139.) $f(x) = \frac{x^2+t}{x-t}$, $t \neq x$, $x_0 = 1$

$f(x)$ streng monoton fallend in $U_\varepsilon(x_0)$

$f(x_0) = f(1) = \frac{1+t}{1-t} \Rightarrow t \neq 1$

Satz: f stetig in $U_\varepsilon(x_0)$, differenzierbar in $U_\varepsilon(x_0)$
falls $f'(x) < 0,\ \forall x \in U_\varepsilon(x_0) \Rightarrow f$ in $U_\varepsilon(x_0)$ streng monoton fallend
(Umkehrung gilt NICHT!, z.B. für $f(x) = x^3$)
$\Rightarrow$ Fall mit $f'(x) = 0$ zusätzlich prüfen!

• $f'(x_0) = \frac{2x_0(x_0-t)-(x_0^2+t)}{(x_0-t)^2} = \frac{x_0^2-2tx_0-t^2}{\underbrace{(x_0-t)^2}_{>0}}$

$f'(x_0) < 0 \iff x_0^2 - 2tx_0 - t^2 < 0$

$\underbrace{t(2x_0+1)}_{>0} > x_0^2$

$\Updownarrow$

$x_0 > -\frac{1}{2}$

✓

$t > \frac{x_0^2}{2x_0+1} = \frac{1}{3}$

$\Rightarrow f$ in $U_\varepsilon(x_0)$ streng monoton fallend für $t > \frac{1}{3}$

$f'(x_0) = 0 \iff t = \frac{x_0^2}{\underbrace{2x_0+1}_{\neq 0}} = \frac{1}{3}$

$\Updownarrow$

$x_0 \neq -\frac{1}{2}$

✓

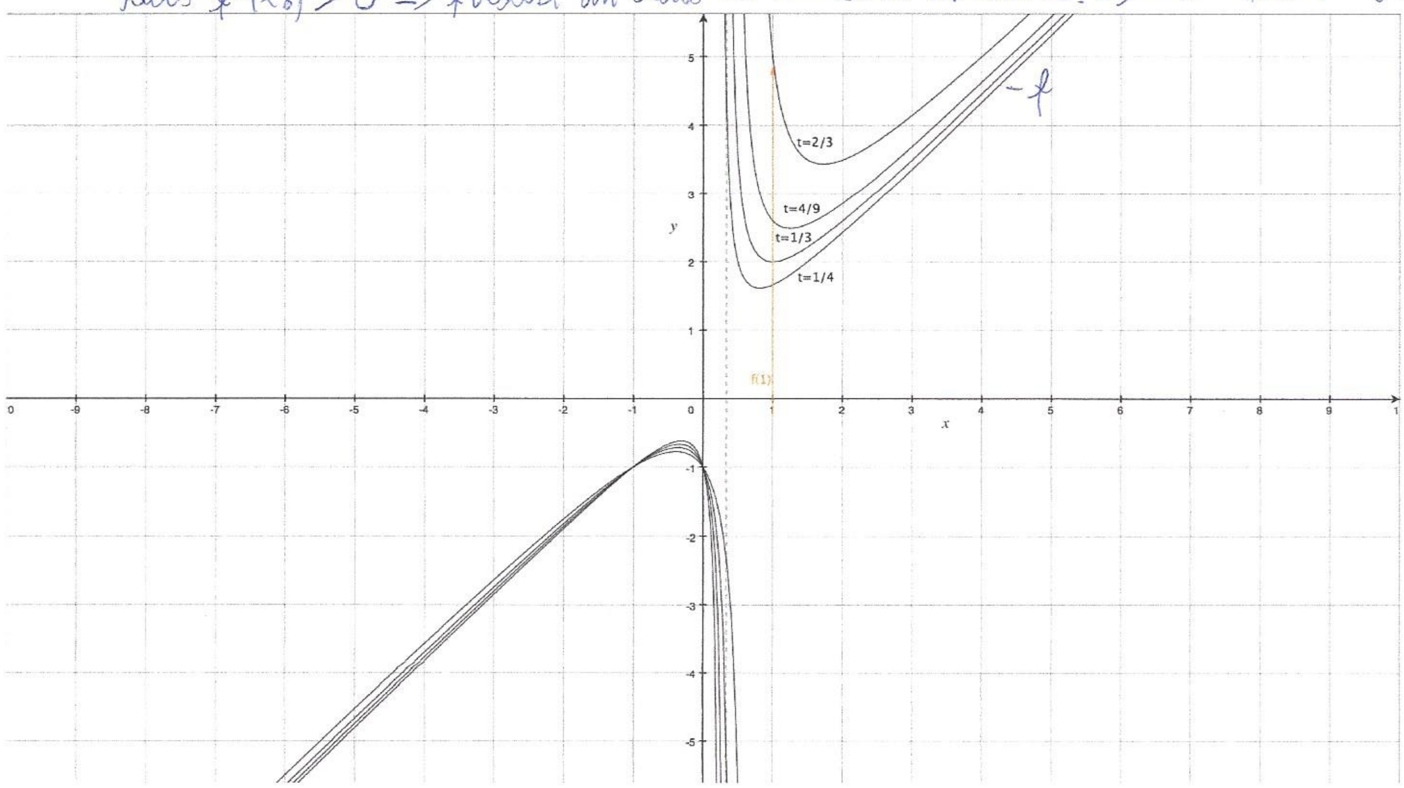
$f''(x_0) = \frac{(2x_0-2t)(x_0-t)^2-(x_0^2-2tx_0-t^2)\cdot 2(x_0-t)}{\underbrace{(x_0-t)^2}_{>0}}$

x_0, t einsetzen

$f''(x_0) = \frac{(2-\frac{2}{3})(1-\frac{1}{3})^2-(1-2\cdot\frac{1}{3}\cdot 1-\frac{1}{9})\cdot 2\cdot(1-\frac{1}{3})}{(1-\frac{1}{3})^2}$

$= \frac{\frac{16}{27}-\frac{8}{27}}{\frac{12}{27}} = \frac{8}{12} = \frac{2}{3} > 0$

Satz: sei f zweimal stetig differenzierbar in $U_\varepsilon(x_0)$.
Sei $x_0 \in U_\varepsilon(x_0)$ mit $f'(x_0) = 0$
Falls $f''(x_0) > 0 \Rightarrow f$ besitzt an Stelle x_0 relatives Minimum. $\Rightarrow$ -‖- für $t = \frac{1}{3}$. (X)



144.) $f(x) = x \cdot e^{-x^2}$

$f: \mathbb{R} \to \mathbb{R}$

Nullstellen:

$$f(x)=0 \Leftrightarrow x \cdot \underbrace{e^{-x^2}}_{>0} = 0 \Leftrightarrow x=0$$

Extremwerte:

$$f'(x) = e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x) =$$

$$= \underbrace{e^{-x^2}}_{>0}(1-2x^2)$$

$$f''(x) = -2x \cdot e^{-x^2}(1-2x^2) + e^{-x^2} \cdot (-4x) =$$

$$= e^{-x^2}(-2x+4x^3-4x) = 2e^{-x^2}(2x^3-3x) =$$

$$= 2x\underbrace{e^{-x^2}}_{>0}(2x^2-3)$$

$$f'''(x) = 2\left[-2x \cdot e^{-x^2} \cdot (2x^3-3x) + e^{-x^2}(6x^2-3)\right] =$$

$$= 2\left[e^{-x^2}(-4x^4+6x^2+6x^2-3)\right] =$$

$$= \underbrace{2e^{-x^2}}_{>0}(-4x^4+12x^2-3)$$

$$f\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} \cdot e^{-\frac{1}{2}} = \frac{1}{\sqrt{2e}}$$

$$f'(x) = 0 \Leftrightarrow 2x^2 = 1 \Leftrightarrow x \in \left\{\pm \underbrace{\frac{\sqrt{2}}{2}}_{=\sqrt{\frac{1}{2}}}\right\}$$

$$f''\left(\pm\frac{\sqrt{2}}{2}\right) = \pm\sqrt{2} \cdot e^{-\frac{1}{2}}(1-3) = \mp 2\sqrt{2} \cdot e^{-\frac{1}{2}} \Rightarrow \begin{cases} \frac{\sqrt{2}}{2} & \text{lok. Maximum} \\ -\frac{\sqrt{2}}{2} & \text{lok. Minimum} \end{cases}$$

$$f\left(-\frac{\sqrt{2}}{2}\right) = -\frac{1}{\sqrt{2e}}$$

Wendepunkte:

$$f''(x) = 0 \Leftrightarrow x = 0 \lor x = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$$

$$f'''(0) = 2 \cdot e^{-0^2}(-4 \cdot 0^4 + 12 \cdot 0^2 - 3) = 2 \cdot (-3) = -6 < 0$$

$$f'''\left(\pm\frac{\sqrt{6}}{2}\right) = 2e^{-\frac{3}{2}}\left(-4 \cdot \frac{9}{4} + 12 \cdot \frac{3}{2} - 3\right) = 2e^{-\frac{3}{2}}\underbrace{(-9+18-3)}_{6} > 0$$

$$f\left(\frac{\sqrt{6}}{2}\right) = \frac{\sqrt{6}}{2} \cdot e^{-\frac{3}{2}} = \sqrt{\frac{3}{2e^3}}$$

konkav → konvex | konvex → konkav

144.) → Grenzwerte:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x\cdot e^{-x^2} \quad („\infty\cdot 0") = \lim_{x\to+\infty} \frac{x}{e^{x^2}} \quad („\frac{\infty}{\infty}") = \lim_{x\to+\infty} \frac{(x)'}{(e^{x^2})'} = \lim_{x\to+\infty} \frac{1}{2x e^{x^2}} = 0$$

$$\lim_{x\to-\infty} f(x) = (\text{analog}) \lim_{x\to-\infty} \frac{1}{2x e^{x^2}} = 0$$

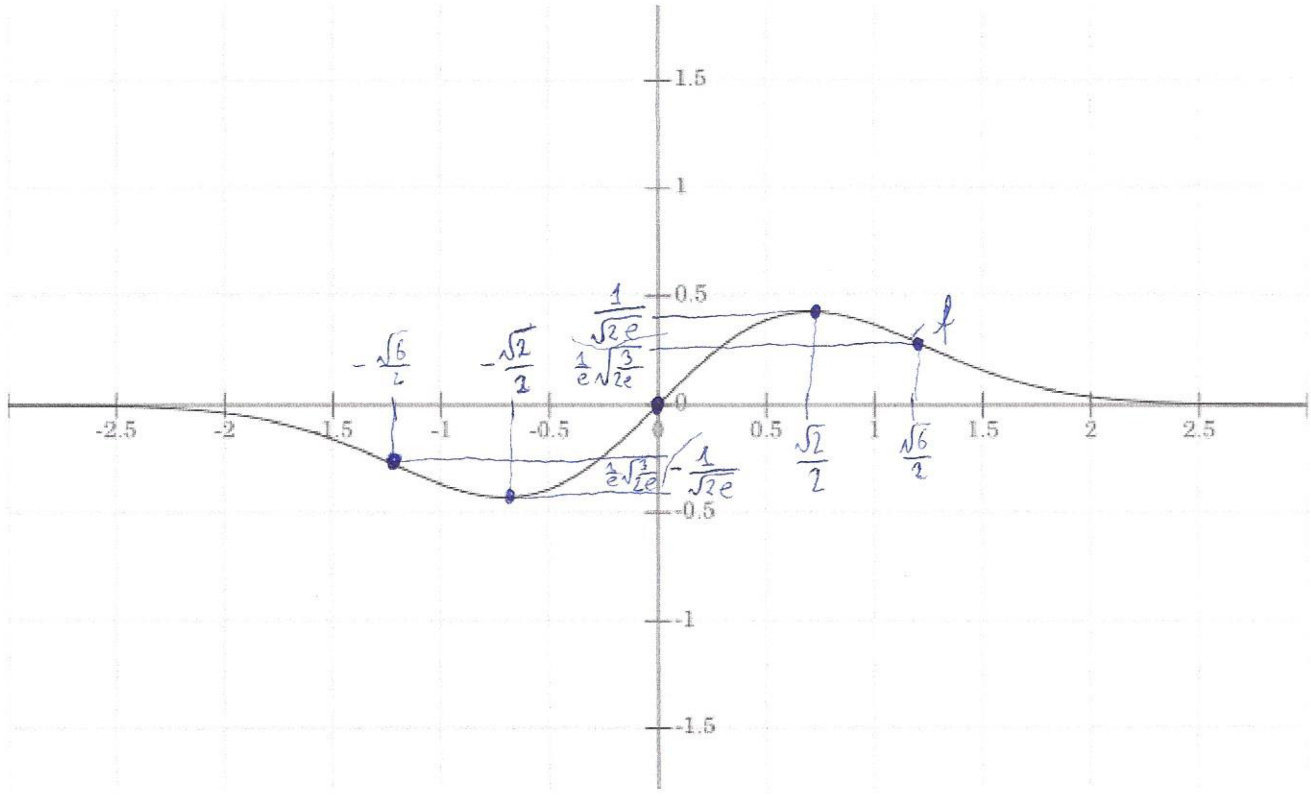
Symmetrieeigenschaften:

- $-f(-x) = -\left[(-x)\cdot e^{-(-x)^2}\right] = x\cdot e^{-x^2} = f(x) \Rightarrow$ f(x) ist eine ungerade Funktion
- $f(-x) = (-x)\cdot e^{-(-x)^2} = -x\cdot e^{-x^2} = -f(x)$

$\Rightarrow$ keine gerade Funktion

· Globale Extrema:

- $\max\{x e^{-x^2}\} = \frac{1}{\sqrt{2e}}$ an Stelle $x = \frac{\sqrt{2}}{2}$
- $\min\{x e^{-x^2}\} = -\frac{1}{\sqrt{2e}}$ an Stelle $x = -\frac{\sqrt{2}}{2}$



183.)

(a) $\lim\limits_{x\to 1^-} \ln(1-x)\cdot\ln(x) = \lim\limits_{x\to 1^-} \frac{\ln(1-x)}{\ln^{-1}(x)}$ = (Regel von de L'Hospital)

(„$-\infty\cdot 0$") (\ „$\frac{-\infty}{-\infty}$" = „$\frac{\infty}{\infty}$")

$$= \lim_{x\to 1^-} \frac{(\ln(1-x))'}{(\ln^{-1}(x))'} = \lim_{x\to 1^-} \frac{-\frac{1}{1-x}}{\frac{1}{x}\cdot(-\ln^{-2}(x))} =$$

$$= \lim_{x\to 1^-} \frac{x\cdot\ln^2(x)}{1-x} \overset{\text{RvdH}}{=} \lim_{x\to 1^-} \frac{(x\ln^2(x))'}{(1-x)'} =$$

(„$\frac{0}{0}$")

$$= \lim_{x\to 1^-} -\left((1\cdot\ln^2(x)) + \left(\cancel{x}\,2\ln(x)\cdot\frac{1}{\cancel{x}}\right)\right) =$$

$$= \lim_{x\to 1^-} \left(2\ln(x) - \ln^2(x)\right) = 0$$

(b)

I) $\lim\limits_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim\limits_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{x^{-1}}$ = (Regel von de L'Hospital)

„$\infty\cdot 0$" „$\frac{0}{0}$"

$$= \lim_{x\to\infty} \frac{(\ln(1+\frac{1}{x}))'}{(x^{-1})'} = \lim_{x\to\infty} \frac{\cancel{-x^{-2}}\cdot\frac{1}{1+\frac{1}{x}}}{\cancel{-x^{-2}}} = \lim_{x\to\infty} \frac{1}{1+\underset{\downarrow\,0}{\frac{1}{x}}} = 1$$

II) $\lim\limits_{x\to\infty} x\cdot\ln\left(1+\frac{1}{x}\right) = \lim\limits_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x = \lim\limits_{x\to\infty} \ln e = \lim\limits_{x\to\infty} 1 = 1$