TECHNISCHE UNIVERSITÄT WIEN

Probability Theory

Introduction

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Counting: Examples

(1) Fair coin

What is the probability of getting exactly 1 head in 3 tosses of a fair coin?

- We list the eight possible outcomes:

$$\{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$$

- Three outcomes have exactly one head

$$\{TTH, THT, HTT\}$$

- All otcomes are equally probable. We have

$$P(\text{one head in 3 tosses}) = \frac{\text{number of outcomes with one head}}{\text{total number of outcomes}} = \frac{3}{8}$$

Counting: Examples

(2) Playing cards: Poker hands

- A full deck of cards consists of 52 cards:
  - 13 values in hierarchical order (ranks): $2, \ldots 9, 10, J, Q, K, A$
  - 4 suits: $\heartsuit$, $\spadesuit$, $\diamondsuit$, $\clubsuit$

- Poker hands
  - consist of 5 cards
  - One pair refers to two cards of the same rank and three others (pairwise different additional rank)
  - Example: $\{5\spadesuit, 5\heartsuit, 8\diamondsuit, 10\clubsuit, Q\heartsuit\}$
    Note: a full house, e.g. $\{5\spadesuit, 5\heartsuit, 8\diamondsuit, 8\clubsuit, 8\heartsuit\}$, is not a hand with exactly one pair

- The probability of a hand with one pair is:

  (a) less than 5%
  (b) between 5% and 10%
  (c) between 10% and 20%
  (d) between 20% and 40%
  (e) bigger than 40%.

We will come back to this question!

Counting: Goal

- In order to **compute** the exact **probability**, we deal with sets. Furthermore, we have to count the number of elements in each of these sets.
  - Our goal is to learn techniques for counting the number of elements in a set.

- We have found that all possible outcomes were equally probable and used this to find a probability by counting.

- The principle:

  Suppose there are n possible outcomes for an experiment and each is equally likely. If there are k desirable outcomes, the probability of a a desirable outcome is k/n.

HW Question: Can you think of an example where the possible outcomes are not equally probable?

Sets

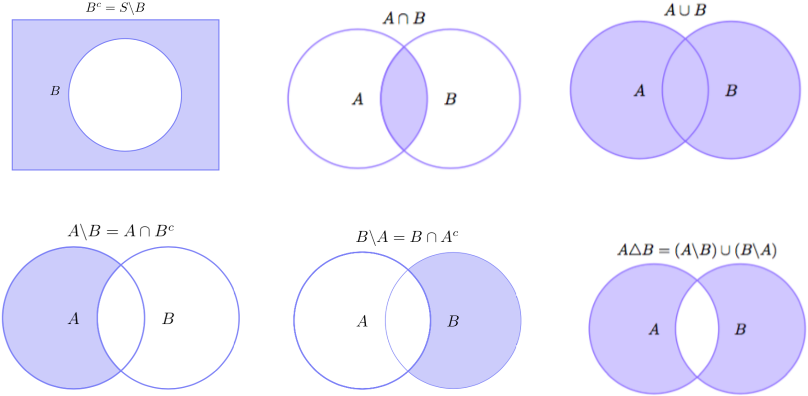
- A set S is a collection of elements.
- $x \in S$ ... an element x in the set S
- $A \subset S$ ... the set A is a subset of S
- $\emptyset$ ... the empty set
- $A^c = S \backslash A = \{x : x \notin A\}$ ... complement
- $A \cap B = \{x : x \in A \text{ and } x \in B\}$ ... intersection
- $A \cup B = \{x : x \in A \text{ or } x \in B\}$ ... union
- $A \backslash B = \{x : x \in A \text{ and } x \notin B\}$ ... difference
- $A \triangle B = (A \backslash B) \cup (B \backslash A)$ ... symmetric difference
- $|A|$ ... number of elements in A

Some properties:

- $A \cup B = B \cup A, \quad A \cap B = B \cap A$
- $A \cap \emptyset = \emptyset, A \cup \emptyset = A, A \cap S = A, \quad A \cup S = S$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C), (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- $(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$
- Complement laws: $A \cup A^c = S,\ A \cap A^c = \emptyset,\ (A^c)^c = A,\ \emptyset^c = S,\ S^c = \emptyset$
- De Morgan's laws: $(A \cup B)^c = A^c \cap B^c,\ (A \cap B)^c = A^c \cup B^c$

Venn Diagrams

- Venn diagrams give an easy way to visualize set operations



Example

- Start with a set S of all natural numbers less than 20. Consider two subsets

$$A = \{\text{the set of all odd numbers}\}$$
$$B = \{\text{the set of all natural numbers divisible by 3}\}$$

  Consider different set operations.
  - Express the set of all numbers divisible by 6 in terms of A and B.
  - What is $A^c \cup B$? State in "words" and as a Venn diagram.

Example: Answer

- Start with a set of S of all natural numbers less than 20. Consider two subsets

$$S = \{1, 2, \ldots, 18, 19\}$$
$$A = \{\text{all odd numbers}\} = \{1, 3, 5, \ldots, 17, 19\}$$
$$B = \{\text{all natural numbers divisible by 3}\} = \{3, 6, 9, 12, 15, 18\}$$

Consider different set operations.

- Write the set of all numbers divisible by 6 in the form of A and B.

$$\begin{aligned}\{\text{all natural numbers divisible by 6}\} \\ &= \{\text{all even numbers divisible by 3}\} \\ &= A^c \cap B\end{aligned}$$

HW other ways: $\underline{A^c \cap B = (A \cup B) \setminus A = B \setminus (A \cap B)}$

- What is $A^c \cup B$? State in "words" and as a Venn diagram.

$$\begin{aligned}A^c \cup B &= \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18\} \\ &= \{\text{all numbers less than 19 which are either even or divisible by 3}\}\end{aligned}$$

The Cartesian product

- The Cartesian product of the sets A and B is the set of the ordered pairs

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

- Example

	$\times$	B 1	2	3	4
	1	(1,1)	(1,2)	(1,3)	(1,4)
A	2	(2,1)	(2,2)	(2,3)	(2,4)
	3	(3,1)	(3,2)	(3,3)	(3,4)

$$A \times B = \{1, 2, 3\} \times \{1, 2, 3, 4\}$$

$$= \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\}$$

Counting

1. The Inclusion-Exclusion Principle

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Example:
  - A band consists of singers and guitar players
    - 7 people sing S
    - 4 play guitar G
    - 2 do both $S \cap G$

  How many people are in the band?

$$\text{Size of the band} = |S| + |G| - |S \cap G| = 7 + 4 - 2 = 9$$

Counting

1. The Product Rule

   - If there are n **ways to perform Action 1** and then by m **ways to perform Action 2**, then there are $n \cdot m$ **to perform Action 1 followed by Action 2**.

   - Let $n = 3$ and $m = 2$

$$\text{To choose } \underbrace{\text{one from } \{a_1, a_2, a_3\}}_{\text{Action1}} \text{ AND } \underbrace{\text{one from } \{b_1, b_2\}}_{\text{Action 2}} \text{ is the same}$$

$$\text{as to choose } \underbrace{\text{one from } \{a_1b_1, a_1b_2, a_2b_1, a_2b_2, a_3b_1, a_3b_2\}}_{\text{Action1, Action2}}$$

- Example:
  - There are 8 participants in the 400m final. In how many ways can gold, silver and bronze medals be awarded? $8 \cdot 7 \cdot 6 = 336$ ways

Questions

HW DNA consists of sequences of nucleotides: Adenine (A), Thymine (T), Guanine (G) and Cytosine (C).

(1) How many DNA sequences of lenght 3 are there?
(2) How many DNA sequences of lenght 3 are there with no repeats?

HW Anna does not want to wear green and red together.
She thinks black and jeans go with everything.
Here is her wardrobe:

- shirts: 4R, 5B, 2G
- sweaters: 3B, 2R, 1G
- trousers: 3J, 2B.

How many different outfits can she wear?

Hint: A tree diagram is an easy way to represent answer.

Permutations

- Lining things up = order is important
  - In how many ways can we do this?
    - *abc* and *cab* are different permutations of $\{a, b, c\}$
    - All possible permutations of $\{a, b, c\}$:

      $$abc,\ acb,\ bac,\ bca,\ cab,\ cba$$

      In total $3! = 6$ possibilities
    - How many permutations of $\{a, b, c, d, e, f, g, h, i\}$ are there?

      There are $9!$ possibilities

- The **product rule** says that the number of permutations of a set of k elements equals

$$k! = k \cdot (k-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

Permutations of k from a set of n

- List all permutations of 3 elements from $\{a, b, c, d\}$.

$$\begin{array}{cccccc} abc & abd & acb & acd & adb & adc \\ bac & bad & bca & bcd & bda & bdc \\ cab & cad & cba & cbd & cda & cdb \\ dab & dac & dba & dbc & dca & dcb \end{array}$$

... 24 permutations

- Question: How many permutations of 8 out of a set of 15 are there?

Combinations

- Selection of subsets - order is not important
  - In how many ways can we do this?
    - All combinations of 3 elements out of $\{a, b, c, d\}$ are
$$\{a, b, c\},\ \{a, b, d\},\ \{a, c, d\},\ \{b, c, d\}.$$
In total $\binom{4}{3} = \frac{4!}{3!\,1!} = 4$ possibilities
- The number of all combinations of k from a set of n elements is $\binom{n}{k}$.
    - How many permutations of 3 out of a set of 4 are there?

<u>permutations and combinations</u>

abc	acb	bac	bca	cab	cba	$\{a, b, c\}$
abd	adb	bad	bda	dab	dba	$\{a, b, d\}$
acd	adc	cad	cda	dac	dca	$\{a, c, d\}$
bcd	bdc	cbd	cdb	dbc	dcb	$\{b, c, d\}$

permutations — combinations

$${}_4P_3 = 3! \cdot \binom{4}{3} = 24 \qquad {}_4C_3 = \binom{4}{3}$$

Permutations and Combinations

- Question: How many permutations of 8 out of a set of 15 are there?
  - Answer:
    - There are $\binom{15}{8}$ ways to choose a subset of 8 elements from a set with 15 elements
    - The number of permutations of elements of a set with 8 elements is $8!$
    - There are $\binom{15}{8} \cdot 8! = \frac{15!}{8!\,7!} \cdot 8! = \frac{15!}{7!} = 259\,459\,200$ permutations of 8 out of a set of 15.

Questions

(1) How many possibilities are there to get exactly 3 heads in 10 tosses of a fair coin? $\binom{10}{3} = \frac{10!}{3!\,7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$

(2) What is is the probability to get exactly 3 heads? $\frac{\binom{10}{3}}{2^{10}} = \frac{120}{1024} = 0.117$

Probability theory

- We use the following notation/interpretation:
  - Random experiment
  - Ω ... The set of all possible outcomes ... sample space
    - $\omega \in \Omega$ ... elementary events
  - $A \subseteq \Omega$ ... event
    - $\emptyset$ ... impossible event
    - Ω ... sure event
    - $A^c = \Omega \setminus A$ ... opposite event
    - $A \cap B$, $A \cup B$ of two events A, B ... are also events
    - $A \cap B = \emptyset$ ... A and B are disjoint (mutually exclusive)
  - Probability (probability mass) P: $\mathcal{F} \to [0, 1]$ ... If Ω is finite then $\mathcal{F}$ is the power set $\mathcal{P}(\Omega)$

    Axioms:
    - $P(\Omega) = 1$
    - $P(A) \geqslant 0$ for all $A \in \mathcal{F}$
    - $P(A \cup B) = P(A) + P(B)$ for all disjoint $A, B \in \mathcal{F}$
  - Properties of P
    - $P(A^c) = 1 - P(A)$, special $P(\emptyset) = 0$
    - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ... more details next week ...

Thank you for your attention!