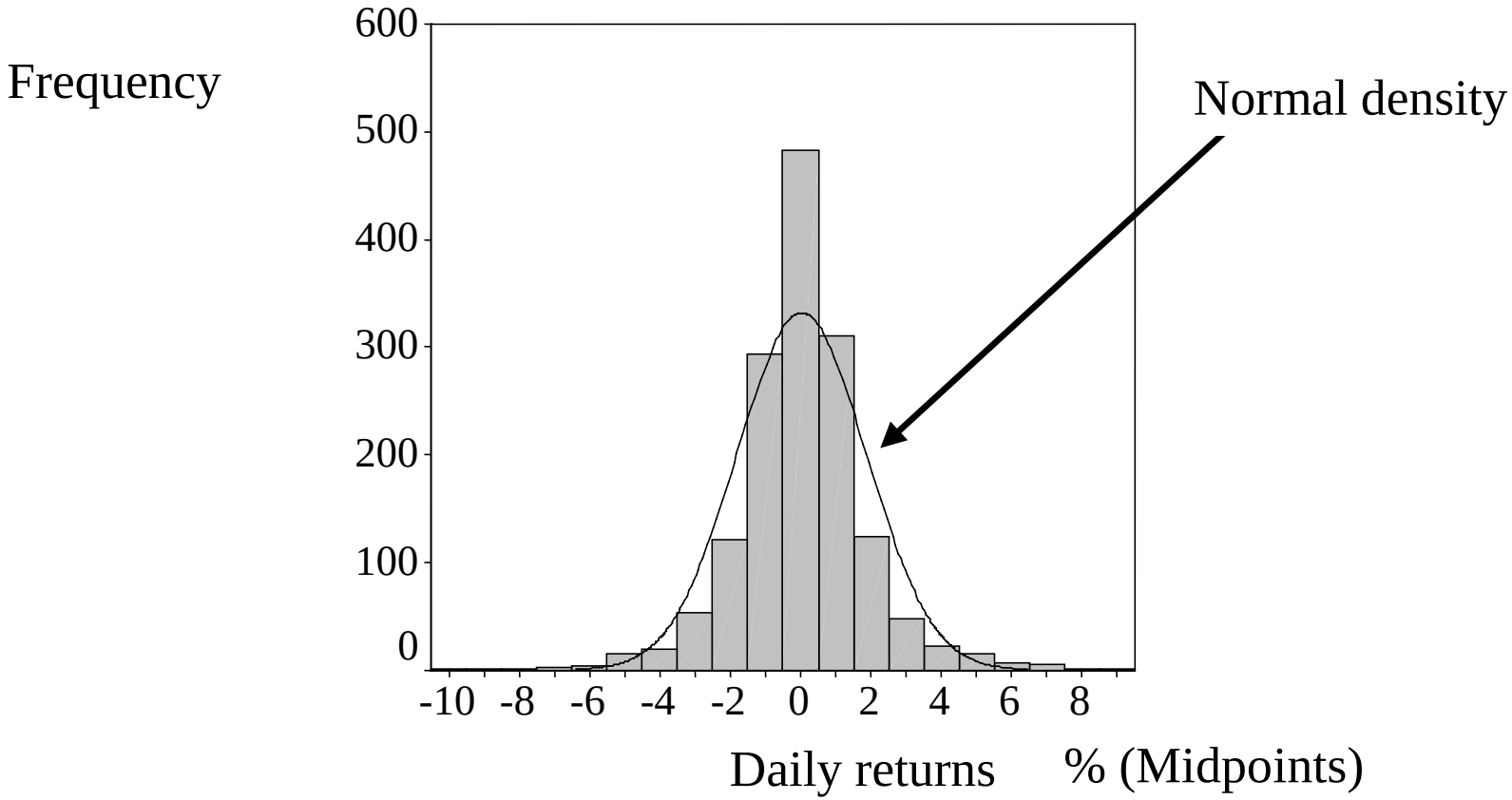
Empirical example of a leptokurtic distribution: 1533 daily OMV returns (2.1.1995 – 16.3.2001)



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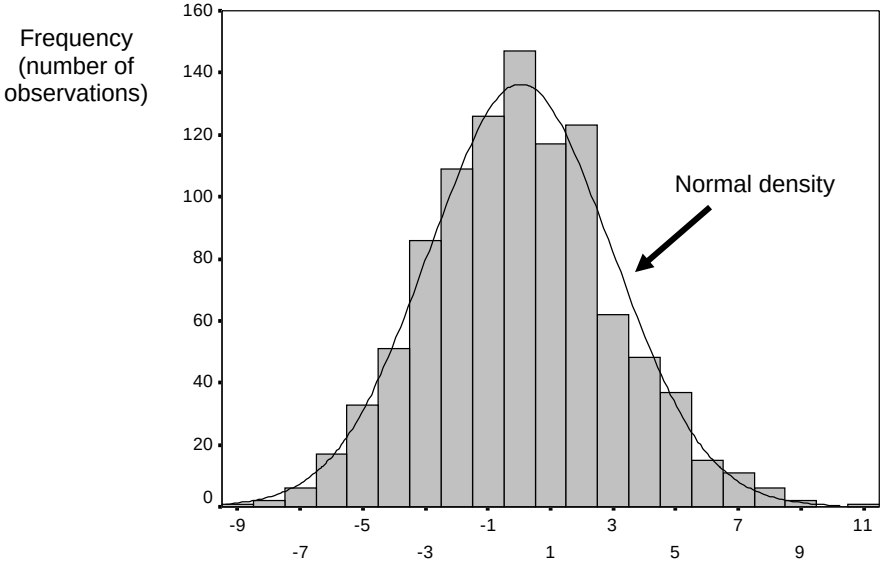
Example

Siemens, 1000 Returns, 26.10.1999 to 16.9.2003, return distribution characteristics:

Mean (%)	0.048
Median (%)	0.000
Mode (%)	0.000
Variance (%^2)	8.532
Standard Deviation (%)	2.921
Skewness	0.167
Kurtosis	0.125

Example: Empirical return distribution and the corresponding **normal density** for 1000 returns of **Siemens** (26.10.1999 until 16.9.2003).



➔ Not significant different from a normal distribution.

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3.3 Value at Risk (VaR) - Basics

- If: Return R $\sim$ N(μ, σ^2), then a probability α exists, so that

$$R \leq [\mu + \sigma \cdot \Phi(\alpha)] \text{ and}$$

$$P[R \leq \mu + \sigma \cdot \Phi(\alpha)] = \alpha \qquad (1)$$

P[.] Probability of the parenthesized expression

E.g. for α = 1% and 5%:

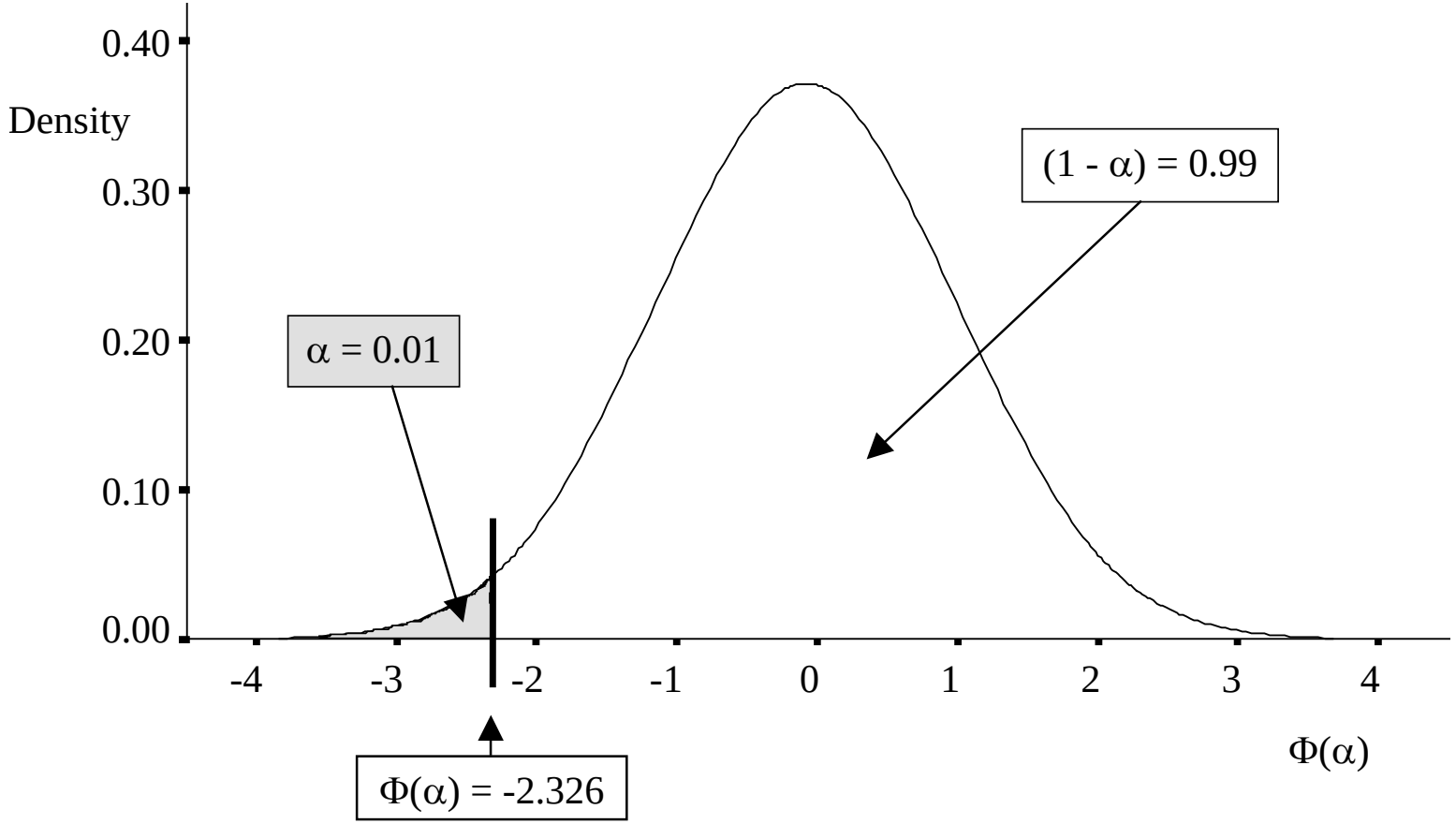
$$P[R \leq \mu + \sigma \cdot (-2.326)] = 0.01$$

$$P[R \leq \mu + \sigma \cdot (-1.645)] = 0.05$$

- Important quantiles $\Phi(\alpha)$ of the standard normal distribution:

α	0.010	0.025	0.050	0.100	0.500	0.900	0.950	0.975	0.990
$\Phi(\alpha)$	-2.326	-1.960	-1.645	-1.282	0.000	1.282	1.645	1.960	2.326

❍ Density of the standard normal distribution and the 1%-Quantile



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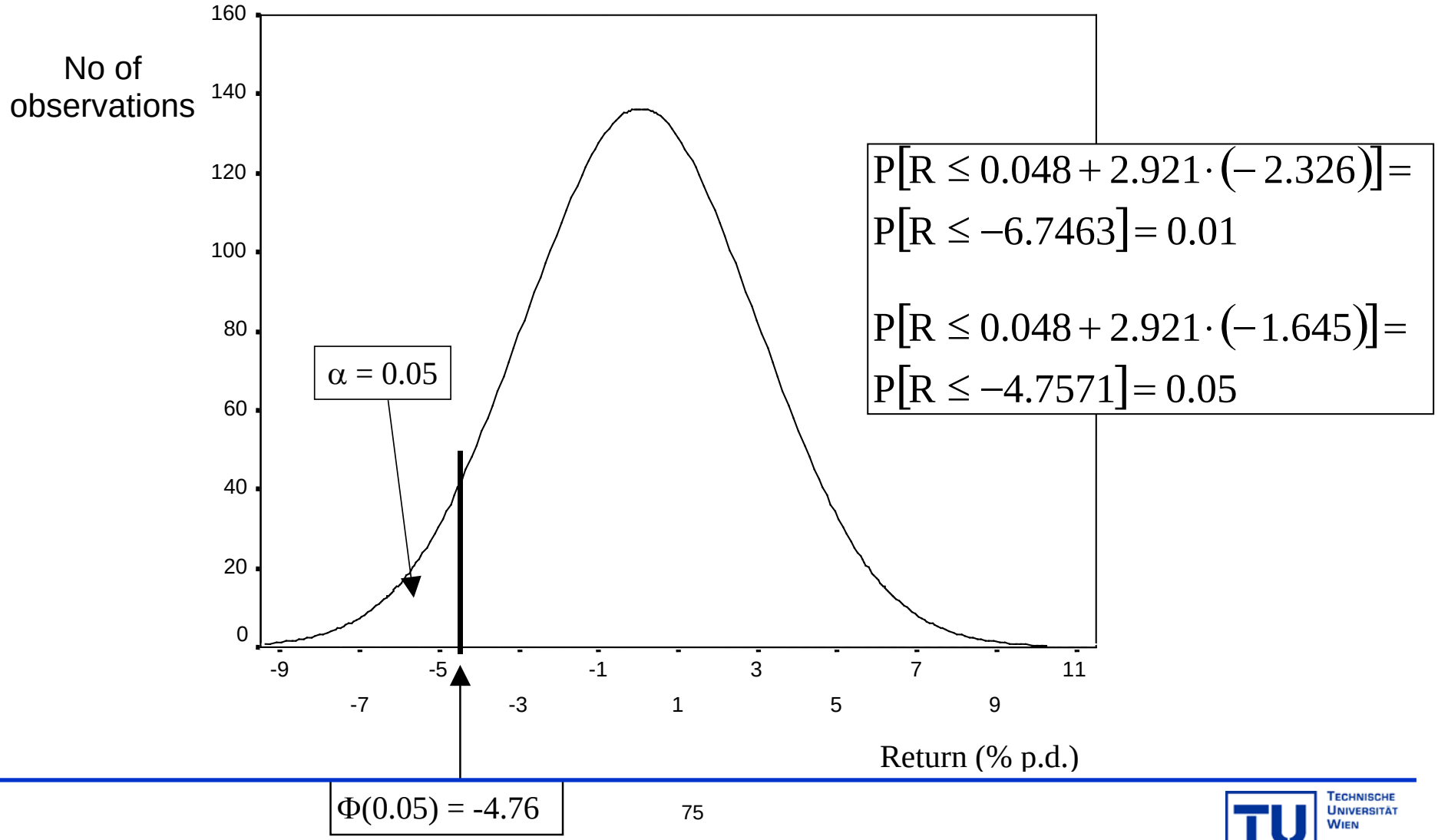
- **Normal distribution of Siemens**, 1000 daily returns, μ = 0.048%, σ = 2.921%



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- VaR = **expected loss** of a position/portfolio that will be **exceeded** with a **prespecified probability**.

$$P\left[\left(MV_{t+1} - MV_t\right) \leq -VaR(\alpha)\right] = \alpha \qquad (2)$$

MV_t Market value of the position/portfolio at time t

- Instead of returns: absolute value changes in € (MV_{t+1} - MV_t) are used
- Negative sign for the VaR, because VaR is defined as loss-variable.
- 3 main methods to calculate VaR:
  - **Delta Normal** (Variance-Covariance) Approach
  - **Historical Simulation** Approach
  - **Monte Carlo Simulation** Approach

(a) Delta Normal Approach

- Assumptions: Returns of the **portfolio** (R_{MV}) are **normal distributed**
  Returns of all **risk factors** (R) are **normal distributed**

- Under the normality assumption eq. (1) is also valid for returns of a portfolio:

$$P\left[R_{t,t+1} \le \mu_{MV} + \sigma_{MV} \cdot \Phi(\alpha)\right] = \alpha$$

$$P\left[\left(\frac{MV_{t+1} - MV_t}{MV_t}\right) \le \left(\mu_{MV} + \sigma_{MV} \cdot \Phi(\alpha)\right)\right] = \alpha$$

$$P\left[\left(MV_{t+1} - MV_t\right) \le MV_t \cdot \left(\mu_{MV} + \sigma_{MV} \cdot \Phi(\alpha)\right)\right] = \alpha$$

- **VaR for quantile α:**

$$VaR(\alpha) = -MV_t \cdot \left[\mu_{MV} + \sigma_{MV} \cdot \Phi(\alpha)\right] \text{ so that } P\left[\left(MV_{t+1} - MV_t\right) \le -VaR(\alpha)\right] = \alpha \qquad (3)$$

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- VaR(α) = expected loss for the next trading day that will be exceeded with probability α, based on the current market value MV_t.

- Delta Normal Approach: **Advantages**
  - **Easy** to understand
  - Fast to implement (for **linear** instruments)
  - **Return series** of market risk factors are not necessary

- Delta Normal Approach: **Disadvantages**
  - Assumptions (**normal** distribution)
  - More complex for **non-linear** instruments (Gamma, Vega)

- For linear instruments (e.g., stocks, bonds, etc.): expected **portfolio** return **(μ_{MV})** and **portfolio** standard deviation **(σ_{MV})** can be replaced by expected **factor** return **(μ)** and **factor** standard deviation **(σ).**

- **VaR** for a **position/portfolio**

$$\mathrm{VaR}(\alpha) = -\mathrm{MV}_t \cdot \left[\mu + \sigma \cdot \Phi(\alpha)\right] \tag{4}$$

Example

Portfolio: 5,000 Siemens stocks
Price = €55.90 (16.9.2003), market value (MV) = €279,500
μ = 0.048% (p.d.), σ = 2.921% (p.d.)

$$\mathrm{VaR}(0.01) = -279{,}500 \cdot [0.00048 + 0.02921 \cdot (-2.326)] = 18{,}855.76€$$

$$\mathrm{VaR}(0.05) = -279{,}500 \cdot [0.00048 + 0.02921 \cdot (-1.645)] = 13{,}295.94€$$

- For a holding period of one trading day:
  - With probability α = 5%: Loss $\geq$ €13,295.94

or

  - With probability α = 95%: Loss < €13,295.94

- For 250 trading days p.a., we can expect a **loss > VaR** on **12 to 13** trading days p.a.

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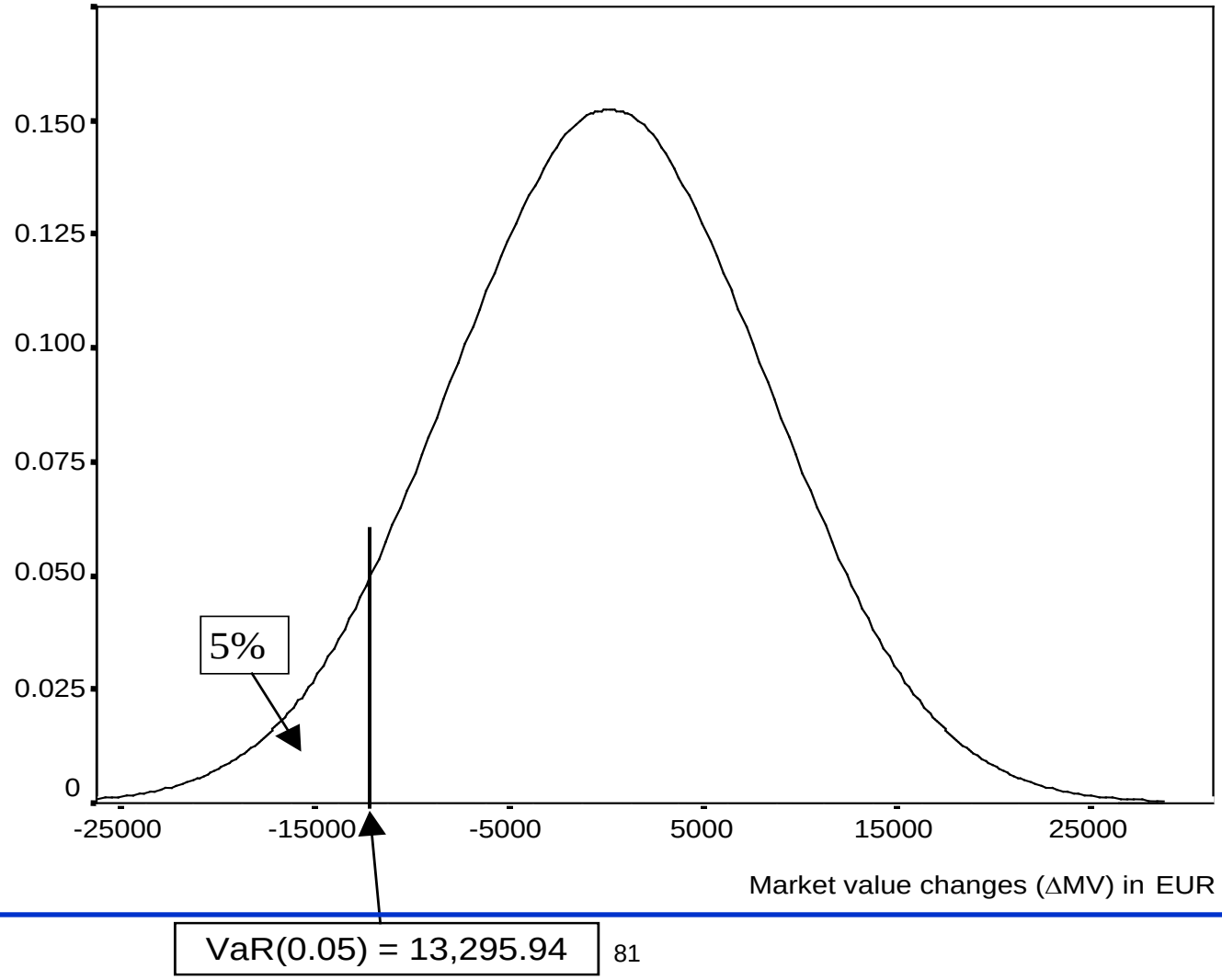
- **Density function** for the (estimated) market value changes in the position of 5,000 Siemens stocks and the corresponding **VaR** for α = 5%.



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(b) VaR: Historical Simulation

- **Advantages**
  - Conceptually **simple** and **easy** to implement
  - **No parametric assumptions** about P/L or market risk factors
  - Automatically accommodates for **fat tails**, **skewness** and any **other non-normal features**
  - Can be used **for any type of position** (linear and non-linear)

- **Disadvantages**
  - It takes **more time** to calculate VaR compared to the delta normal approach
  - Results are **completely dependent on the data** set
    - ➔ **Volatility clustering** effects can negatively affect the VaR
    - ➔ VaR estimates are affected by the **largest losses** in the historical data set

- **Steps:**

1. Specify all necessary **market risk factors**.
2. Select a **historical period** of T observations (e.g. the last 250 or 500 trading days).
3. Calculate a **historical time series of returns** for each market risk factor.
4. Use the returns in (3.) to construct a **distribution** of **synthetic risk factor values** (for each risk factor).
5. Use the distribution in (4.) to **revalue all positions** in the portfolio (for each historical trading day (scenario)). This yields a **distribution of synthetic market values** for each position.
6. Aggregate the distribution of synthetic (potential) market values over all positions to get the **distribution of synthetic (potential) market values for the total portfolio**.
7. Use the distribution in (6.) and the current market value of the portfolio to calculate the **distribution of synthetic (potential) market value changes**.
8. The VaR equals the α-quantile of the distribution in (7).

Example:

Portfolio of 5,000 Siemens stocks: VaR(0.05)

Historical period: 26.10.1999 to 16.9.2003

T = 1000 daily returns

Closing price (16.9.2003): €55.90

Market value of Siemens position (16.8.2003) = €279,500.

❍ Distribution of synthetic (potential) market value changes

History			Forecast (in €)		
	Risk factor		Siemens	Position	
Date	Price (€)	Return (%)	Price	Market value	Market value changes
16.9.2003			**55.90**	**279,500.00**	
25.10.1999	54.40				
26.10.1999	55.47	1.967	56.99950	284,997.52	5,497.52
27.10.1999	55.80	0.595	56.23256	281,162.79	1,662.79
28.10.1999	56.63	1.487	56.73149	283,657.44	4,157.44
⋮	⋮	⋮	⋮	⋮	⋮
11.09.2003	55.40	0.819	56.35778	281,788.90	2,288.90
12.09.2003	54.08	-2.383	54.56809	272,840.43	-6,659.57
15.09.2003	54.60	0.962	56.43750	282,187.50	2,687.50
16.09.2003	55.90	2.381	57.23095	286,154.76	6,654.76

Sorted market value changes

Rank	Market value changes
1	30,454.00
2	25,941.62
⋮	⋮
949	-13,065.38
950	**-13,127.20**
951	**-13,254.71**
952	-13,302.41
953	-13,447.42
⋮	⋮
989	-16,933.04
990	**-16,992.66**
991	**-17,828.75**
992	-18,294.55
993	-18,712.92
994	-18,786.07
995	-18,963.79
996	-19,542.46
997	-20,204.82
998	-21,646.06
999	-22,893.03
1000	-26,007.37

95%

99%

1%

5%

VaR(0.05)
= €13,190.96

VaR(0.01)
= €17,410.71

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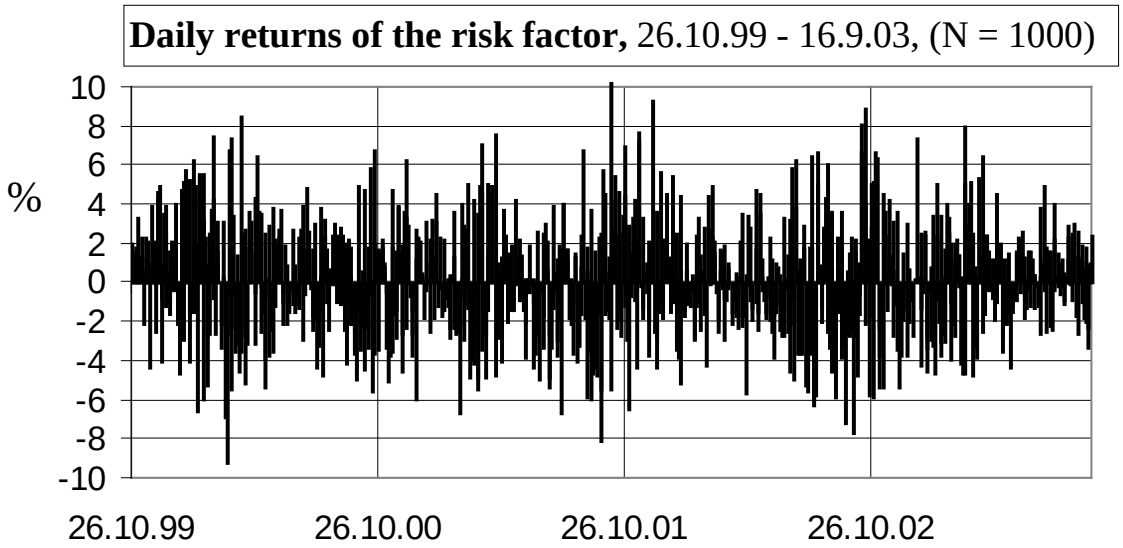
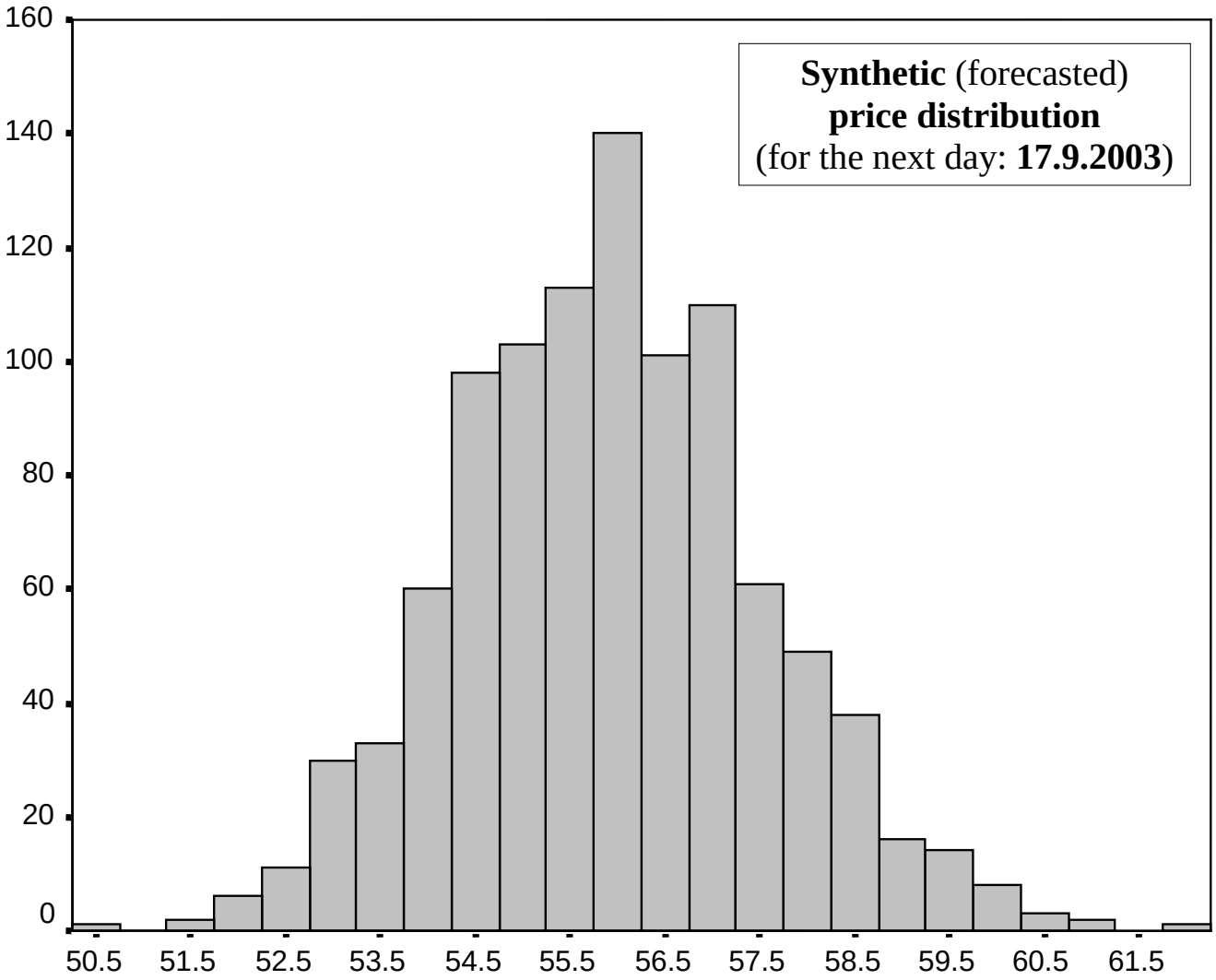
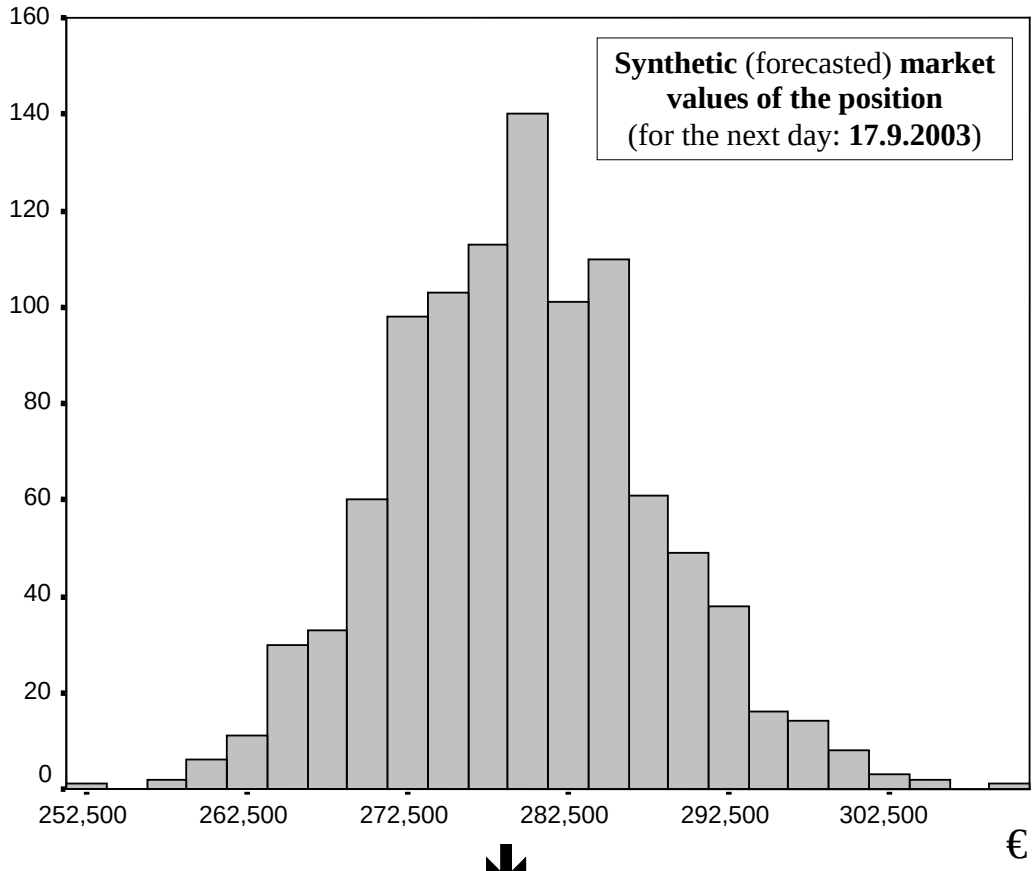
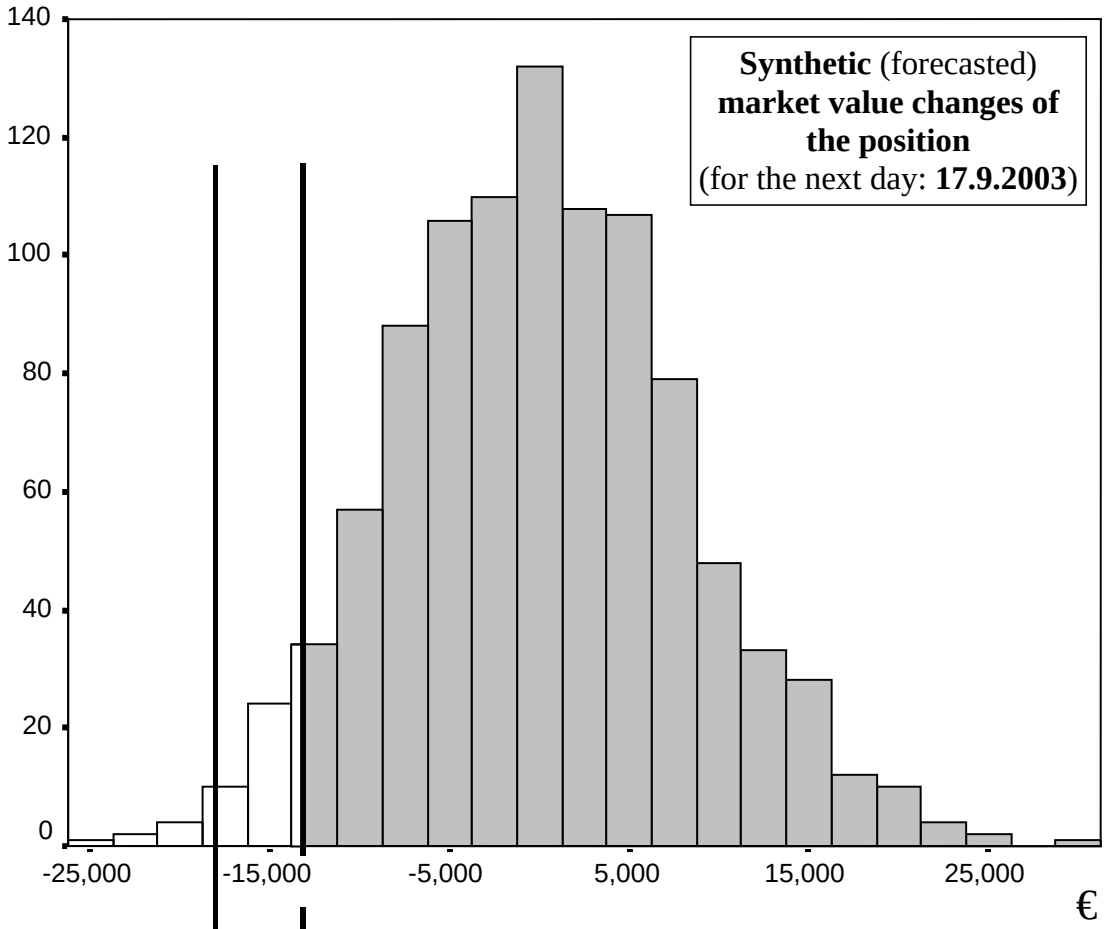

Daily returns of the risk factor, 26.10.99 - 16.9.03, (N = 1000)
%
10
8
6
4
2
0
-2
-4
-6
-8
-10
26.10.99
26.10.00
26.10.01
26.10.02


Synthetic (forecasted)
price distribution
(for the next day: **17.9.2003**)
160
140
120
100
80
60
40
20
0
50.5
51.5
52.5
53.5
54.5
55.5
56.5
57.5
58.5
59.5
60.5
61.5
Siemens





VaR(0.01) = 17,410.71

VaR(0.05) = 13,190.96

Scaling of Volatilities

- When assuming a normal distribution:

  Variance growths linear with time

  Standard deviation growths with the square root of time

- Volatility transformation:

$$\sigma_{p.a.} = \sqrt{250} \cdot \sigma_{p.d.}$$

$$\sigma_{p.a.} = \sqrt{52} \cdot \sigma_{p.w.}$$

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3.4 Relationship between return time series

➔ Characterization via covariance and correlation

- **Covariance** is a measure of the degree to which **two variables move together** in a systematic or predictable way, either positively or negatively.

- **Covariance** for 2 return time series A and B:

$$\mathrm{Cov}(R_A, R_B) = \frac{1}{T-1} \cdot \sum_{t=1}^{T} (R_{A,t} - \overline{R}_A) \cdot (R_{B,t} - \overline{R}_B)$$

$R_{A,t}$ Return of stock A in period t
$R_{B,t}$ Return of stock A in period t
N Number of periods

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MS-Excel: COVAR(starting cell:ending cell)

- **Correlation**:

Correlation is a **standardized version** of covariance where values range from -1 (perfect negative covariance) to +1 (perfect positive covariance):

$$\rho(R_A, R_B) = \frac{Cov(R_A, R_B)}{\sigma_A \cdot \sigma_B}$$

$\rho(R_A,R_B)$	.....	Correlation coefficient between *A* and *B*
σ_A	.....	Standard deviation of return time series *A*
σ_B	.....	Standard deviation of return time series *B*

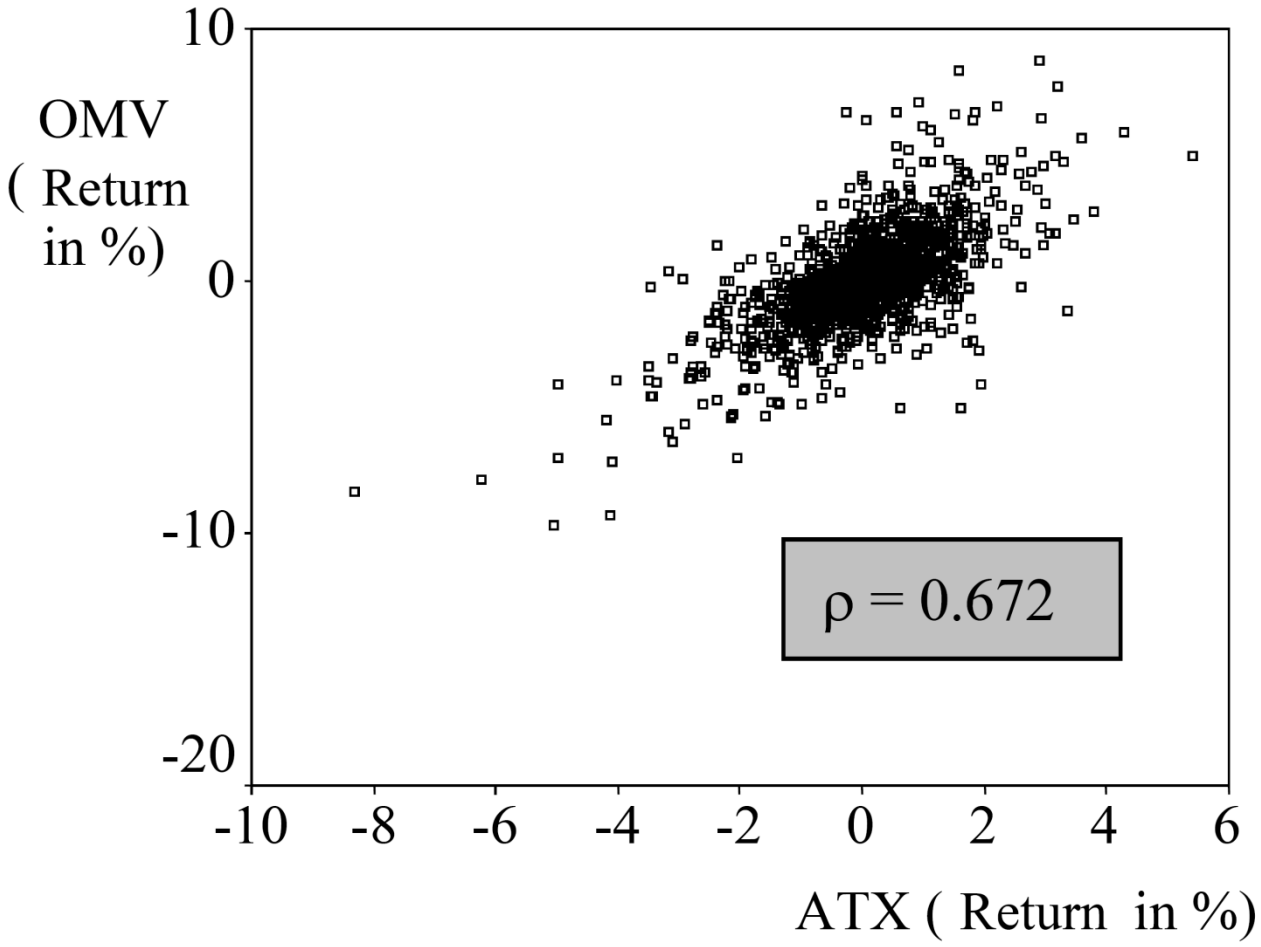
MS-Excel: CORR(starting cell:ending cell)

Correlation between real stock returns: ~ +0.1 - +0.9

- **Positively** correlated assets tend to **move** up and down **together**, while **negatively** correlated assets tend to **move** in **opposite directions**.

- If two assets are completely **independent**, showing no systematic relationship, their correlation and covariance are **zero**.

Example: OMV versus ATX (1.1.1995 to 16.3.2001) (Scatterplot)



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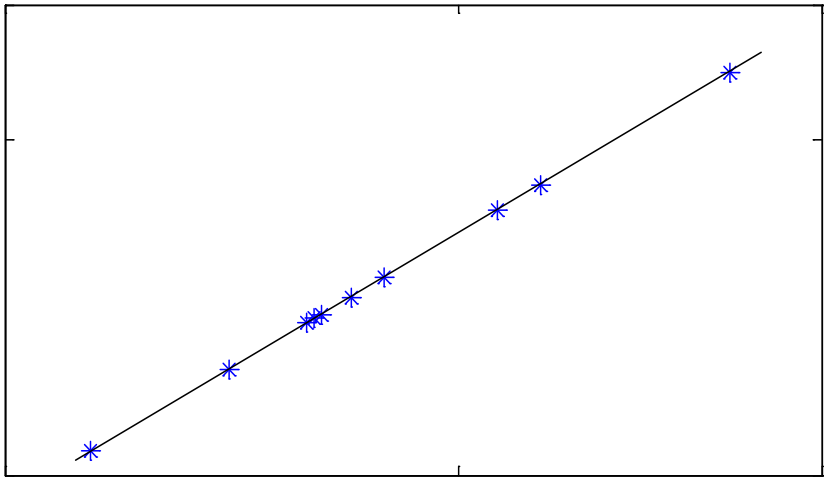
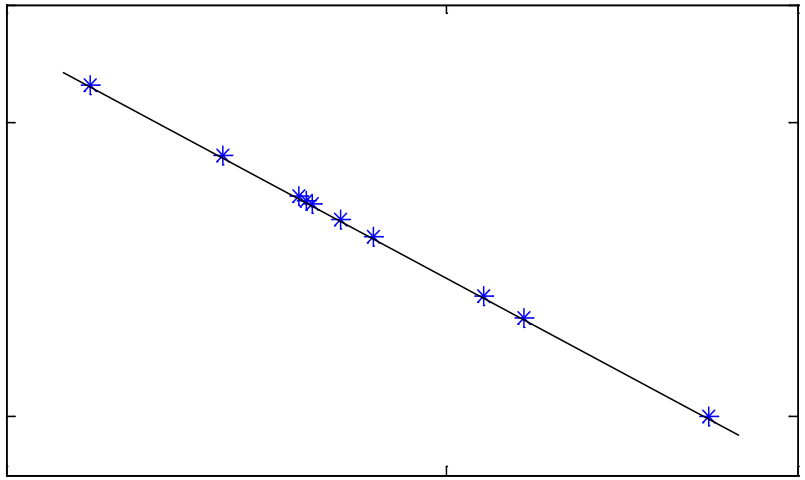
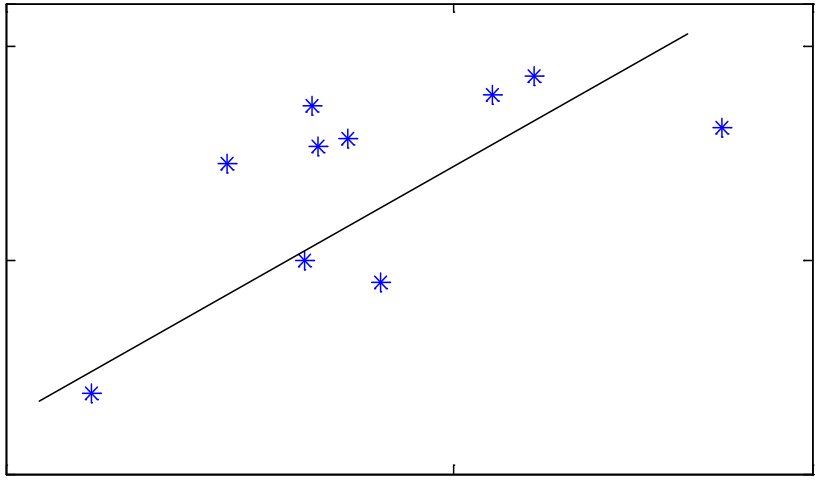
perfect positive correlation (ρ = 1)



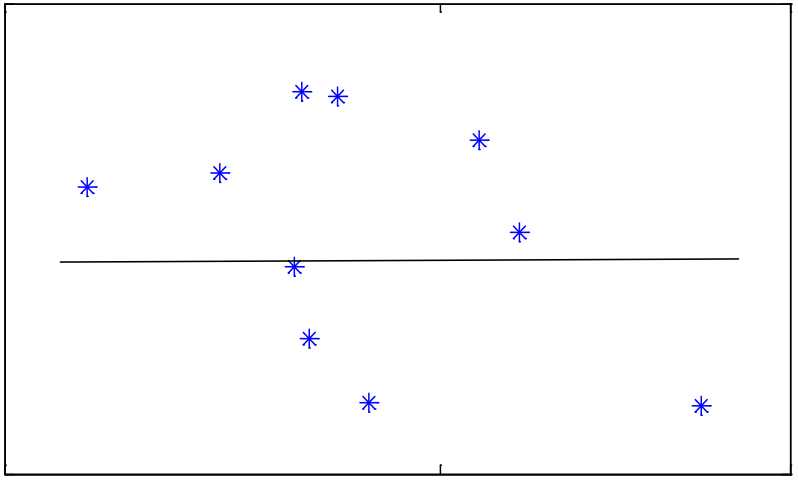
perfect negative correlation (ρ = -1)



imperfect positive correlation (ρ = 0.5)



zero correlation



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4. Basics of Modern Portfolio Theory

4.1 Portfolio Selection of Markowitz

The portfolio selection model of Markowitz (1952) is often viewed as the origin of modern portfolio theory.[11]

Markowitz used for the first time the return variance as risk measure for an asset portfolio.

[11] Markowitz, H., 1952, Portfolio Selection, Journal of Finance, Vol. 7, No. 1, S. 77-91.

4.1.1 Risk Return Profile

Expected return and **risk** are the basis for any investment decision.

Question: In **which assets** should one invest? The answer depends on:

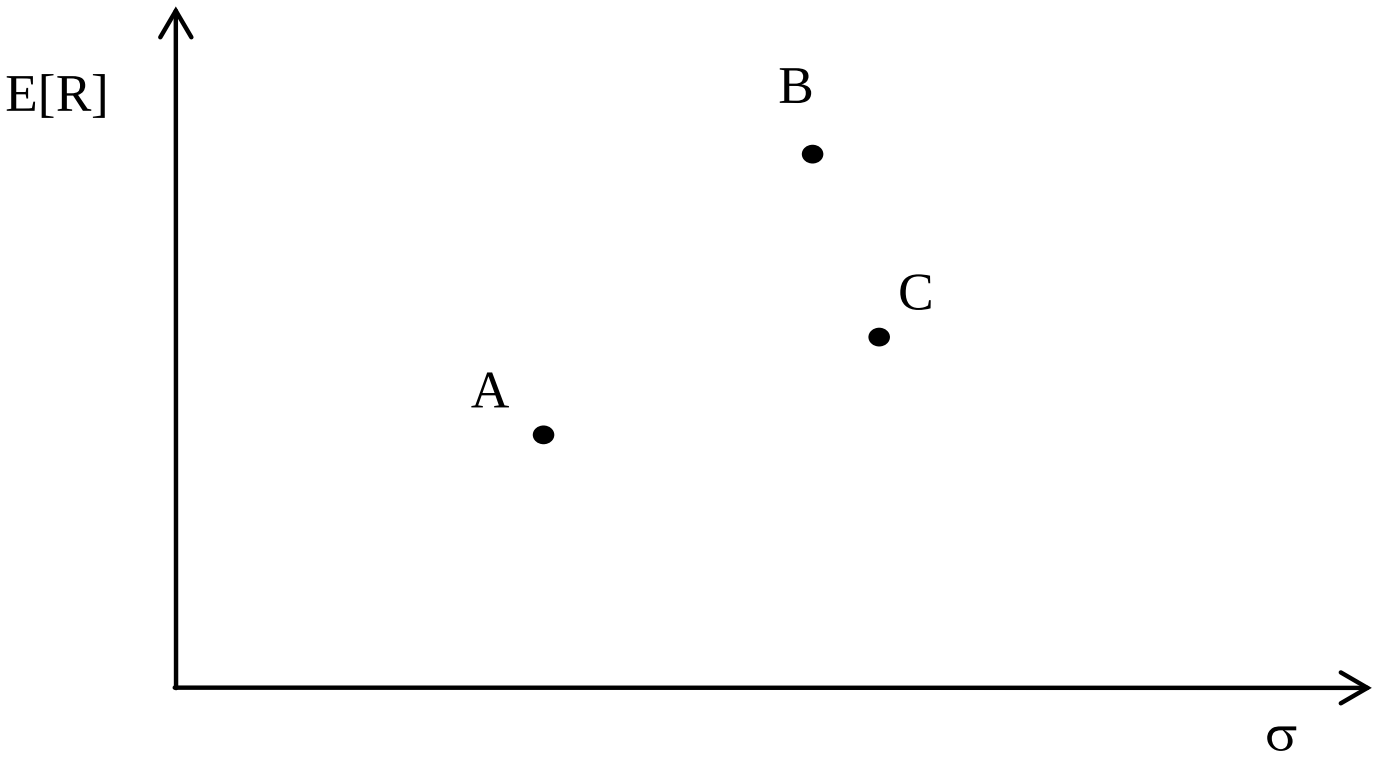
- Characteristics of the securities (E[R], σ, ρ)
- Risk preferences of the investor

A rational investor chooses:

- If stocks are equally risky: The one with the **highest E[R]**
- If stocks have equally E[R]: The one with the **lowest risk**

➔ Implicit assumption: **Risk aversion**

Risk return profile of assets A, B, and C



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Example

The table “TR Index“ in the MS-Excel file *MPT2019_PEF.xlsx* contains daily total return indices of 5 data sets for the 25 year period 21.2.1994 to 21.2.2019:

(1) BMW (Bayerische Motoren Werke) AG
(2) E.ON AG
(3) Münchner Rückversicherungs AG
(4) Siemens AG
(5) DAX 30

(a) Calculate (discrete) daily returns for all 5 time series.

(b) Calculate (arithmetic) mean return and return standard deviation on (i) daily and (ii) yearly basis.

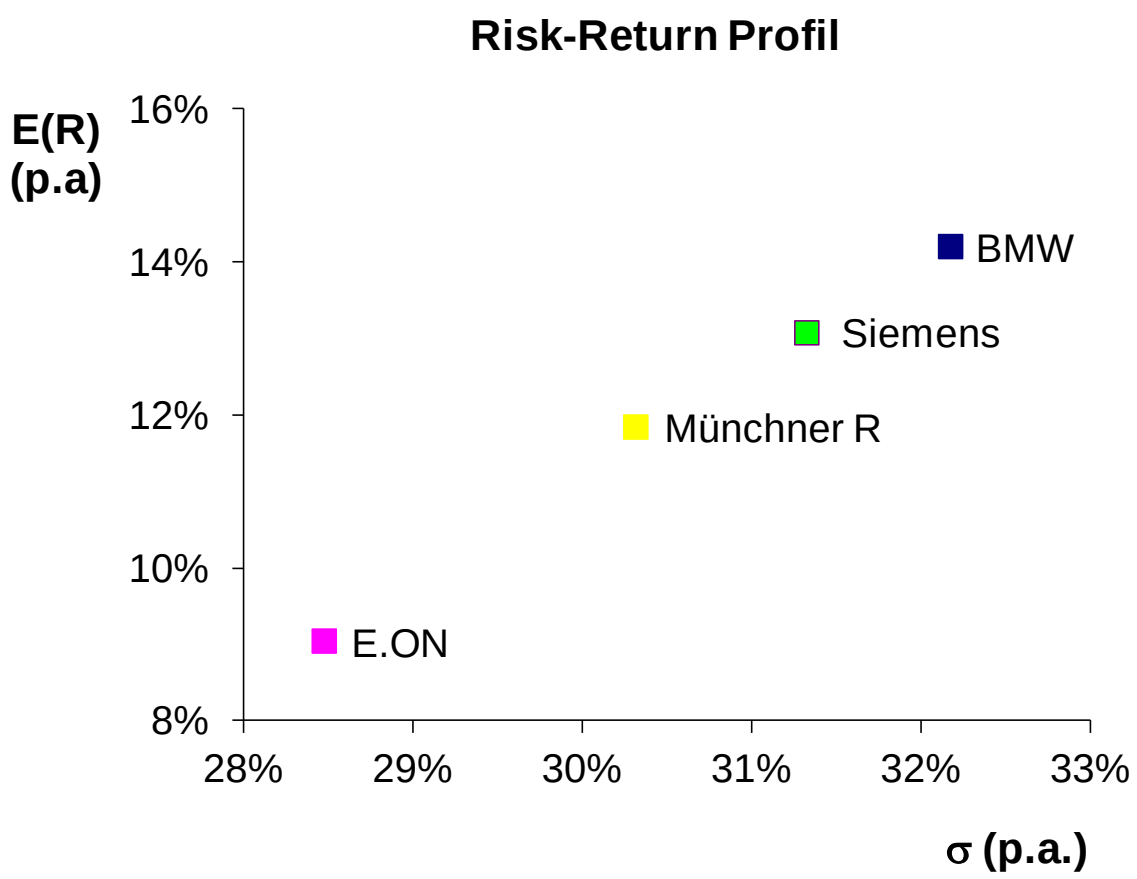
Sample mean p.a = N * sample mean p.d. (N=261 trading days p.a.)

Standard deviation p.a. = $\sqrt{N}$ * standard deviation p.d.

(c) Compute covariances (p.d. and p.a.) and correlations between the 5 time series.

Covariance p.a. = N * covariance p.d.

For (a) and (b) one gets:



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For (c) one gets:

Correlations	BMW	E.ON	Münchner R	Siemens	DAX
BMW		0.41	0.47	0.52	**0.67**
E.ON			0.43	0.44	**0.60**
Münchner R				0.50	**0.68**
Siemens					**0.75**

Covariances p.a.	BMW	E.ON	Münchner R	Siemens	DAX
BMW	**0.1035**	0.0373	0.0454	0.0529	0.0495
E.ON	0.0373	**0.0811**	0.0374	0.0396	0.0390
Münchner R	0.0454	0.0374	**0.0919**	0.0476	0.0471
Siemens	0.0529	0.0396	0.0476	**0.0982**	0.0538
DAX	0.0495	0.0390	0.0471	0.0538	**0.0523**

4.1.2 Risk preferences

Different investors value the relationship between risk and return differently. We can distinguish between:

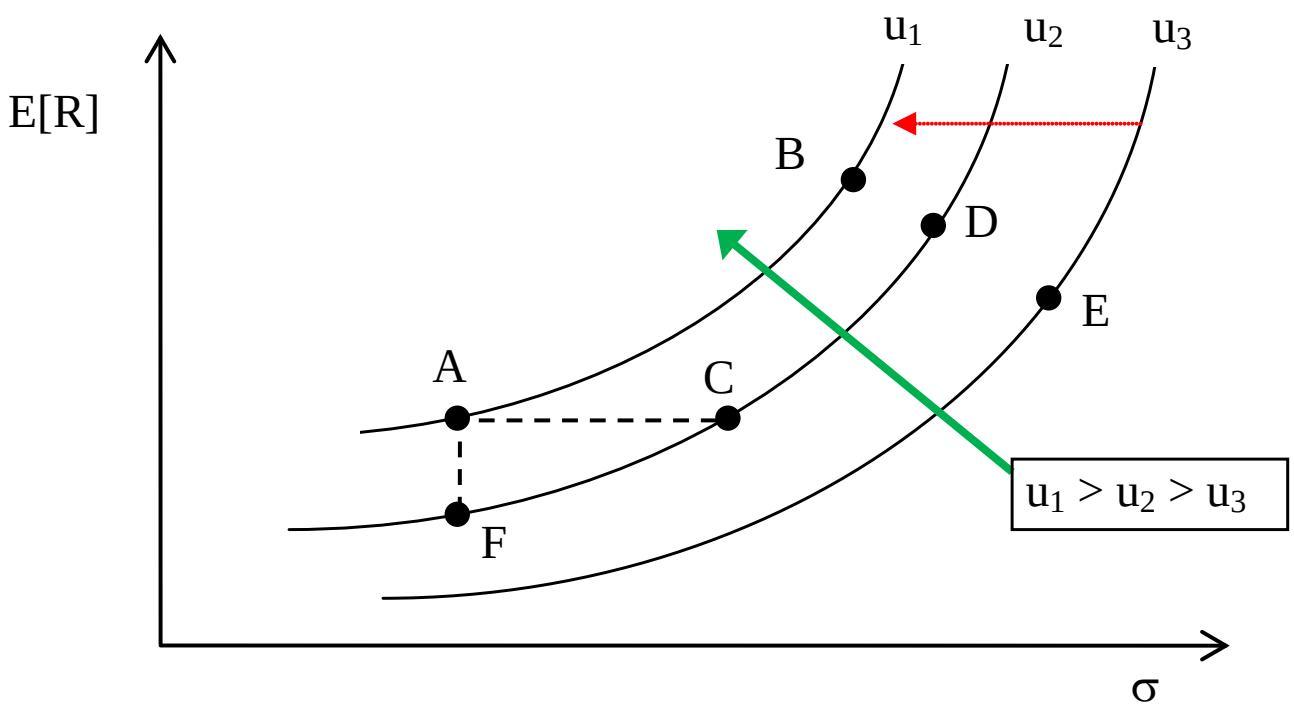
(a) risk aversion

(b) risk neutrality

(c) risk seeking behavior

(a) Risk aversion

Starting point: Each investor tries to **maximize** his/her **utility**.



Indifference curve: Portfolio combinations with equal utility
Dominance considerations: A is better than C; A is better than F

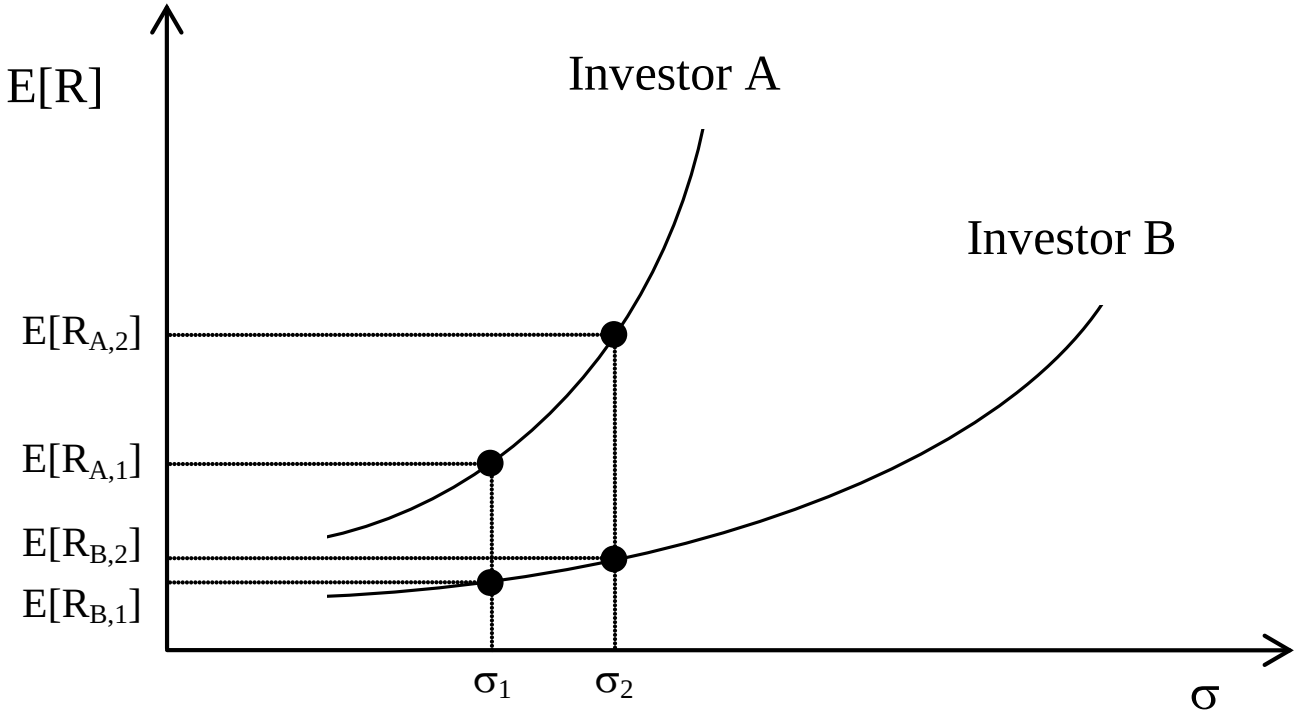
An indifference curve is defined as all portfolio combinations that generate for a particular investor the same utility.

Thus, portfolios *A* and *B* generate the same utility (u_1) and the corresponding investor is indifferent between them.

But portfolios *A* and *C* are not equally ranked. *A* has the same expected return as *C* but lower risk. The investor will, therefore, prefer *A* to *C*.

For a risk averse investor the utility increases along the green arrow. Portfolios B, D, E: Portfolio *B* generates for the corresponding (risk averse) investor the highest utility level and has, therefore, a higher ranking compared to *D* or *E*.

Different levels of risk aversion



Investor A has a higher risk aversion than investor B.

A requires for bearing the same amount of additional risk (σ_2 - σ_1) a higher extra return (or risk premium). ➔ Steeper indifference curve

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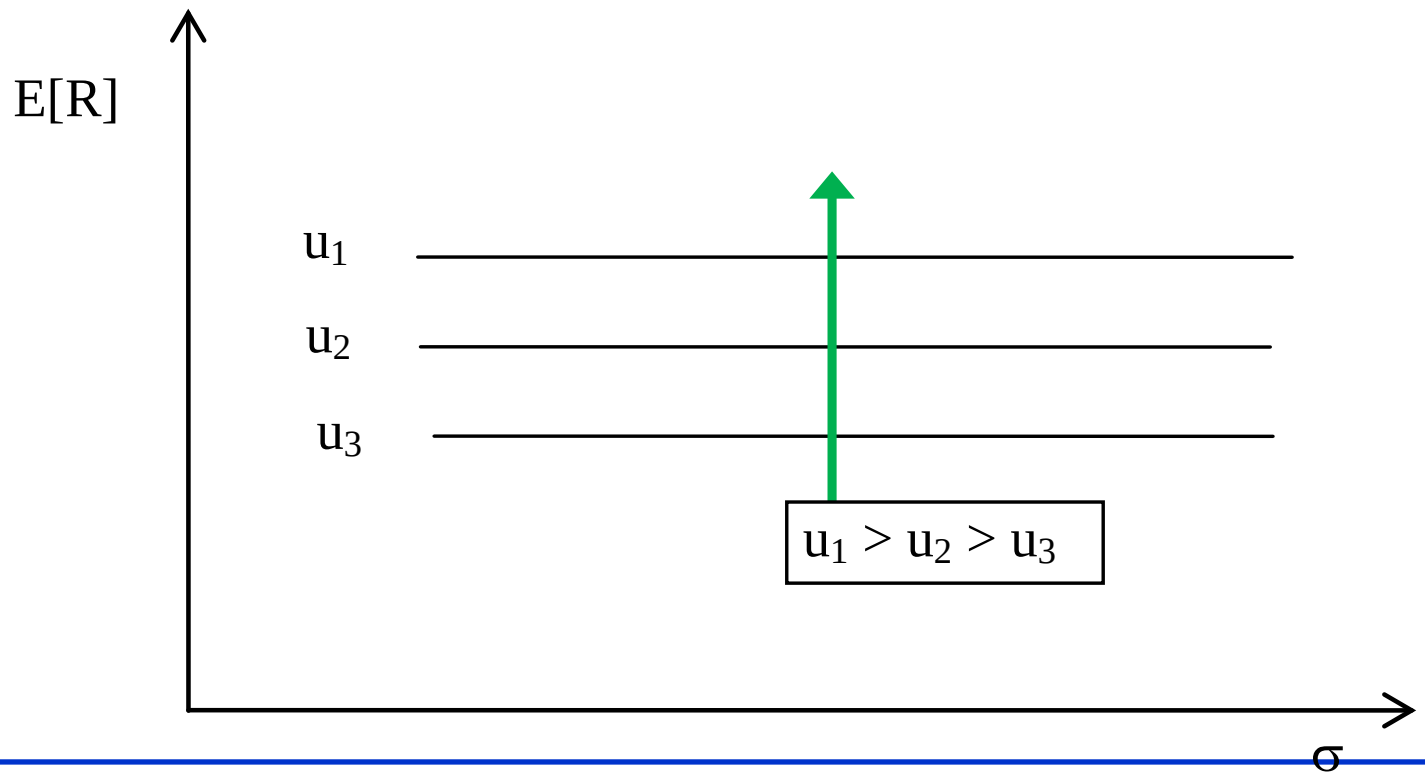
(b) Risk neutrality

Risk neutrality is characterized by the fact that the utility only increases or decrease when the expected return increases or decreases. The corresponding level of risk does not matter for a risk neutral investor.



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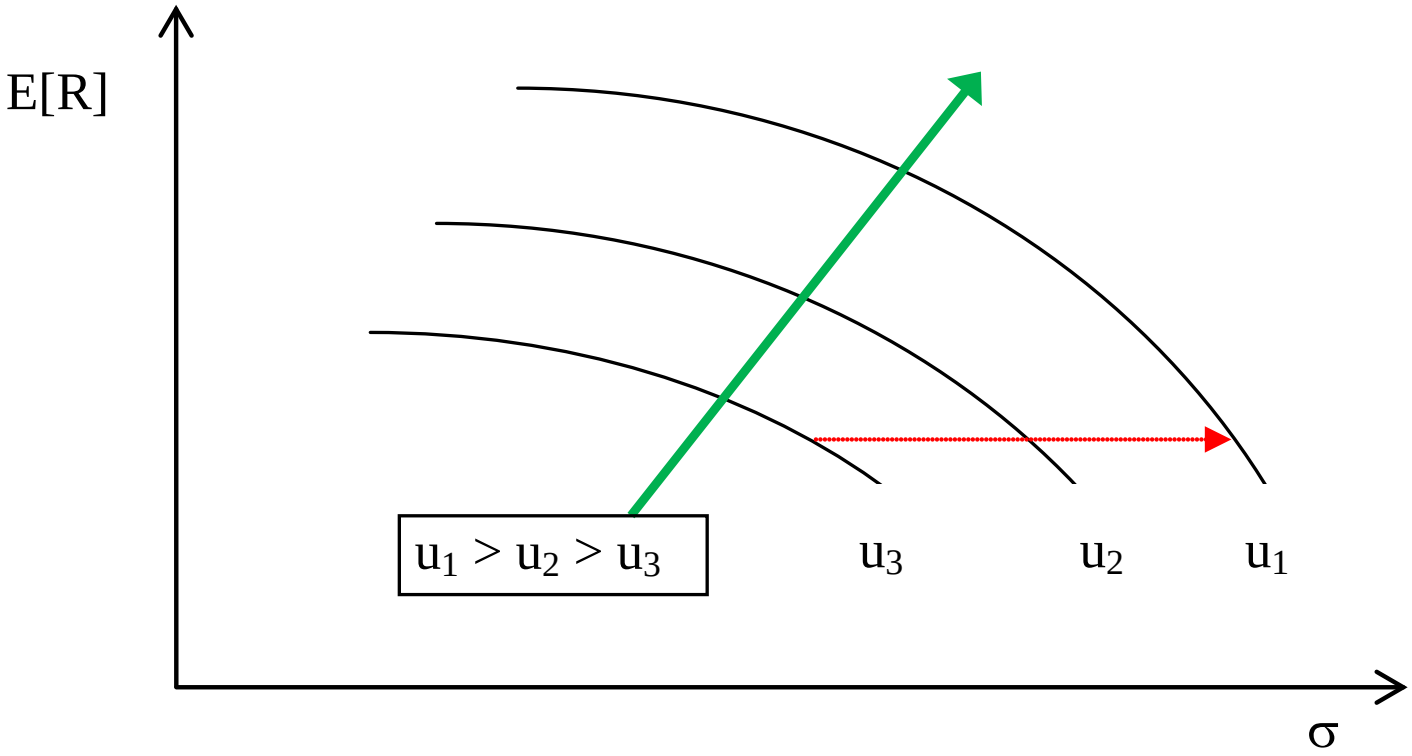
(c) Risk seeking behavior

A **risk seeking** investor is special, as the utility increases for the same expected return but a higher risk level. He/she weights scenarios on the right side (up-side) of the return distribution more than the left side (downside).



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❍ An example for risk aversion

A risk-averse person is not willing to take part in a game where she/he can win and lose money with an expected value (to win/lose money) of zero.

Example: Flipping a coin

- Head (50% probability): + €100,000.-
- Tail (50% probability): - €100,000.-

Expected profit = 0.5·100,000 + 0.5·(-100,000) = **0**

➔ **Risk-averse** persons do not like this: they **weight negative outcomes more**

➔ **Risk neutral** persons: they are **indifferent**

➔ **Risk lovers**: they like this game and **weight positive outcomes more**

❍ Risk aversion (1)

- We **usually** need to take **more risk** in order to get **more return**, so there must be some **trade-off** between risk and return.
- **Risk aversion** means that for investments having the **same expected return**, **lower volatility** is **preferred** to higher volatility.
- **Most people** are risk-averse, at least concerning large sums of money.
- **Higher levels** of risk aversion lead to larger proportions in **risk-free** assets.
- **Lower levels** of risk aversion lead to larger proportions in the **risky asset** portfolio.
- **Theories of capital markets** assume (normally) that investors are **risk-averse**.

❍ Risk Aversion (2)

How can we **measure** risk aversion?

One often used way to represent risk aversion is via a **quadratic utility function**:

$$U(R) = E(R) - 0.5 \cdot A \cdot \sigma^2$$

- **More** expected **return** (E[R]) is always **better**
- **The utility** decreases with more risk (σ)
- "***A***" is the **risk aversion parameter**: the <u>larger</u> "***A***" the <u>more return</u> will be <u>required</u> for a particular risk level (higher risk aversion)
- Many studies have concluded that investors' average risk aversion is between $A = 2$ and $A = 4$

Example: Risk aversion

- Use the 4 individual stocks in *MPT2019_PEF.xlsx* and assume a **risk aversion parameter *A* of *2***.

- Compute the **utility values** for the return series. **Which** stock would you **choose** in this case?

You get (see *MPT2019_PEF.xlsx*):

	BMW	E.ON	Münchner R	Siemens
Utility (A=2)	**0.0384**	0.0091	0.0263	0.0324

4.1.3 Diversification

An investor has typically the **possibility** to **invest** in **several assets** at the same time.

He/she can choose individual assets or can form a **portfolio** of several assets.

If a portfolio of several assets is formed we have to solve the problem of how the portfolio **components** should be **weight**.

Are all possible combinations efficient?

The simplest approach is the **2-risky-asset approach**.

Example:

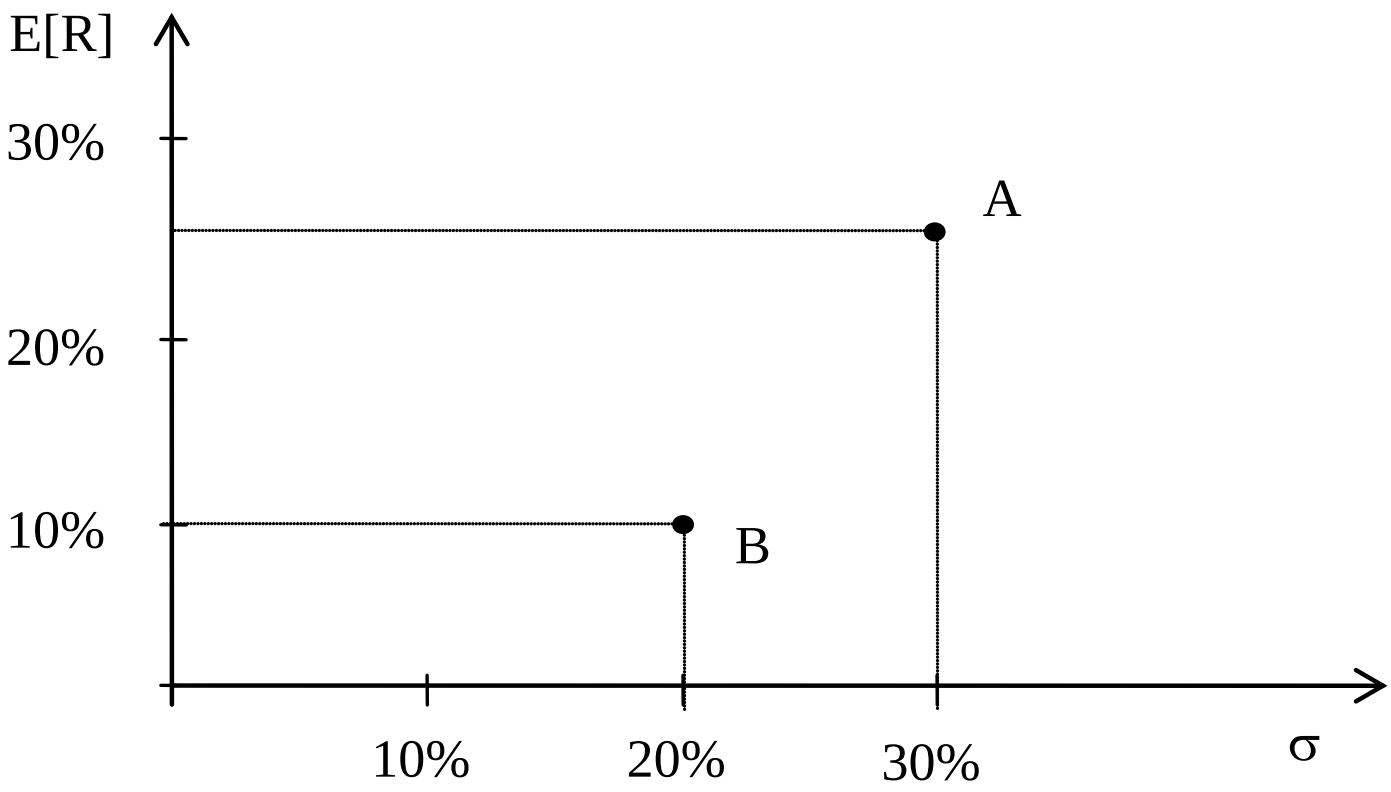
Let us **assume** that an investor can choose between asset A and B with the following characteristics:

Asset A: $E[R_A] = 25\%$, $\sigma_A = 30\%$

Asset B: $E[R_B] = 10\%$, $\sigma_A = 20\%$

Summary of notations:

- $E[R_i]$ expected return of asset i
- σ_i volatility (= standard deviation of returns) of asset i
- $\rho_{i,j}$ correlation between assets i and j
- x_i portfolio weight of asset i
- R_P portfolio return
- $E[R_P]$ expected portfolio return
- σ_P portfolio volatility (risk)



If, **e.g.**, the investor **chooses** a weight of **70% for A** and **30% for B**, the expected return of this portfolio easily can be calculated as:

$$E[R_P] = 0.7 \cdot E[R_A] + 0.3 \cdot E[R_B] = 0.7 \cdot 25 + 0.3 \cdot 15 = 22\%$$

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Without fixed weights:

$$E[R_P] = x_A \cdot E[R_A] + x_B \cdot E[R_B]$$

x_A and x_B are the weights invested in asset A and B, respectively.

Furthermore, it is assumed that all weights sum up to 1 (or 100%). Thus, the investor is investing 100% of a particular amount of money, not more and not less:

$$x_A + x_B = 1$$

In general:

The **expected return of a portfolio** (consisting of N individual securities) is a **linear combination** of the expected returns of the individual components in that portfolio:

$$E[R_P] = \sum_{i=1}^{N} x_i \cdot E[R_i]$$

with

$$\sum_{i=1}^{N} x_i = 1$$

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Variance of the **portfolio** consisting of assets A and B:

$$\sigma_P^2 = E[R_P - E(R_P)]^2 = E[x_A \cdot R_A + x_B \cdot R_B - (x_A \cdot E(R_A) + x_B \cdot E(R_B))]^2$$

$$\sigma_P^2 = x_A^2 \cdot E[R_A - E(R_A)]^2 + x_B^2 \cdot E[R_B - E(R_B)]^2 + 2 \cdot x_A \cdot x_B \cdot E[(R_A - E(R_A)) \cdot (R_B - E(R_B))]$$

$$\sigma_P^2 = x_A^2 \cdot \sigma_A^2 + x_B^2 \cdot \sigma_B^2 + \underline{2 \cdot x_A \cdot x_B \cdot Cov(R_A, R_B)}$$

Thus, the **covariance** between the portfolio components is an **additional important determinat** for the portfolio risk.

The covariance is defined as: $Cov(R_A, R_B) = \rho_{A,B} \cdot \sigma_A \cdot \sigma_B$

The volatility of the portfolio is the square root of the portfolio variance:

$$\sigma_P = \sqrt{x_A^2 \cdot \sigma_A^2 + x_B^2 \cdot \sigma_B^2 + 2 \cdot x_A \cdot x_B \cdot \rho_{A,B} \cdot \sigma_A \cdot \sigma_B}$$

3 important cases:

$\rho_{A,B} = +1$:

$$\sigma_P = \sqrt{x_A^2 \cdot \sigma_A^2 + x_B^2 \cdot \sigma_B^2 + 2 \cdot x_A \cdot x_B \cdot \sigma_A \cdot \sigma_B}$$
$$= \sqrt{(x_A \cdot \sigma_A + x_B \cdot \sigma_B)^2}$$
$$= x_A \cdot \sigma_A + x_B \cdot \sigma_B$$

$\rho_{A,B} = -1$:

$$\sigma_P = \sqrt{x_A^2 \cdot \sigma_A^2 + x_B^2 \cdot \sigma_B^2 - 2 \cdot x_A \cdot x_B \cdot \sigma_A \cdot \sigma_B}$$
$$= \sqrt{(x_A \cdot \sigma_A - x_B \cdot \sigma_B)^2}$$
$$= x_A \cdot \sigma_A - x_B \cdot \sigma_B$$

$\rho_{A,B} < +1$:

$$\sigma_P = \sqrt{x_A^2 \cdot \sigma_A^2 + x_B^2 \cdot \sigma_B^2 + 2 \cdot x_A \cdot x_B \cdot \rho_{A,B} \cdot \sigma_A \cdot \sigma_B}$$
$$< (x_A \cdot \sigma_A + x_B \cdot \sigma_B) = \sigma_{P,\rho_{A,B}=+1}$$

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Fundamental proposition:

Different to expected portfolio returns, the return standard deviation of a portfolio is only a linear combination of σ_A and σ_B, if the correlation between A and B is exactly +1 ($\boldsymbol{\rho_{A,B} = +1}$).

On the other hand, if $\boldsymbol{\rho_{A,B} < +1}$, a **diversification effect** takes place:

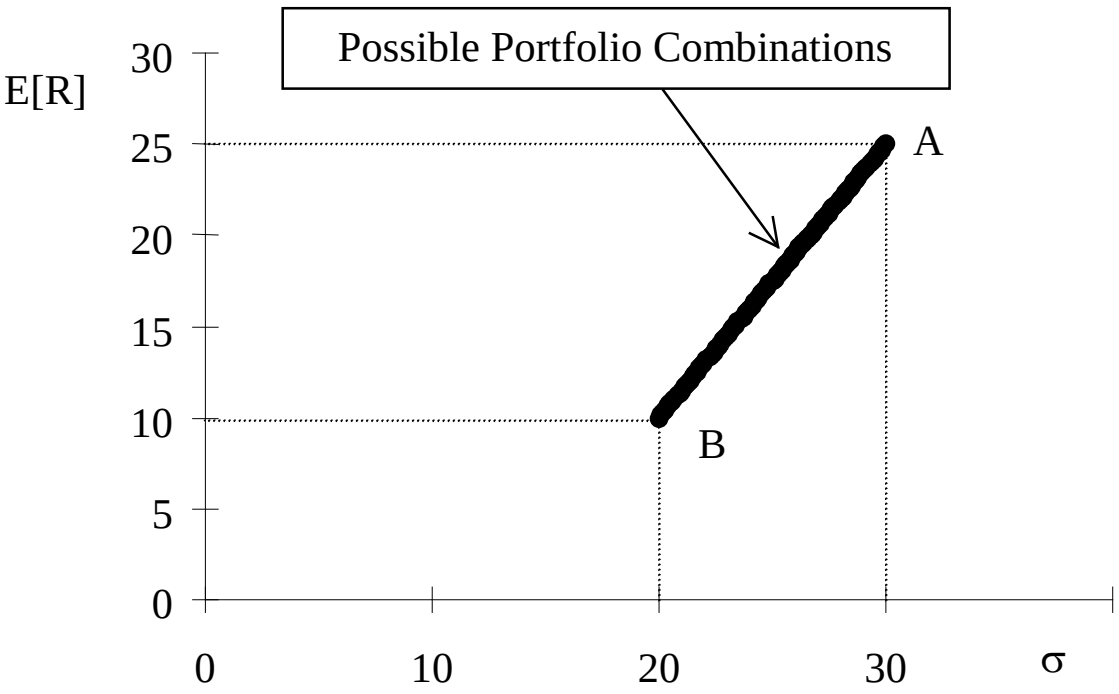
Some **risks** can be **eliminated** by generating a **portfolio**.

In the special case of $\boldsymbol{\rho_{A,B} = -1}$, a combination of A and B (i.e. weights x_A and x_B) exists that eliminates the portfolio risk ($\sigma_P = 0$).

Example (continued)

Case A: $\rho_{A,B} = +1$

x_A	x_B	$\rho_{A,B}$	σ_P	$E[R_P]$
1	0	1.0	30.00	25.00
0.9	0.1	1.0	29.00	23.50
0.8	0.2	1.0	28.00	22.00
0.7	0.3	1.0	27.00	20.50
0.6	0.4	1.0	26.00	19.00
0.5	0.5	1.0	25.00	17.50
0.4	0.6	1.0	24.00	16.00
0.3	0.7	1.0	23.00	14.50
0.2	0.8	1.0	22.00	13.00
0.1	0.9	1.0	21.00	11.50
0	1	1.0	20.00	10.00



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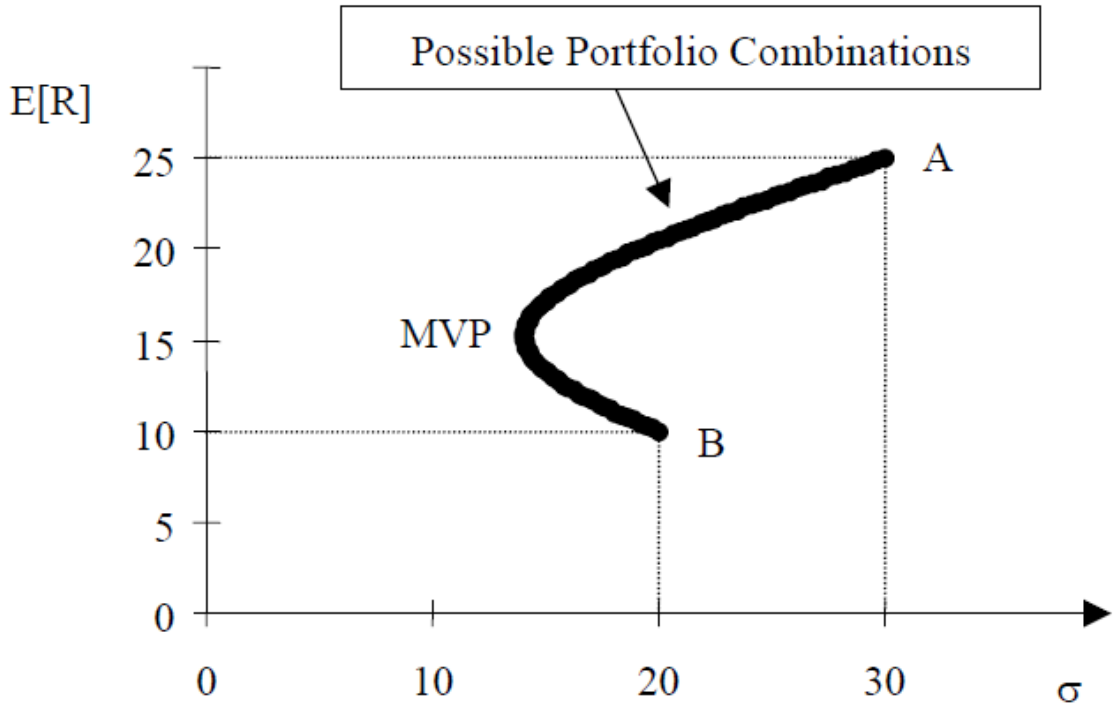
Case B: $\rho_{A,B} = -0.3$

x_A	x_B	$\rho_{A,B}$	σ_P	$E[R_P]$
1	0	-0.3	30.00	25.00
0.9	0.1	-0.3	26.47	23.50
0.8	0.2	-0.3	23.12	22.00
0.7	0.3	-0.3	20.03	20.50
0.6	0.4	-0.3	17.37	19.00
0.5	0.5	-0.3	15.33	17.50
0.4	**0.6**	**-0.3**	**14.20**	**16.00**
0.3	**0.7**	**-0.3**	**14.19**	**14.50**
0.2	0.8	-0.3	15.31	13.00
0.1	0.9	-0.3	17.34	11.50
0	1	-0.3	20.00	10.00



Minimum variance portfolio (**MVP**): Lowest possible portfolio risk

❍ Weights for the MVP:

$$\frac{\partial \sigma_P^2}{\partial x_A} = \frac{\partial\left(x_A^2 \cdot \sigma_A^2 + x_B^2 \cdot \sigma_B^2 + 2 \cdot x_A \cdot x_B \cdot \rho_{A,B} \cdot \sigma_A \cdot \sigma_B\right)}{\partial x_A}$$

$$= \frac{\partial\left(x_A^2 \cdot \sigma_A^2 + (1 - x_A)^2 \cdot \sigma_B^2 + 2 \cdot x_A \cdot (1 - x_A) \cdot \rho_{A,B} \cdot \sigma_A \cdot \sigma_B\right)}{\partial x_A}$$

$$= 2 \cdot x_A \cdot \sigma_A^2 - 2 \cdot \sigma_B^2 + 2 \cdot x_A \cdot \sigma_B^2 - 4 \cdot x_A \cdot \rho_{A,B} \cdot \sigma_A \cdot \sigma_B + 2 \cdot \rho_{A,B} \cdot \sigma_A \cdot \sigma_B = 0$$

$$x_A = \frac{\sigma_B^2 - \rho_{A,B} \cdot \sigma_A \cdot \sigma_B}{\sigma_A^2 + \sigma_B^2 - 2 \cdot \rho_{A,B} \cdot \sigma_A \cdot \sigma_B}$$

$x_B = 1 - x_A$

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MVP weights for assets A and B:

$$x_A = \frac{0.2^2 - (-0.3) \cdot 0.3 \cdot 0.2}{0.3^2 + 0.2^2 - 2 \cdot (-0.3) \cdot 0.3 \cdot 0.2} = 0.3494$$

$$x_B = 1 - x_A = 1 - 0.3494 = 0.6506$$

Portfolio standard deviation of the MVP:

$$\sigma_P = \sqrt{(0.3494)^2 \cdot (0.3)^2 + (0.6506)^2 \cdot (0.2)^2 + 2 \cdot 0.3494 \cdot 0.6506 \cdot (-0.3) \cdot 0.3 \cdot 0.2}$$
$$= 14.048\%$$

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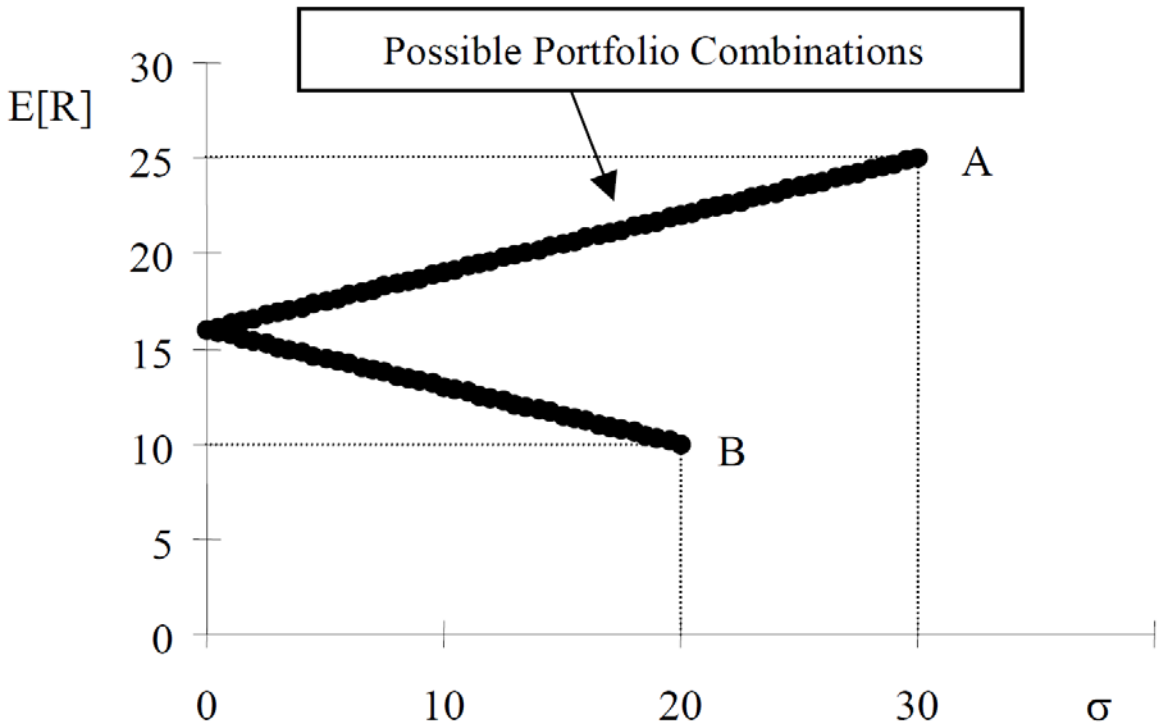
Case C: $\rho_{A,B} = -1$

x_A	x_B	$\rho_{A,B}$	σ_P	$E[R_P]$
1	0	-1.0	30.00	25.00
0.9	0.1	-1.0	25.00	23.50
0.8	0.2	-1.0	20.00	22.00
0.7	0.3	-1.0	15.00	20.50
0.6	0.4	-1.0	10.00	19.00
0.5	0.5	-1.0	5.00	17.50
0.4	**0.6**	**-1.0**	**0.00**	**16.00**
0.3	0.7	-1.0	5.00	14.50
0.2	0.8	-1.0	10.00	13.00
0.1	0.9	-1.0	15.00	11.50
0	1	-1.0	20.00	10.00



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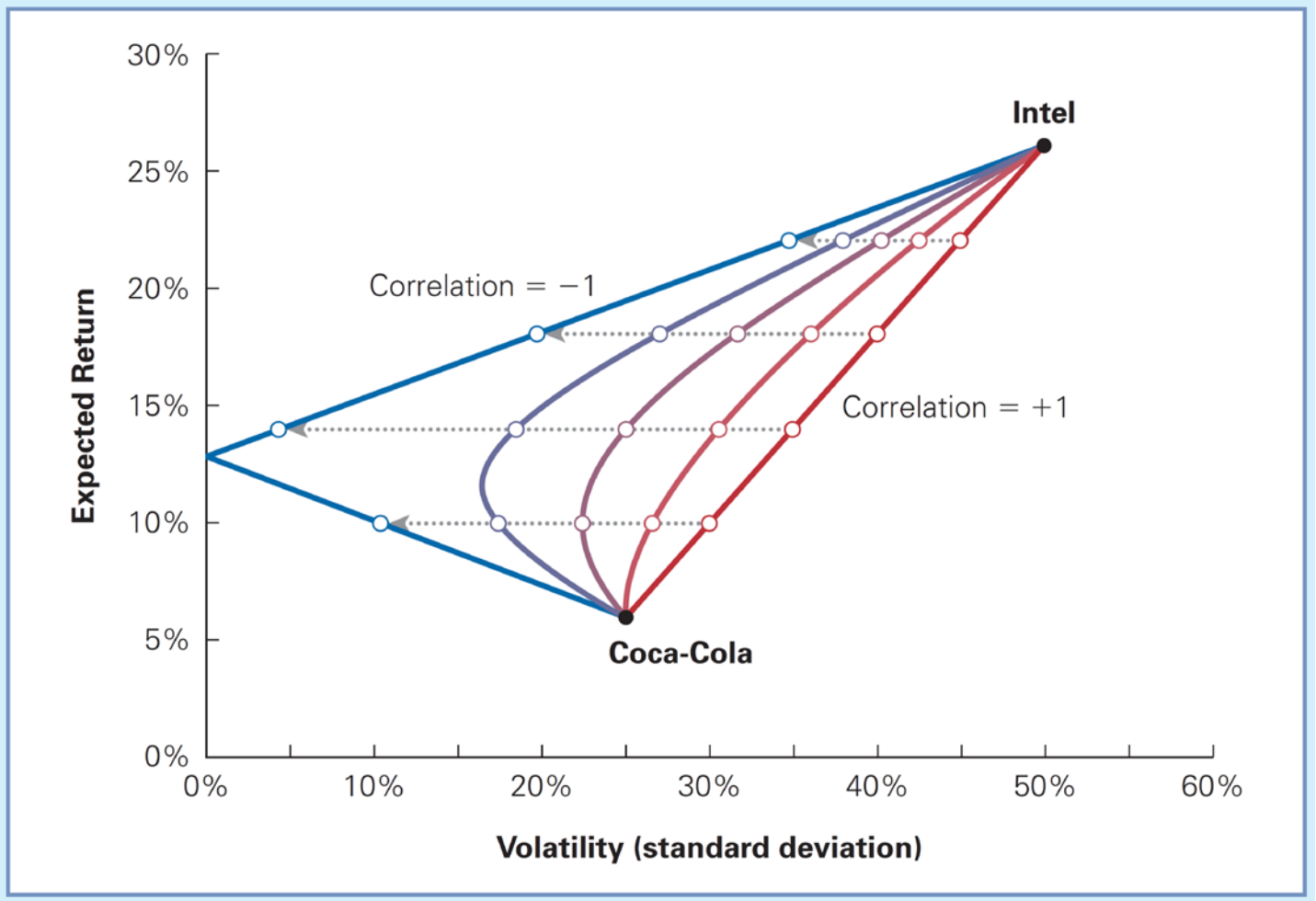
Changing the correlation: Effect on volatility and expected return[12]



[12] Source: Berk and DeMarzo (2014), Corporate Finance, 3rd edition, p. 366.

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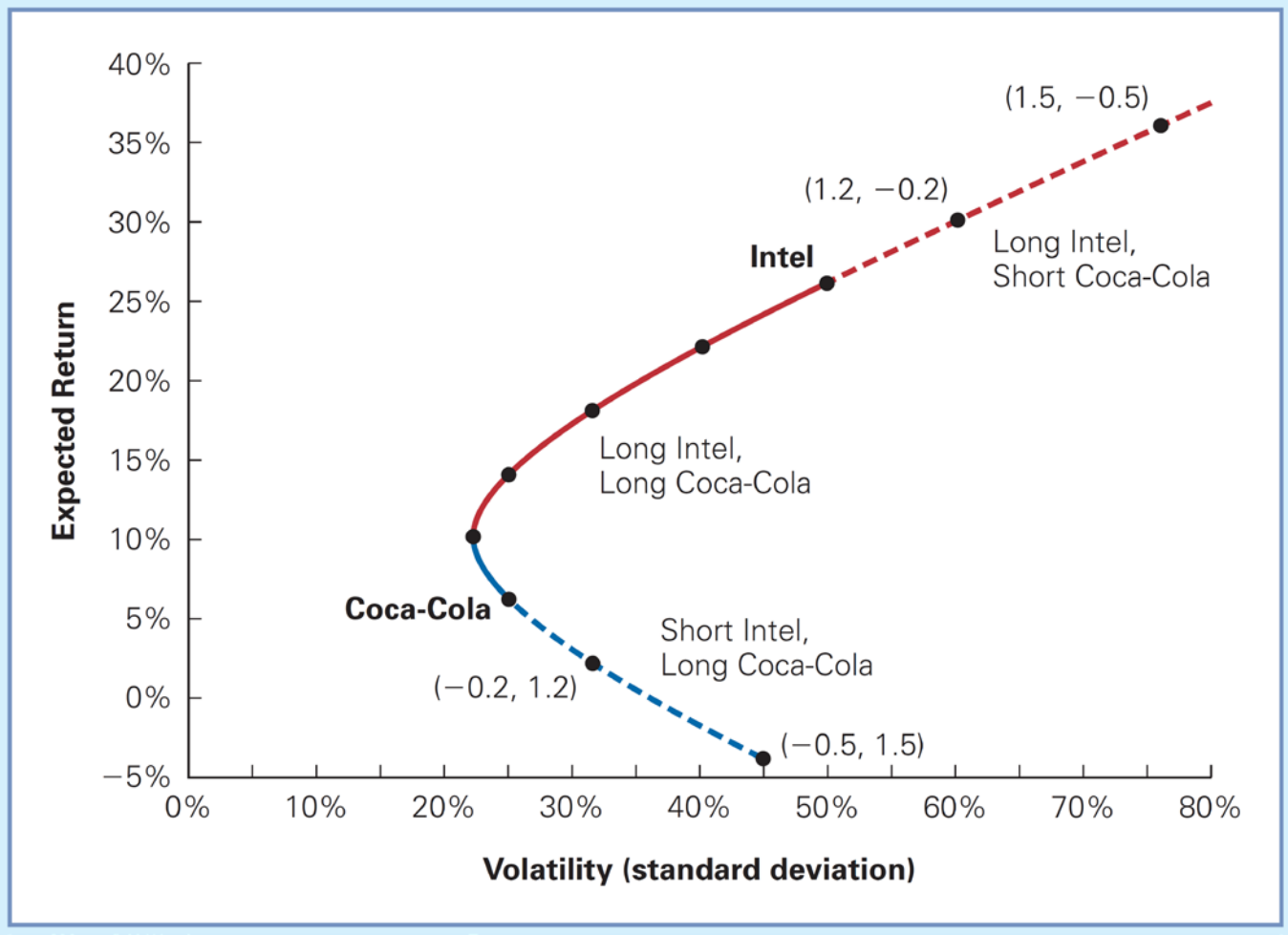
When short sales are allowed[13]



[13] Source: Berk and DeMarzo (2014), Corporate Finance, 3rd edition, p. 368.

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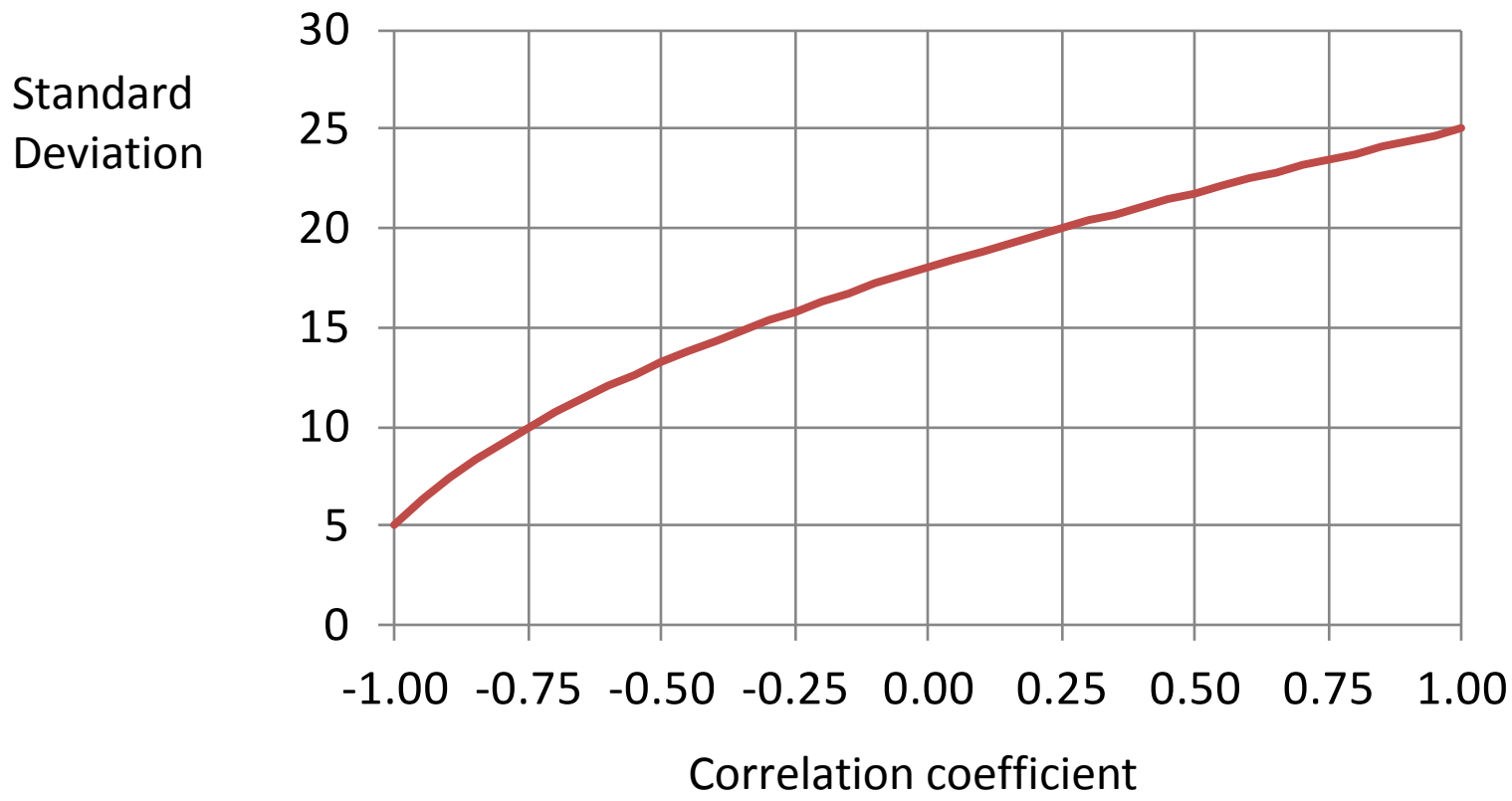
Correlation effect: Portfolio standard deviation as a function of $\rho_{A,B}$ for constant portfolio weights of $\mathbf{x_A = x_B = 0.5}$:



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Example: Portfolio of **E.ON** AG and **BMW** AG

Now let's consider portfolios consisting of **2 stocks**: **E.ON** AG and **BMW** AG (see *MPT2019_PEF.xlsx*).

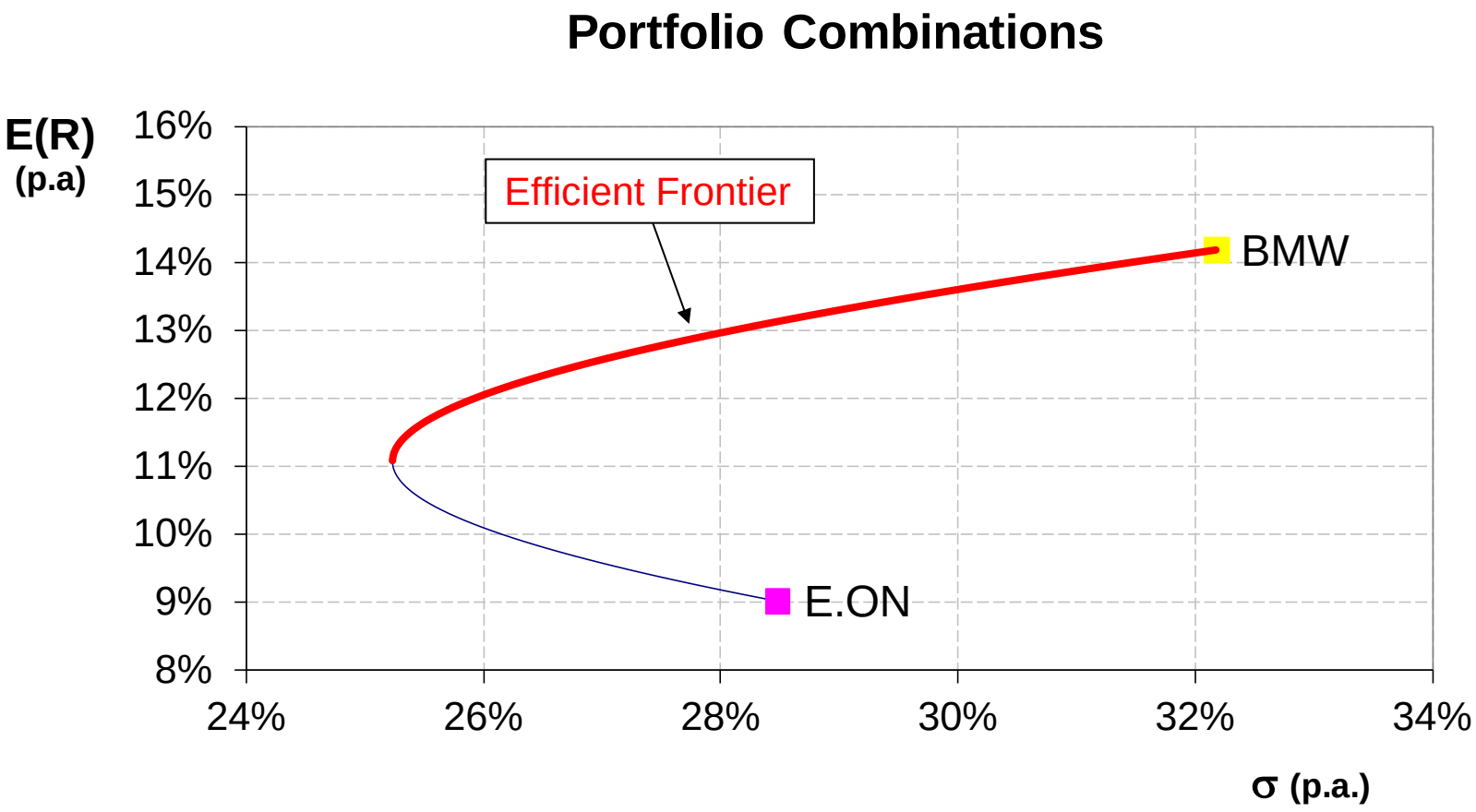
(a) Create a worksheet '**2-Asset PF'** and copy for both stock expected return (p.a.) and standard deviation (p.a.) from the worksheet '**Returns**'.

(b) **Vary portfolio weights** and compute <u>expected return</u> and the <u>standard deviation</u> for all portfolios. Consider only positive portfolio weights (<u>no short selling</u> permitted).

Note: Do not forget the condition $x_1 + x_2 = 1$.
You can account for it by defining: $x_{BMW} = 1 - x_{E.ON}$
Then you have only one weight to vary.

(c) Plot the **investment opportunity set** and specify the **efficient frontier** on the graph.

You get (see *MPT2019_PEF.xlsx*)**:**



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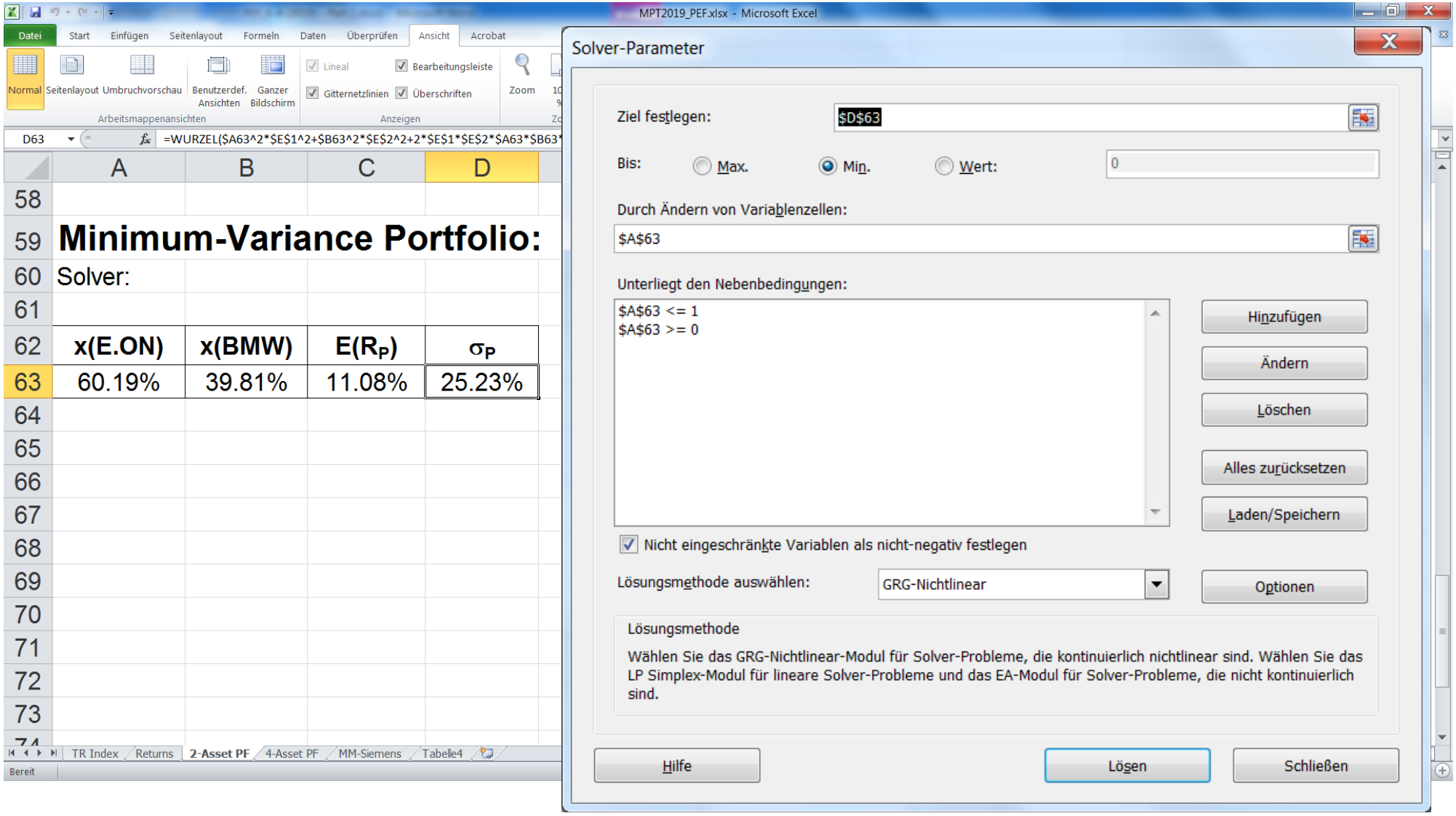
(d) Use the **Solver** to compute the **minimum-variance portfolio (MPV)**. Find the corresponding portfolio weights, as well as the portfolio expected return and risk.

You get:

x(E.ON)	x(BMW)	$E(R_P)$	σ_P
60.19%	39.81%	11.08%	25.23%

❍ Using the Solver for Portfolio Optimization

- Make sure that under Excel-Options/**Add-Ins** the ‘Solver’ is included
- Choose ‘**Solver**’ from the Data menu (if you have not done the previous step, this choice will not appear)
- **Target cell** (*Ziel festlegen*)**: Minimize** the **portfolio risk**
- **Changing cells** (*Durch Änderung von Variablenzellen*)**:** Cell with **unknown portfolio weight**; the second weight is defined as 1 minus the first one
- Add a **constraint** (*Unterliegt den Nebenbedingungen*) that the unknown weight is >=0 and <=1.


MPT2019_PEF.xlsx - Microsoft Excel
D63
=WURZEL($A63^2*$E1^2+B63^2*E2^2+2*E1*E2*$A63*$B63*
Minimum-Variance Portfolio:
Solver:
x(E.ON) | x(BMW) | E(RP) | σP
60.19% | 39.81% | 11.08% | 25.23%
TR Index | Returns | 2-Asset PF | 4-Asset PF | MM-Siemens | Tabelle4
Bereit
Solver-Parameter
Ziel festlegen: D63
Bis: Max. Min. Wert: 0
Durch Ändern von Variablenzellen:
A63
Unterliegt den Nebenbedingungen:
A63 <= 1
A63 >= 0
Hinzufügen
Ändern
Löschen
Alles zurücksetzen
Laden/Speichern
Nicht eingeschränkte Variablen als nicht-negativ festlegen
Lösungsmethode auswählen: GRG-Nichtlinear
Optionen
Lösungsmethode
Wählen Sie das GRG-Nichtlinear-Modul für Solver-Probleme, die kontinuierlich nichtlinear sind. Wählen Sie das LP Simplex-Modul für lineare Solver-Probleme und das EA-Modul für Solver-Probleme, die nicht kontinuierlich sind.
Hilfe
Lösen
Schließen

(e) Compute the optimal portfolio for an investor with a risk aversion parameter A of 0.5, 2, and 4 (use the quadratic utility function: $U(R) = E(R) - 0.5 \cdot A \cdot \sigma^2$)

You get (see *MPT2019_PEF.xlsx*)**:**

Risk Aversion	A	$U(R_P)$	x(E.ON)	x(BMW)	$E(R_P)$	$\sigma(R_P)$
lower	**0.5**	0.11602	0.00%	100.00%	14.19%	32.17%
	2	0.05319	36.00%	64.00%	12.33%	26.48%
higher	**4**	-0.01352	48.00%	52.00%	11.71%	25.55%

4.1.4 Risk of a well diversified portfolio

Consider a portfolio with N assets (stocks) and equal weights: $x_i = \frac{1}{N}$

Adapted variance-covariance matrix as basis for the variance computation:

Stock	1	2		N
1	$x_1^2 \cdot \sigma_1^2$	$x_1 \cdot x_2 \cdot Cov(R_1, R_2)$	…	$x_1 \cdot x_N \cdot Cov(R_1, R_N)$
2	$x_2 \cdot x_1 \cdot Cov(R_2, R_1)$	$x_2^2 \cdot \sigma_2^2$		
	$\vdots$		$\ddots$	
N	$x_N \cdot x_1 \cdot Cov(R_N, R_1)$			$x_N^2 \cdot \sigma_N^2$

Portfolio variance: $\sigma_P^2 = \sum_{i=1}^{N} x_i^2 \cdot \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} x_i \cdot x_j \cdot Cov(R_i, R_j)$

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or alternatively with equal weights:

$$\sigma_P^2 = \underbrace{\sum_{i=1}^{N}\left(\frac{1}{N^2}\cdot\sigma_i^2\right)}_{A} + \underbrace{\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}\frac{1}{N}\cdot\frac{1}{N}\cdot Cov(R_i,R_j)}_{B}$$

$$A = \frac{1}{N}\cdot\overline{\sigma^2}$$

$$B = 2\cdot\frac{1}{N^2}\cdot\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} Cov(R_i,R_j) = \frac{N-1}{N}\cdot\left[\frac{2}{N\cdot(N-1)}\cdot\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} Cov(R_i,R_j)\right]$$

$$B = \left(1-\frac{1}{N}\right)\cdot\overline{Cov}$$

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The decomposition of the portfolio **variance** into a pure variance component (**A**, main diagonal) and a pure **covariance** component (**B**) shows that the pure variance component becomes more and more irrelevant for the portfolio risk with increasing N.

For **N** $\rightarrow$ ∞: **A** $\rightarrow$ **0** and **B** $\rightarrow$ $\overline{\text{Cov}}$, and for the portfolio risk it follows:

$$\sigma_P^2 \rightarrow \overline{\text{Cov}}$$

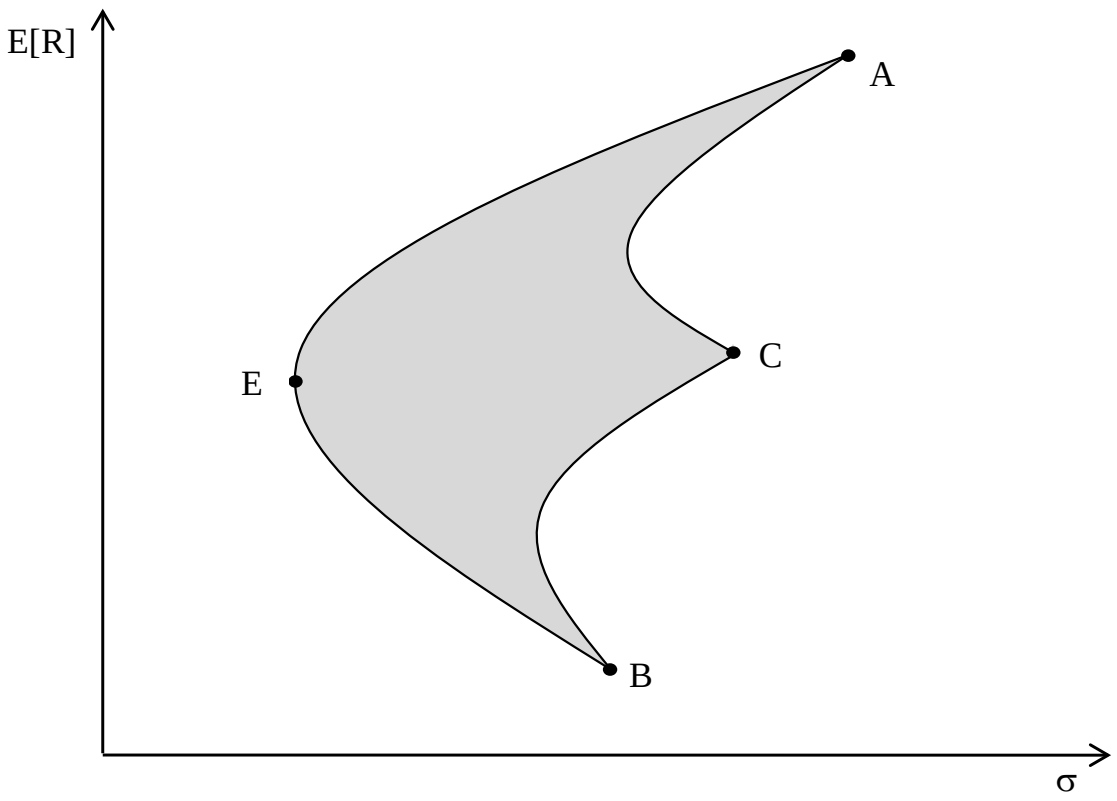
Thus, the risk level of individual securities within a portfolio is of decreasing importance with increasing number of securities! The risk of a well-diversified portfolio converges to the **average covariance** level of the individual securities.

4.1.5 Efficient Set

Starting from the results for a portfolio consisting of 2 assets we now want to analyze the portfolio compositions for a portfolio with more than 2 assets.

If a portfolio contains 3 risky assets *A*, *B*, and *C* all (endless many) portfolios in the grey area in the figure below can be generated.

Portfolio E is special, as its variance is lowest. E is, therefore, the MVP of all possible combinations of the three risky assets.



However, not all possible portfolios (grey area) are relevant. Relevant (efficient) portfolios have the following characteristics:

(a) **Minimal risk** for all possible expected returns.

(b) **Maximal expected return** for all possible risk levels.

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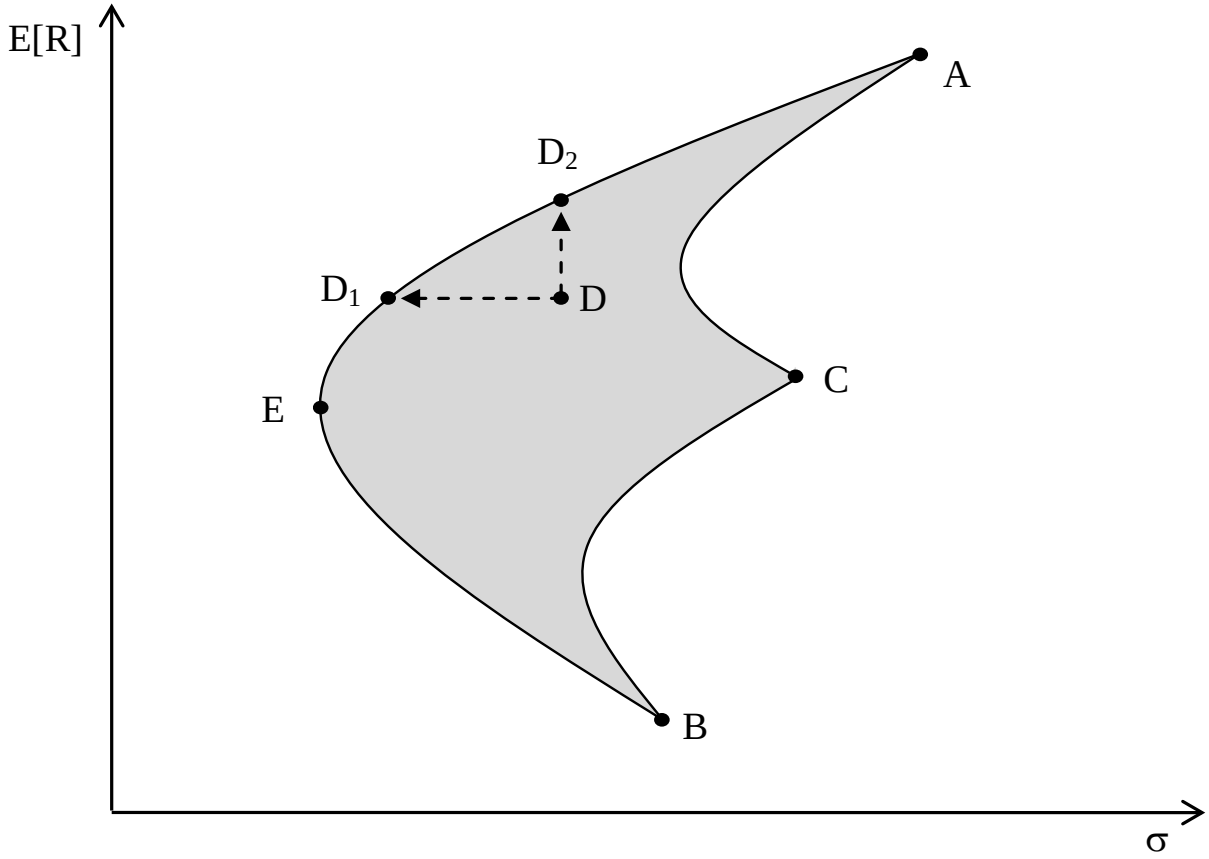

E.g., portfolio D is clearly dominated by portfolio D_1 as well as portfolio D_2. Thus, only portfolios on the line between E and A are efficient. They dominant all other possible portfolios (grey area).

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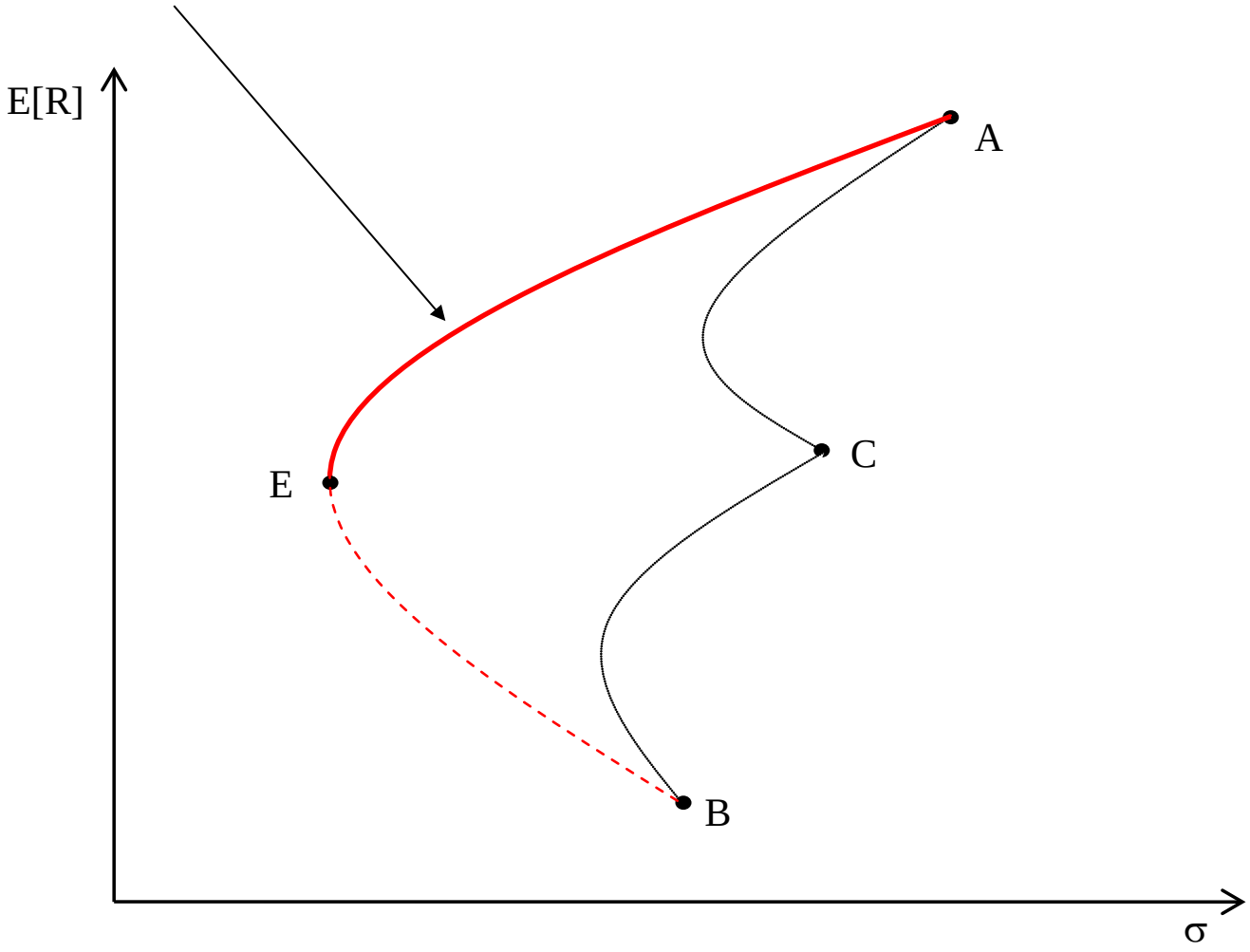
Efficient Frontier:



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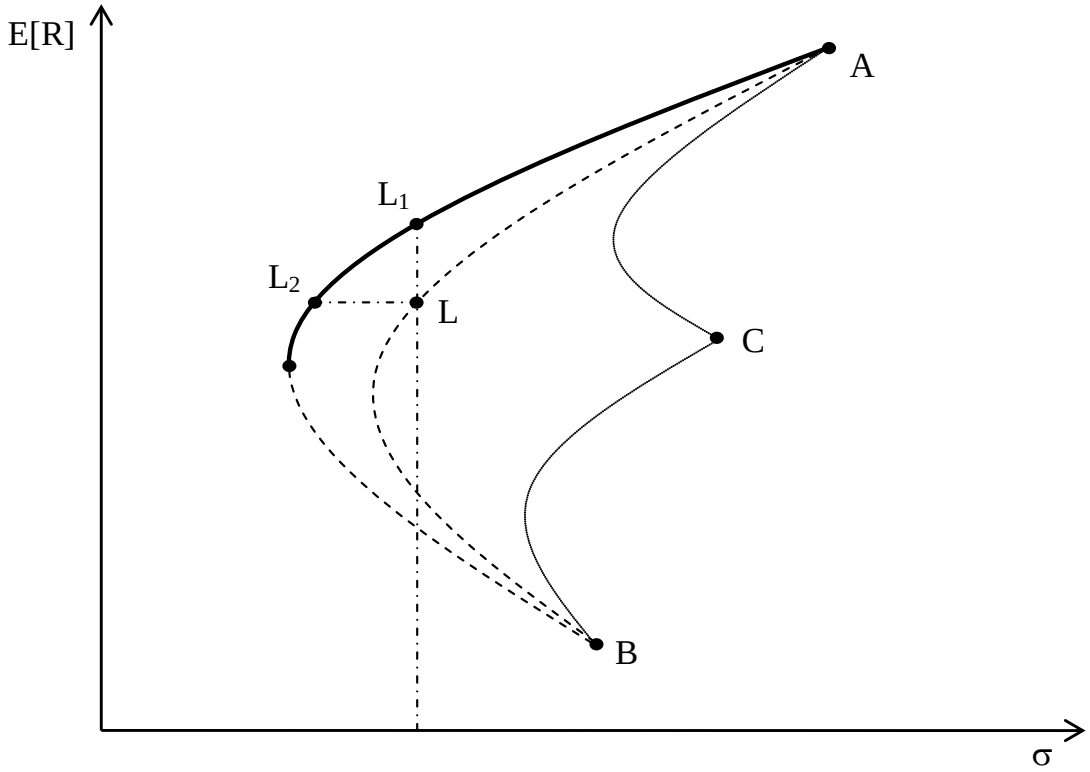
Diversification effect due to additional assets



Portfolio *L* is efficient when only *A* and *B* can be used to generate portfolios. If *C* is added, the Efficient Frontier moves to the left. E.g., *L* is then clearly dominated by L_1 and L_2.

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Portfolio selection steps by Markowitz:

Based on Markowitz, investors should choose those portfolios that maximize the portfolio performance for a given level of risk (variance).

Steps:

(a) Selection of a **set of assets** for possible investment.

(b) Estimation of expected return, volatility and correlation of in (a) specified assets.

(c) Calculation of the **Efficient Frontier** based on the information generated in (b).

(d) Selection of the **optimal portfolio** for the corresponding investor based on the in (c) specified **Efficient Frontier**.

❍ Summary for a portfolio of N securities:

(i) In total the sum of all portfolio weights equals one: $\sum_{i=1}^{N} x_i = 1$

(ii) **The expected return of the portfolio** is a linear combination of the expected returns on the stocks in that portfolio:

$$E(R_P) = \sum_{i=1}^{N} x_i \cdot E(R_i)$$

(iii) Variance of the portfolio: $$\sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_i \sigma_i \rho_{i,j}$$

The volatility of the portfolio is the square root of the portfolio variance.

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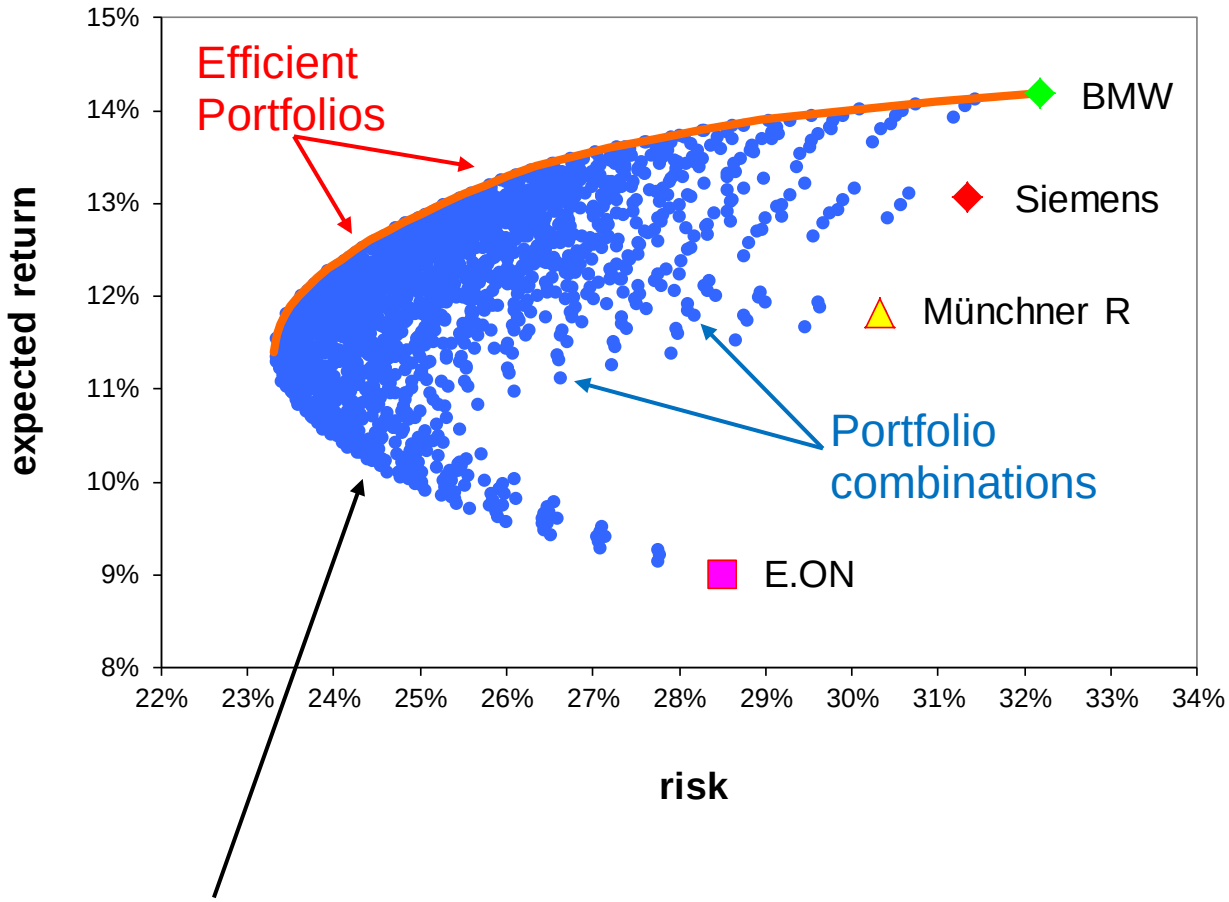
Example: Risk-Return profil for portfolios consisting of all securities (see also *MPT2019_PEF.xlsx*)



Diversification: One gets less risk than by the least risky stock

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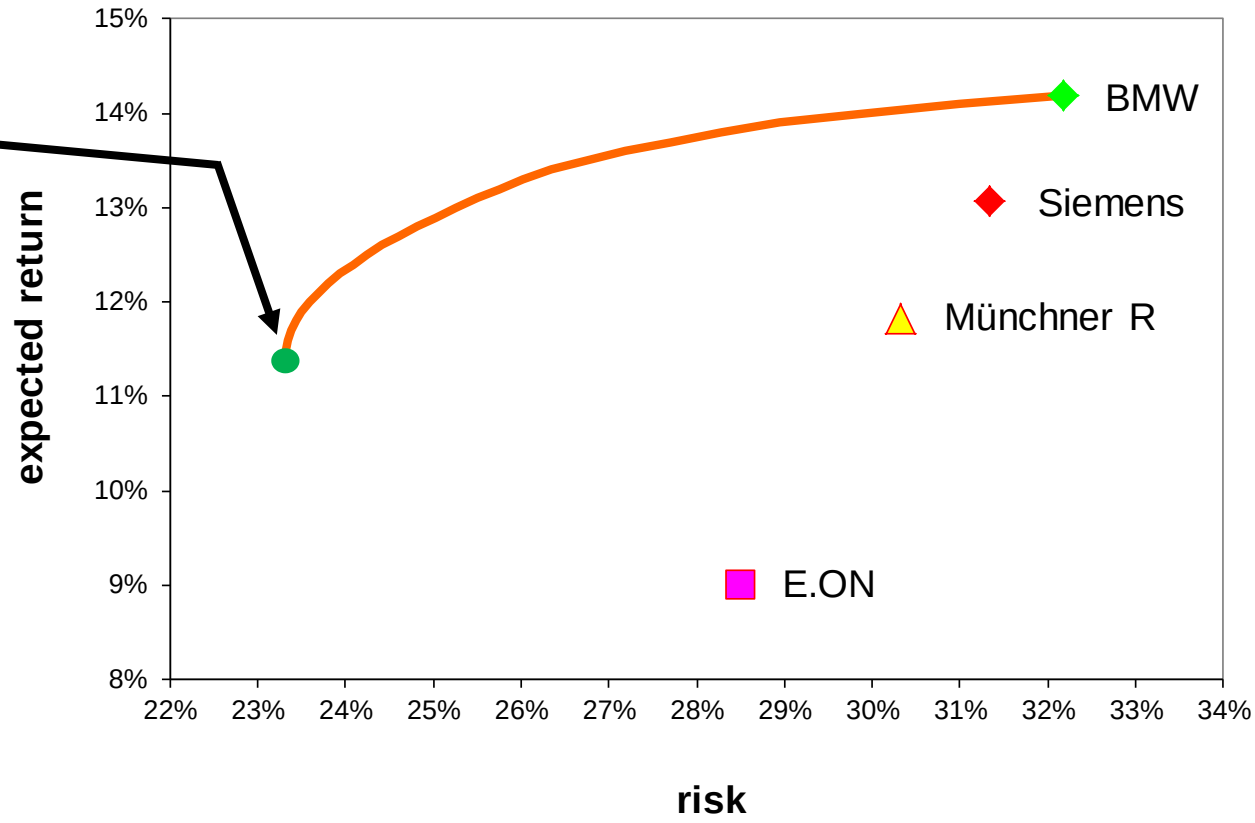
Minimum-Variance Portfolio (MPV): The portfolio on the efficient frontier with the lowest risk level

BMW	18.38%
E.ON	38.36%
Münchner R	24.94%
Siemens	18.33%
E(RP)	11.41%
$\sigma(R_P)$	23.31%



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❍ Diversification effect if more assets are included

Compare:

- Minimum-variance portfolio of **two** securities:

  Portfolio volatility is 25.23%

- Minimum-variance portfolio of **four** securities:

  Portfolio volatility is 23.31%

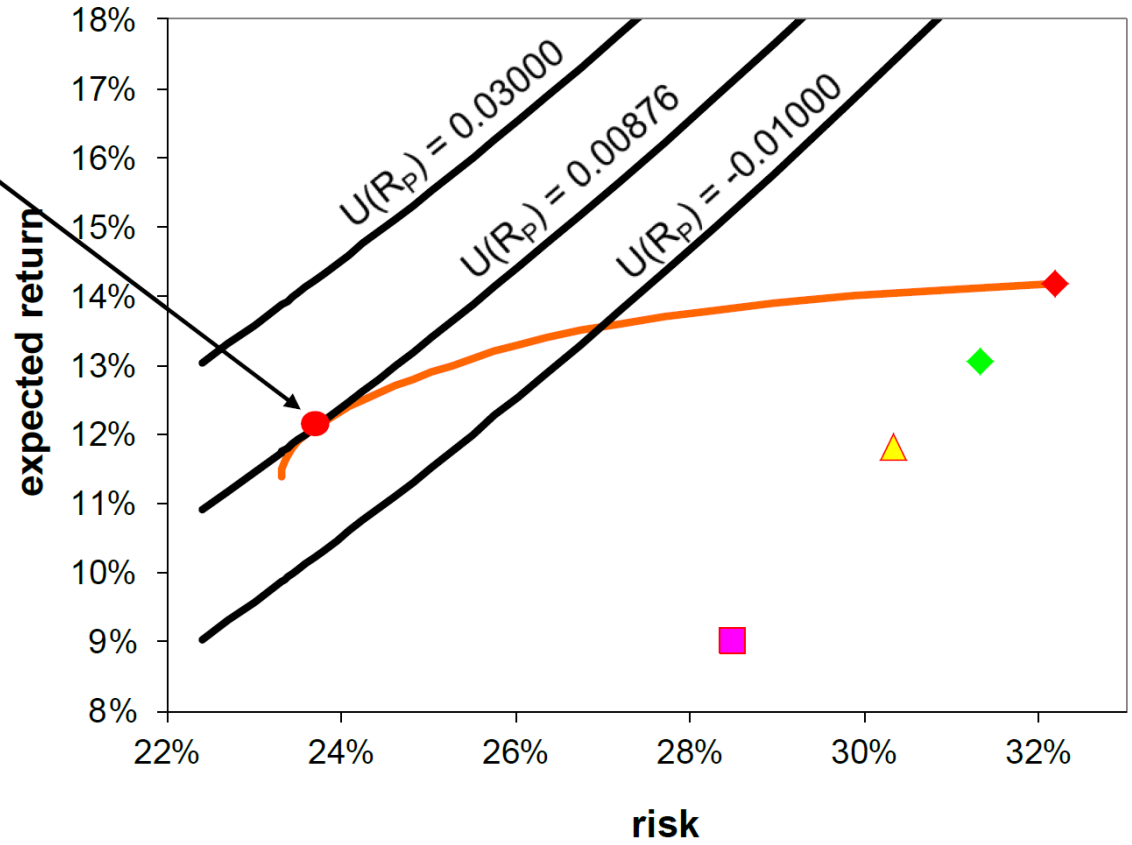
- If we continue to increase the number of securities in a portfolio, the diversification effect will be even more pronounced.

❍ Efficient frontier and risk preferences ($U(R_P) = E(R_P) - 0.5 \cdot A \cdot \sigma^2_P$)

- **Higher risk aversion (*A* = 4):**

The optimal portfolio:

x_{BMW}	= 28.1%
$x_{Siemens}$	= 23.2%
$x_{Münchner\ R}$	= 24.6%
x_{EON}	= 24.2%
$E(R_P)$	**= 12.10%**
σ_P	**= 23.69%**



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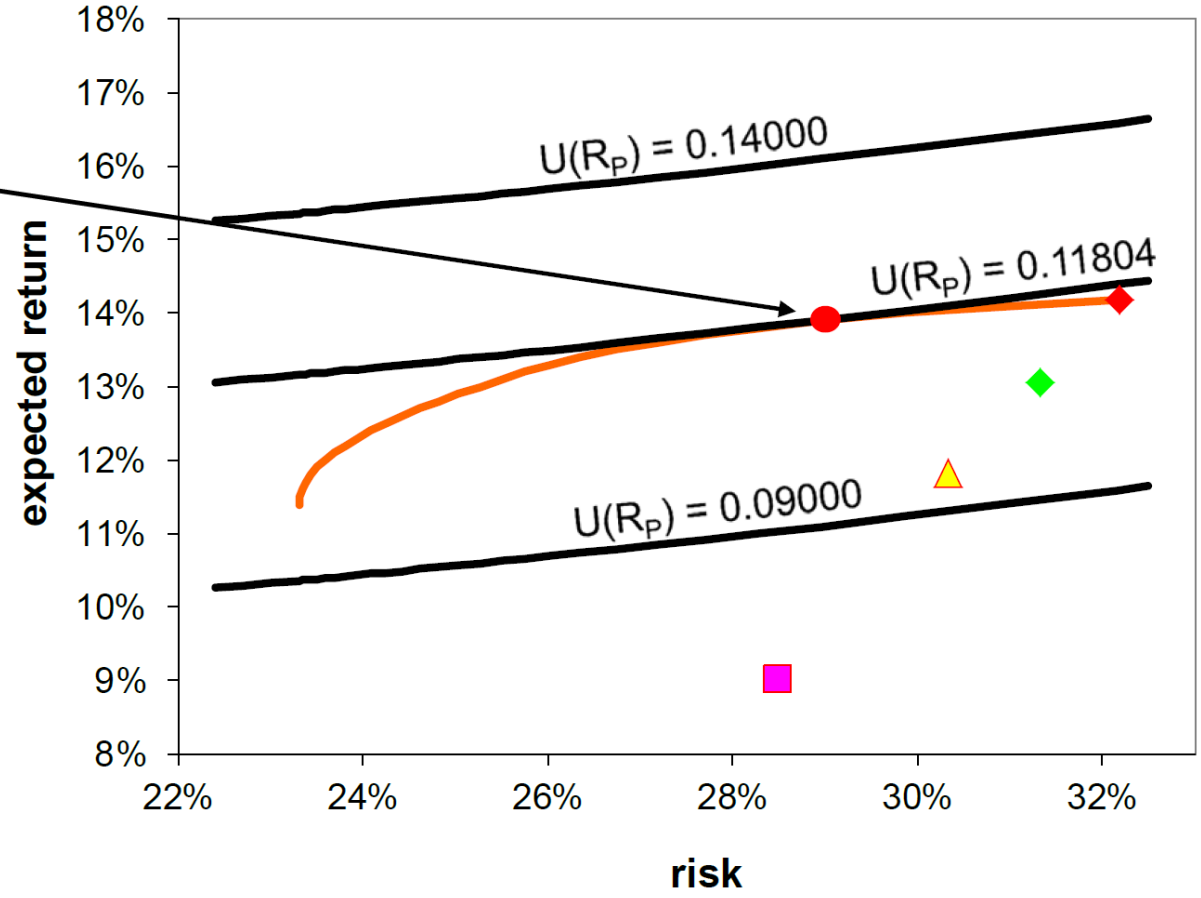
- **Less risk aversion (*A* = 0.5)**

The optimal portfolio:

x_{BMW}	= 74.3%
$x_{Siemens}$	= 25.7%
$x_{Münchner\ R}$	= 0.0%
x_{EON}	= 0.0%
$E(R_P)$	**= 13.90%**
σ_P	**= 28.95%**



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❍ Summary

- An **indifference curve** corresponds to **portfolio combinations** with the **same utility** value.
- For **more risk-averse** investor, as volatility increases, he or she will demand sharply higher expected returns to hold the portfolio. That's why the indifference curves in this case **increase steeper** than of a less risk-averse investor.
- These different curves will result in **different optimal portfolio choices** for investors.
- The **optimization procedure** simply takes the efficient frontier and finds its **point of tangency** with the **highest utility** curve in the investor set. In other words, it identifies the single point that provides the investor with the highest level of utility.

4.2 Value at Risk - More than one Risk Factor

- So far (see chapter 3.3 below) only one risk factor has been considered.

- If there are two or more sources of risk:
  - ➔ **Diversification effects** in calculating VaR
  - ➔ Most important: **Correlation** structure between factor returns

- **First step:** For each risk factor in a portfolio a delta-equivalent has to be calculated:

$$\Delta_k = \sum_{i=1}^{N} \Delta_{i,k} \qquad (k = 1, \ldots , K)$$

N Number of instruments in the portfolio

$\Delta_{i,k}$ Delta-equivalent of instrument i relative to market risk factor k, with $\Delta_{i,k} = \delta_{i,k} \cdot MV_i$ ($\delta = 1$ for a stock when the stock is used as risk factor)

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❍ **Second step:** For each market risk factor *k* a VaR has to be calculated

$$\mathrm{VaR}(\alpha)_k = -\Delta_k \cdot [\mu_k + \sigma_k \cdot \Phi(\alpha)]$$

or (8)

$$\mathrm{VaR}(\alpha)_k = -\Delta_k \cdot \sigma_k \cdot \Phi(\alpha) - \Delta_k \cdot \mu_k$$

- If k > 1:

  ➔ VaR has to be divided into 2 components:

$$\mathrm{VaR}(\alpha) = \mathrm{VaR}_{\sigma,\rho} + \mathrm{VaR}_{\mu} \tag{9}$$

Component 1: $\mathrm{VaR}_{\sigma,\rho}$ Depends on the **volatility** of the risk factors (σ_1, ..., σ_K) and the **correlation structure** between risk factors.

Component 2: VaR_{μ} Depends on the **expected return** (μ_1, μ_2, ... , μ_K) of the risk factors.

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- $\mathrm{VaR}_{\sigma,\rho}$ is defined as: $$\mathrm{VaR}_{\sigma,\rho} = \sqrt{V^T \cdot C \cdot V} \qquad (10)$$

  with $V^T = [V_1 \quad V_2 \quad \ldots \quad V_K]$ and $V_k = -\Delta_k \cdot \sigma_k \cdot \Phi(\alpha)$

  Δ_k Delta-equivalent of risk factor *k*
  σ_k Volatility of risk factor *k*

  If we assume for all μ_k (k = 1, ... , K) to be 0: V^T = vector of VaR numbers (all risk factors)

- Correlation matrix of risk factor returns:

$$C = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1K} \\ \rho_{21} & 1 & \cdots & \rho_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{K1} & \rho_{K2} & \cdots & 1 \end{pmatrix}$$

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- VaR_μ is defined as: $$\mathrm{VaR}_\mu = -\Delta^T \cdot \mu \qquad (11)$$

with

$$\Delta^T = \begin{bmatrix} \Delta_1 & \Delta_2 & \dots & \Delta_K \end{bmatrix} \quad \text{Vector of Delta-equivalents}$$
$$\mu^T = \begin{bmatrix} \mu_1 & \mu_2 & \dots & \mu_K \end{bmatrix} \quad \text{Vector of expected factor returns}$$

- For 2 risk factors the VaR is given by:

$$\mathrm{VaR}(\alpha) = \sqrt{\begin{bmatrix} V_1 & V_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & \rho_{1,2} \\ \rho_{1,2} & 1 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}} - \begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix} \cdot \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

and (12)

$$\mathrm{VaR}(\alpha) = \sqrt{(V_1)^2 + (V_2)^2 + 2 \cdot \rho_{1,2} \cdot V_1 \cdot V_2} - \Delta_1 \cdot \mu_1 - \Delta_2 \cdot \mu_2$$

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Example:

Portfolio of 2 stock positions: 50,000 T-Online stocks (long position)
70,000 Infineon stocks (short position)
α = 5%, H = 1 trading day

Market data (10.10.2003):

	T-Online	**Infineon**
Closing price	€10.03	€12.19
σ (p.d.)	3.29653%	4.39294%
μ (p.d.)	0.05232%	0.00304%

Correlation	**T-Online**	**Infineon**
T-Online	1.000	0.543
Infineon	0.543	1.000

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Step 1: Vector of delta-equivalents

- $\Delta_{T-Online} = \text{price} \cdot \text{amount} = 10.03 \cdot 50{,}000 = 501{,}500$ (long)
- $\Delta_{Infineon} = \text{price} \cdot \text{amount} = 12.19 \cdot (-70{,}000) = -853{,}300$ (short)

➔ Important: short positions have a **negative sign**!

Step 2: Vector of VaRs for all $\mu = 0$

- $V_{T-Online} = -501{,}500 \cdot 0.03297 \cdot (-1.645) = 27{,}199.18$ (long)
- $V_{Infineon} = -(-853{,}300) \cdot 0.04393 \cdot (-1.645) = -61{,}663.60$ (short)

Step 3: VaR component 1

$$\mathrm{VaR}_{\sigma,\rho} = \sqrt{(27{,}199.18)^2 + (-61{,}663.60)^2 + 2 \cdot 0.543 \cdot 27{,}199.18 \cdot (-61{,}663.60)}$$

= **52,160.87**

Step 4: VaR component 2

$\mathrm{VaR}_{\mu} = -\Delta_1 \cdot \mu_1 - \Delta_2 \cdot \mu_2$ = -501,500·0.0005232 - (-853,300)·0.0000304 = **-236.44**

Step 5: Total VaR

$\mathrm{VaR}(\alpha) = \mathrm{VaR}_{\sigma,\rho} + \mathrm{VaR}_{\mu}$ = 52,160.87 - 236.44 = **51,924.42**

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- **Using matrix notation:**

$$\mathrm{VaR}(\alpha) = \sqrt{V^T \cdot C \cdot V} - \Delta^T \cdot \mu$$

$V^T = [V_1 \quad V_2 \quad \ldots \quad V_K]$ Vector of VaRs for all $\mu = 0$

$\Delta^T = [\Delta_1 \quad \Delta_2 \quad \ldots \quad \Delta_K]$ Vector of Delta-equivalents

$\mu^T = [\mu_1 \quad \mu_2 \quad \ldots \quad \mu_K]$ Vector of expected factor returns

MS-Excel:

```
{=SQRT(MMULT(MMULT(MTRANS(V_Start:V_End), C_Start:C_End), V_Start:V_End))}
```

minus

```
{=MMULT(MTRANS(Δ_Start:Δ_End), μ_Start:μ_End)}
```


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- VaR calculation using the **portfolio volatility**

$$\mathrm{VaR}_P = -\delta_p \cdot \sigma_P \cdot \Phi(\alpha) - \delta_P \cdot \mu_P$$

with

δ_P Total exposure of the portfolio (sum of the delta-equivalents of all market risk factors)

σ_P Volatility of the portfolio (of risk factors)

μ_P Expected return of the portfolio (of risk factors)

- **For a portfolio of two risk factors we get:**

$$\sigma_P = \sqrt{w_1^2 \cdot \sigma_1^2 + w_2^2 \cdot \sigma_2^2 + 2 \cdot w_1 \cdot w_2 \cdot \sigma_1 \cdot \sigma_2 \cdot \rho_{12}}$$

$$\mu_P = w_1 \cdot \mu_1 + w_2 \cdot \mu_2$$

with $w_1 = \frac{\delta_1}{\delta_P}; \quad w_2 = \frac{\delta_2}{\delta_P}$

w_k	weight of market risk factor k (k = 1, 2)
δ_k	Delta-equivalent (exposure) against risk factor k (k = 1, 2)
σ_k	Volatility (standard deviation) of risk factor k (k = 1, 2)
$\rho_{1,2}$	Correlation between risk factor 1 and 2
μ_k	Expected returns of risk factor k (k = 1, 2)

Example:

Coupon bond:
Time to maturity = 2 years, face value = €10 Mio, coupon = 5.5%p.a.

Market data: Discount factors (=risk factors): P(1) = 0.96, P(2) = 0.915,
$\mu_1 = \mu_2 = 0$, $\sigma_1 = 0.10\%$ p.d., $\sigma_2 = 0.13\%$ p.d., $\rho_{12} = 0.96$

➔ Calculate the VaR for $\alpha = 5\%$

Step 1: Valuation

➔ The coupon bond consist of two zerobonds:

	time	
	1 year	2 years
Zerobond 1 (long position)	550,000	
Zerobond 2 (long position)		10,550,000

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- **VaR of zerobond 1:**

$$V(0.05)_1 = -550{,}000 \cdot 0.96 \cdot 0.0010 \cdot (-1.645) = 868.56€$$

- **VaR of zerobond 2:**

$$V(0.05)_2 = -10{,}550{,}000 \cdot 0.915 \cdot 0.0013 \cdot (-1.645) = 20{,}643.48€$$

- **VaR of the coupon bond:**

$$VaR(0.05) = \sqrt{[V(0.05)_1]^2 + [V(0.05)_2]^2 + 2 \cdot \rho_{1,2} \cdot V(0.05)_1 \cdot V(0.05)_2}$$

$$VaR(0.05) = \sqrt{[868.56]^2 + [20{,}643.48]^2 + 2 \cdot 0.96 \cdot 868.56 \cdot 20{,}643.48} = 21{,}478.67\ €$$

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- **General concept:** Representation of the total portfolio in a **risk matrix** (matrix of all Delta-equivalents):

	Risk factors (RF)			
	RF 1	RF 2	...	RF K
Position 1	δ_{11}	δ_{12}	...	δ_{1K}
Position 2	δ_{21}	δ_{22}	...	δ_{2K}
⋮	⋮	⋮	...	⋮
Position N	δ_{N1}	δ_{N2}	...	δ_{NK}
Total	δ_1	δ_2	...	δ_K

- **Advantage:**
  - Reduction of the risk measurement problem to a few risk factors
  - Exposure netting at the risk factor level (before the VaR is calculated)

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Example

A portfolio consists of 3 government bonds (10.10.2003):

7%-Bond, face value: €2 Mio, time to maturity: 2 years
4%-Bond, face value: €4 Mio, time to maturity: 5 years
5%-Bond, face value: €3 Mio, time to maturity: 3 years

Market data for the risk factors (5.11.2001 to 10.10.2003):

	3M	6M	9M	1Y	2Y	3Y	4Y
$\sigma_{P(T)}$	0.01314%	0.01823%	0.02737%	0.03880%	0.11065%	0.17735%	0.23335%
P(T)	0.99438	0.98891	0.9833	0.97722	0.9473	0.91221	0.87425
	5Y	6Y	7Y	8Y	9Y	10Y	
$\sigma_{P(T)}$	0.28809%	0.32649%	0.36201%	0.41095%	0.44715%	0.48645%	
P(T)	0.83519	0.79583	0.75625	0.71758	0.68034	0.64582	

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Correlation matrix

	3M	6M	9M	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
3M	1.000	0.614	0.445	0.358	0.158	0.195	0.186	0.165	0.118	0.110	0.096	0.084	0.090
6M	0.614	1.000	0.706	0.616	0.351	0.361	0.364	0.319	0.272	0.271	0.246	0.234	0.218
9M	0.445	0.706	1.000	0.861	0.477	0.479	0.475	0.440	0.377	0.377	0.371	0.353	0.329
1Y	0.358	0.616	0.861	1.000	0.537	0.549	0.530	0.513	0.451	0.462	0.443	0.425	0.401
2Y	0.158	0.351	0.477	0.537	1.000	0.944	0.913	0.912	0.880	0.852	0.835	0.807	0.804
3Y	0.195	0.361	0.479	0.549	0.944	1.000	0.950	0.950	0.889	0.892	0.869	0.851	0.846
4Y	0.186	0.364	0.475	0.530	0.913	0.950	1.000	0.969	0.893	0.896	0.873	0.857	0.854
5Y	0.165	0.319	0.440	0.513	0.912	0.950	0.969	1.000	0.902	0.910	0.882	0.868	0.873
6Y	0.118	0.272	0.377	0.451	0.880	0.889	0.893	0.902	1.000	0.958	0.953	0.945	0.938
7Y	0.110	0.271	0.377	0.462	0.852	0.892	0.896	0.910	0.958	1.000	0.974	0.969	0.976
8Y	0.096	0.246	0.371	0.443	0.835	0.869	0.873	0.882	0.953	0.974	1.000	0.974	0.966
9Y	0.084	0.234	0.353	0.425	0.807	0.851	0.857	0.868	0.945	0.969	0.974	1.000	0.973
10Y	0.090	0.218	0.329	0.401	0.804	0.846	0.854	0.873	0.938	0.976	0.966	0.973	1.000

➔ How large is the VaR(0.05) of this portfolio?

Solution:

	Risk factors and Cash Flows				
Positions	**1Y**	**2Y**	**3Y**	**4Y**	**5Y**
7%-Bond	140,000	2,140,000			
4%-Bond	160,000	160,000	160,000	160,000	4,160,000
5%-Bond	150,000	150,000	3,150,000		
Sum (face value, €)	**450,000**	**2,450,000**	**3,310,000**	**160,000**	**4,160,000**
P(T)	0.97722	0.9473	0.91221	0.87425	0.83519
Delta-equivalent (€)	**439,749.00**	**2,320,885.00**	**3,019,415.10**	**139,880.0**	**3,474,390.4**
$\sigma_{P(T)}$	0.00038796	0.00110648	0.00177354	0.00233349	0.00288093
V(0.05) in €	**280.65**	**4,224.37**	**8,809.04**	**536.94**	**16,465.59**

$$\mathrm{VaR}(0.05)_P = \sqrt{V^T \cdot C \cdot V}$$

$$V^T = \begin{bmatrix} 280.65 & 4{,}224.37 & 8{,}809.04 & 536.94 & 16{,}465.59 \end{bmatrix}$$

$$C = \begin{pmatrix} 1.000 & 0.537 & 0.549 & 0.530 & 0.513 \\ 0.537 & 1.000 & 0.944 & 0.913 & 0.912 \\ 0.549 & 0.944 & 1.000 & 0.950 & 0.950 \\ 0.530 & 0.913 & 0.950 & 1.000 & 0.969 \\ 0.513 & 0.912 & 0.950 & 0.969 & 1.000 \end{pmatrix}$$

$\mathrm{VaR}(0.05)_P$ = 29,646.25 €

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5. Market Model

5.1 Basics

The market model describes the **relationship** between the return of an **individual stock i** and a **market index M**:

$$R_i = \alpha_i + \beta_i \cdot R_m + \varepsilon_i$$

R_i = Return of stock i (in period t)

R_m = Return of market index *m* (in period t) (e.g. DAX, S&P500, FT100)

α_i = Alpha, intercept

β_i = Beta, slope

ε_i = Residual return, random variable, $E[\varepsilon_i] = 0$, σ_{ε_i}, $\rho_{\varepsilon,m} = 0$, is not explained by general market movements, Covariance(ε_i, ε_j) = 0

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- Only one factor explains stock returns (single-index model).

- The market index functions as a proxy for potential general factors influencing the economy (e.g., interest rates and changes in the form of the term structure, commodities and currencies, inflation, unemployment rate, general economic growth, ...)

- Three return components:

  (i) α_i: Expected performance if $E[R_m] = 0$

  (ii) $\beta_i \cdot R_{m,t}$: Systematic return component

  (iii) ε_i: Unsystematic, firm specific return component; **cannot** be explained by the index (the market)

The market model in expected return formulation:

$$E[R_i] = \alpha_i + \beta_i \cdot E[R_m]$$

Example:

- For a stock i with: $\alpha_i = 2\%$, $\beta_i = 1.2$
- Market model equation: $R_i = 2\% + 1.2 \cdot R_m + \varepsilon_i$

If the market rises by +10%: $E[R_i] = 2\% + 1.2 \cdot 10\% = 14\%$

If the realized return is only 11%: $\varepsilon_{i,M} = R_i - E[R_i] = 11\% - 14\% = -3\%$

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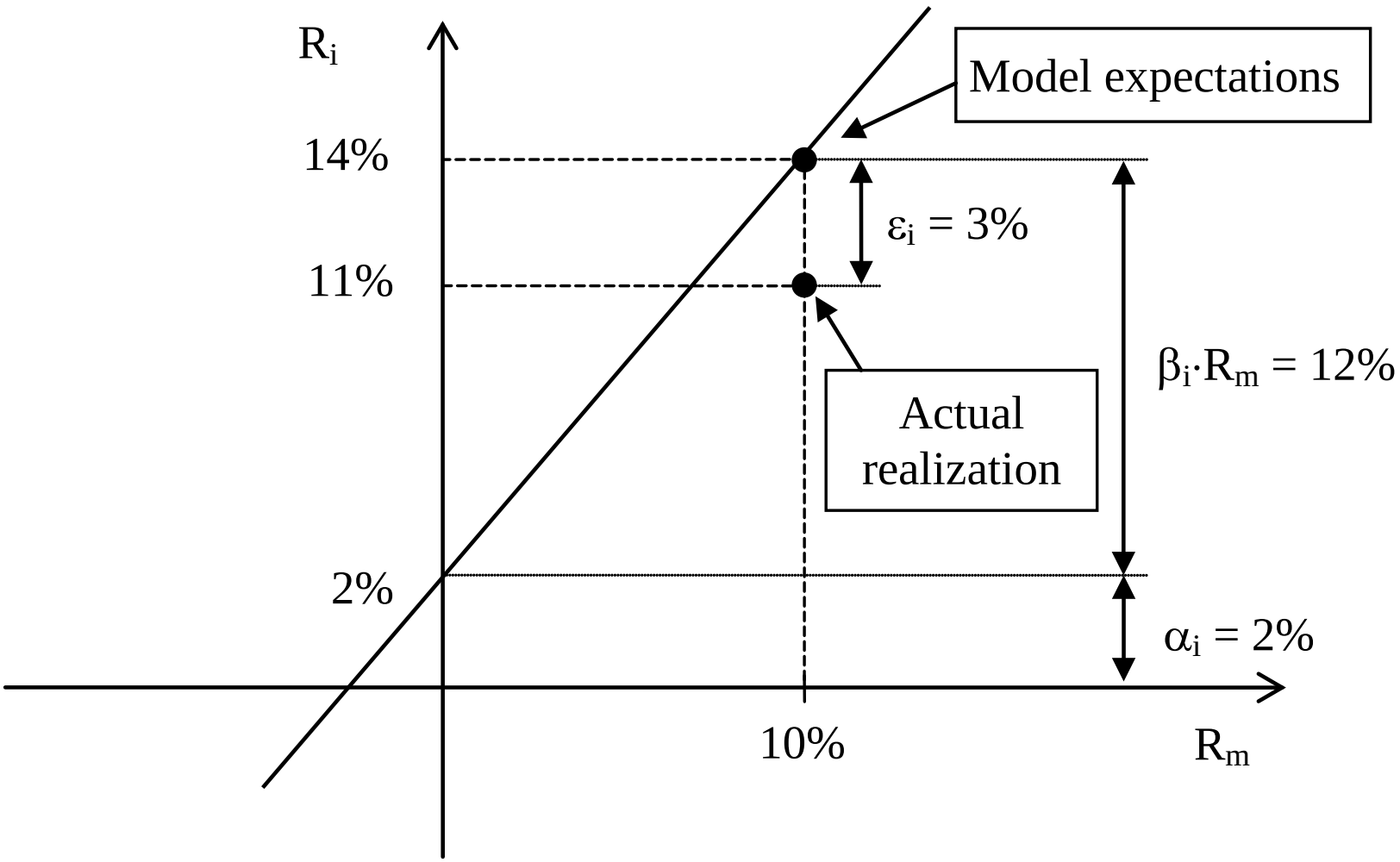

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- Beta β_i is defined as: $\beta_i = \dfrac{\sigma_{i,m}}{\sigma_m^2}$

$\sigma_{i,m}$: Covariance between stock *i* and market index *m*

σ_m^2: Variance of market index returns.

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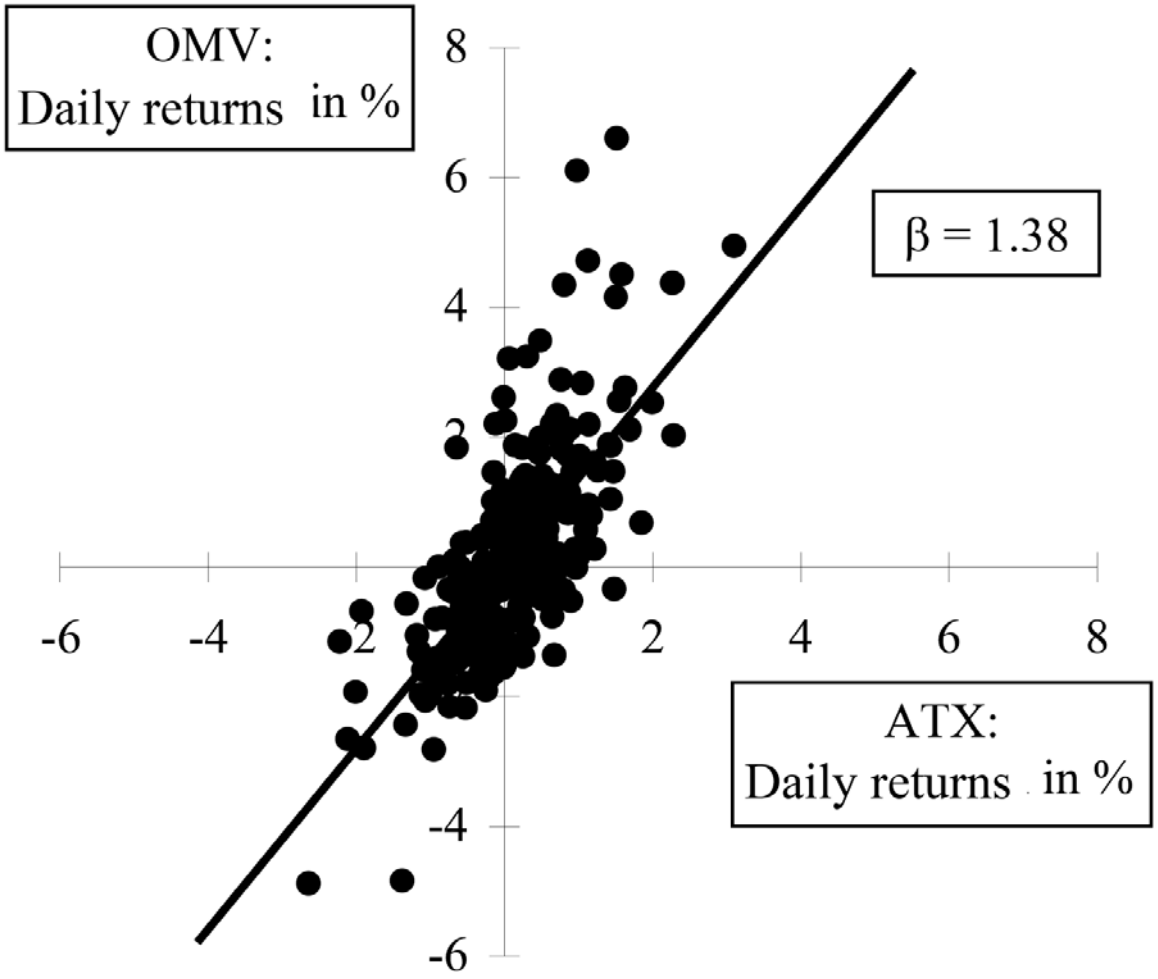
Example: OMV and ATX



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Example: Market model parameter estimation for Siemens AG

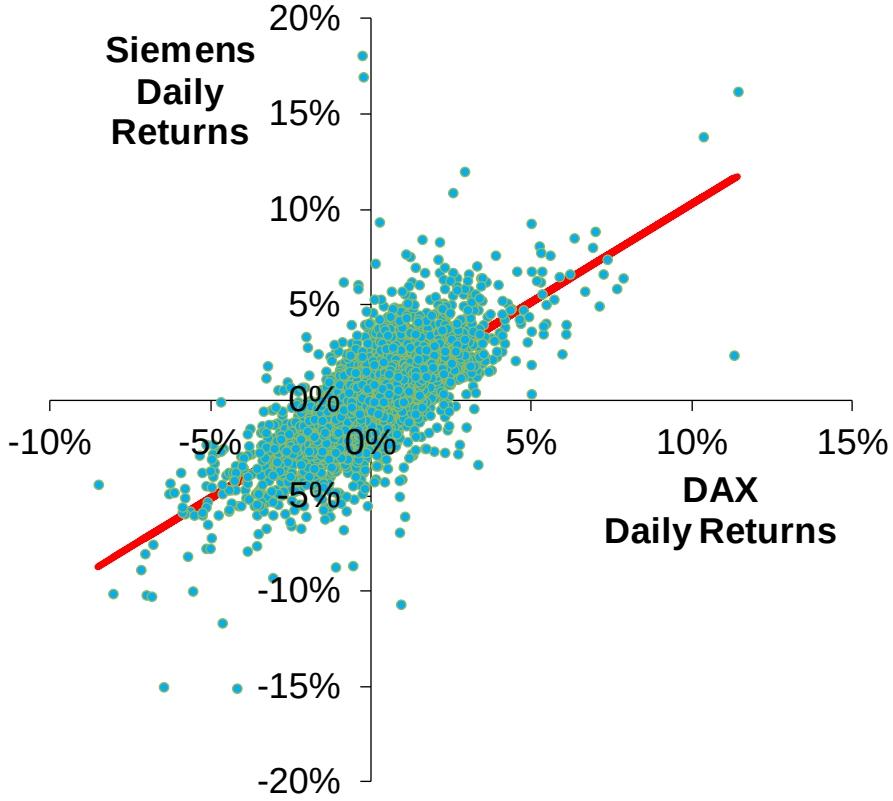
Use the data in the MS-Excel file *MPT2019_PEF.xlsx* to **estimate α_i and β_i** for Siemens AG using the DAX as market index.

You get:

Regression line:

$R_{Siemens} = 0.0131\% + 1.0295\ R_{DAX}$

Beta: $\beta_{Siemens} = 1.0295$



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❍ Beta-Estimation: Regression Analysis

Step 1: We can use the Excel function **LINEST()** to estimate a regression of the form

$$R_i = \alpha_i + \beta_i R_m$$

As an output from this function, you get estimated values for β_i, a constant term (α_i), as well as additional regression statistics.

Annex: LINEST()

➔ Array function

Array functions are functions that produce **results** that fill **several cells**.

Step A: Type in your data in two columns, one for variable x and one for variable y:

	A	B	C
1			
2			
3		x	y
4		2	2.3
5		3	4.5
6		4	6.7
7		5	9.8
8		6	12.3
9		7	15.4
10			

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Step B: Select the area that will hold the output of the array formula. For "LINEST" you should drag to form a 5 row by 2 column data array.

	A	B	C
1			
2			
3		x	y
4		2	2.3
5		3	4.5
6		4	6.7
7		5	9.8
8		6	12.3
9		7	15.4
10			
11			
12			
13			
14			
15			
16			
17			

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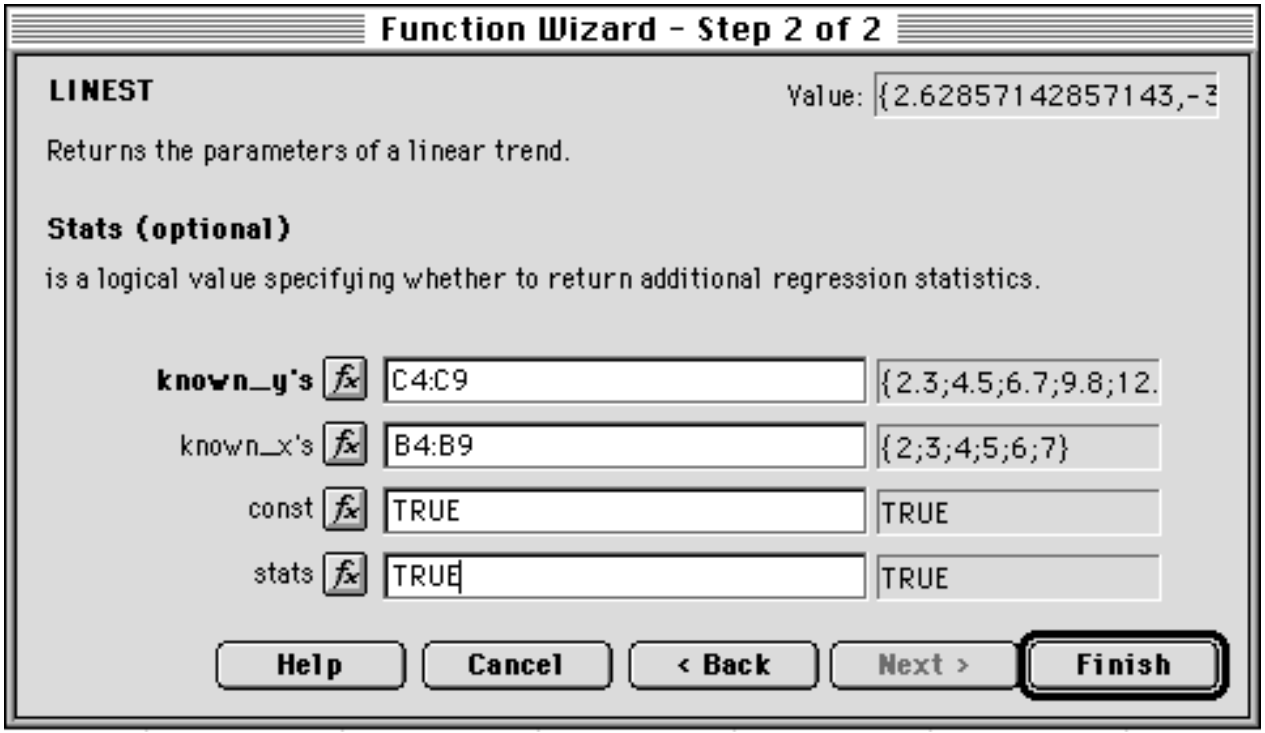
Step C: Press the wizard button ("**fx**"; formula section); Select "**Statistical**" + "**LINEST**":



Select **y** and **x** values; type in "TRUE" in the last two dialog boxes.

Step D: Click on "Finish."

Step E: Now here is the important step. LINEST is an array function, which means that when you enter the formula in one cell, multiple cells will be used for the output of the function.

To specify that LINEST is an array function do the following. **Highlight** the entire formula, including the "=" sign, hold down the "**Ctrl**" and "**Shift**" keys and **press** "**Enter**." **Excel adds "{ }" brackets** around the formula, to show that it is an array. Note that you cannot type in the "{ }" characters yourself.

Output:

slope	2.6286	-3.3286	intercept
se(slope)	0.084997	0.409106	se(intercept)
R^2	0.9958	0.35557	s(y)
F	956.38	4	degrees of freedom
regression ss	120.91429	0.50571	residual ss

se: standard error

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$$\text{t}-\text{statistic(slope)} = \frac{\text{slope}}{\text{se(slope)}} = \frac{2.6286}{0.084997} = 30.926$$

$$\text{t}-\text{statistic(intercept)} = \frac{\text{intercept}}{\text{se(intercept)}} = \frac{-3.3286}{0.409106} = -8.136$$

The t-statistic follows a t-distribution with n-2 degrees of freedom. P-values can be calculated using TDIST and/or NORMSDIST in excel:

P-value(slope) = TDIST(t-statistic, degrees of freedom, tail)

P-value(slope) = TDIST(30.926, 4, 2) = 0.000

2-sided test: tail = 2
1-sided test: tail = 1

or under normal approximation (2-sided test):

P-value(slope) = 2·(1 - NORMSDIST(t-statistic))

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❍ Beta-Estimation: Regression Analysis (2)

Step 2: We can now analyze **how reliable** the Beta **estimate** for Siemens (provided above) is.

From the **regression statistics**, the coefficient of determination (**R^2**) is of special interest. It measures the **proportion** of a security's **total variability explained** by the market index.

Values of R^2 range between 0 and 1, where 0 represents the least explanatory power and 1 shows full correlation between the security and the index.

The closer to 1, the more the estimated Beta should be trusted and vice-versa.

Example:

- R^2 of the Siemens Beta estimate is 0.5647. Quite good!!

- This implies that DAX returns explain 56.47% of the movements in the Siemens return series.

5.2 Systematic and unsystematic risk

- The total risk of an asset or portfolio can be divided into two parts:

  (1) **Market risk** (systematic risk) and (2) **residual risk** (unsystematic risk):

$$\sigma_i^2 = \beta_i^2 \cdot \sigma_m^2 + \sigma_{\varepsilon_i}^2$$

$\beta_i^2 \cdot \sigma_m^2$ = **market risk**, $\sigma_{\varepsilon_i}^2$ = **residual risk**

- **Portfolio of N assets:**

$$R_P = \sum_{i=1}^{N} x_i \cdot R_i$$

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When R_i is replaced by its market model equation, we get for the portfolio:

$$R_P = \sum_{i=1}^{N} x_i \cdot (\alpha_i + \beta_i \cdot R_m + \varepsilon_i)$$

$$= \sum_{i=1}^{N} x_i \cdot \alpha_i + \left(\sum_{i=1}^{N} x_i \cdot \beta_i \right) \cdot R_m + \sum_{i=1}^{N} x_i \cdot \varepsilon_i$$

$$= \alpha_P + \beta_P \cdot R_m + \varepsilon_P$$

The **total risk** of a portfolio: $\sigma_P^2 = \beta_P^2 \cdot \sigma_m^2 + \sigma_{\varepsilon_P}^2$

$$\beta_P^2 = \left(\sum_{i=1}^{N} x_i \cdot \beta_i \right)^2$$

$$\sigma_{\varepsilon_P}^2 = \sum_{i=1}^{N} x_i^2 \cdot \sigma_{\varepsilon_i}^2 , \quad \text{as } Cov(\varepsilon_i, \varepsilon_j) = 0 \quad \forall i \neq j$$

➔ $\sigma_P{}^2$ = systematic risk ($\beta_P^2 \cdot \sigma_m^2$) + unsystematic risk ($\sigma_{\varepsilon_P}^2$)

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❍ Unsystematic risk:

- Unsystematic risk of firm i = **company specific**
- Affects only firm i
- Portfolio: ➔ Firms with **good news** (e.g. discovery of a new oil field, development of a new product)

  Firms with **bad news** (e.g. strike, fire, loss of production)

- **With growing N**: Firm specific good and bad news will **cancel out**.

  ➔ The more a portfolio is **diversified**, the **lower** is the unsystematic risk

- Thus, this kind of risk can be diversified.

- If we invest in each stock the same amount of money with $x_i = \frac{1}{N}$,

  ↳ Unsystematic risk of the portfolio:

$$\sigma^2_{\varepsilon_P} = \sum_{i=1}^{N} \left(\frac{1}{N}\right)^2 \cdot \sigma^2_{\varepsilon_i}$$

$$= \frac{1}{N} \cdot \left[\frac{\sigma^2_{\varepsilon_1} + \sigma^2_{\varepsilon_2} + \ldots + \sigma^2_{\varepsilon_N}}{N} \right]$$

$$= \frac{1}{N} \cdot \sigma^2_{\varepsilon_{Average}}$$

➔ with N ↑: Unsystematic risk becomes unimportant

❍ Systematic risk:

- Economy-wide sources of risk that affects the overall stock market. Also called “market risk”. This risk is not diversifiable.

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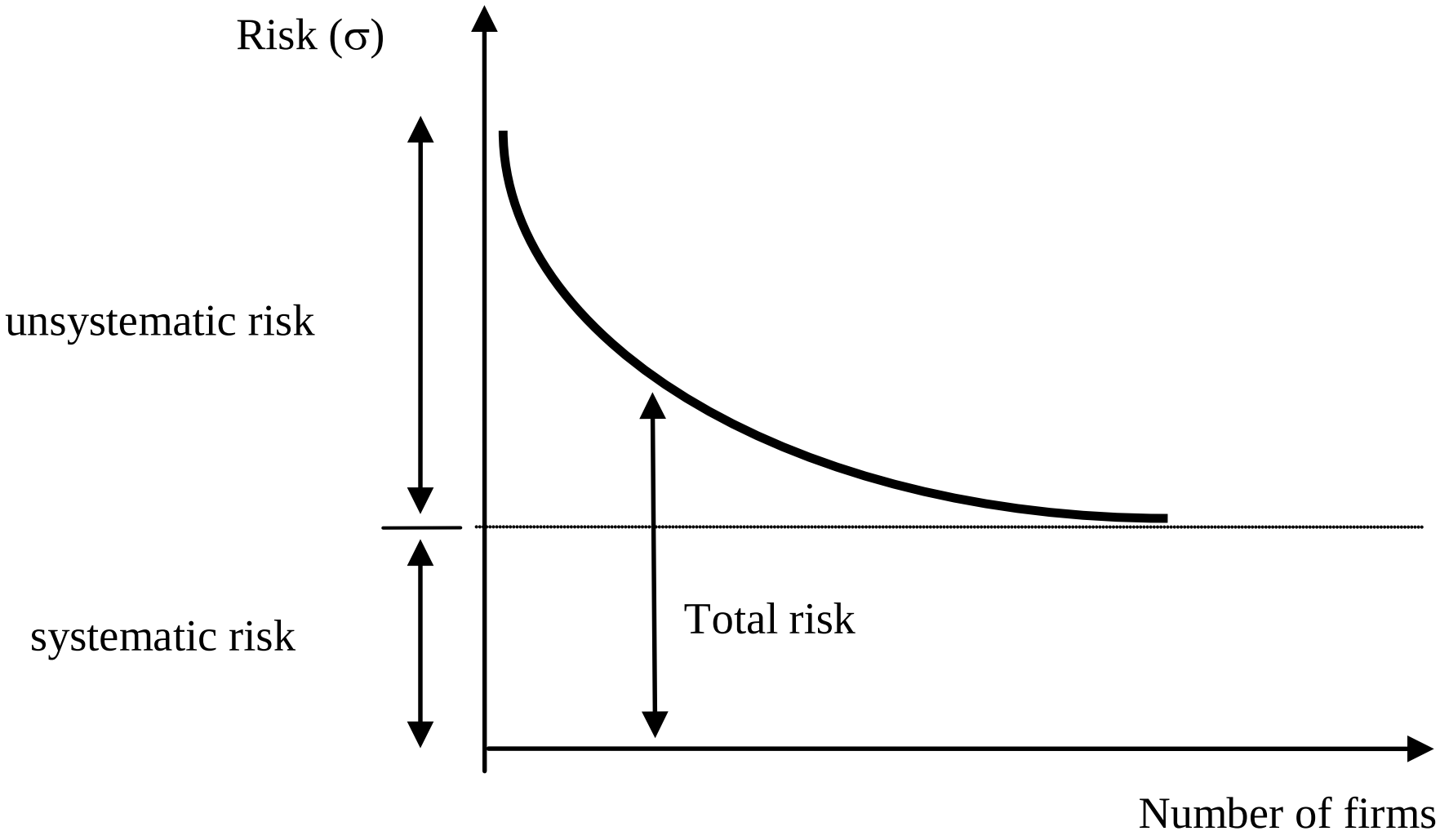

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