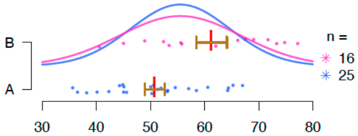
TECHNISCHE UNIVERSITÄT WIEN

Surrounding the two-sample t-test



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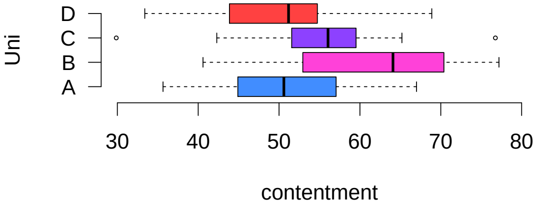
Motivation

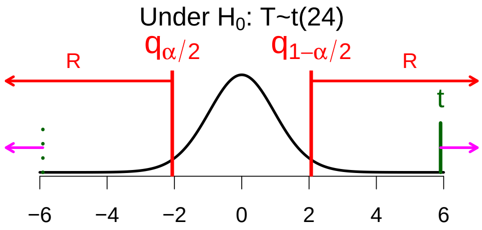
At four universities students of a certain study program were interviewed regarding the level of their satisfaction with the study situation. An extensive survey had to be filled out. Subsequently, for every respondent a global value of ’contentment’ was evaluated.



Question?

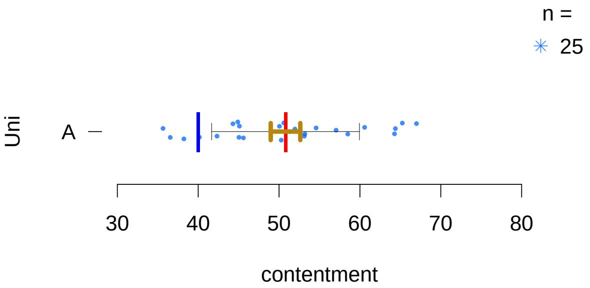
- The median value of contentment of the respondents from Uni A was about the same as the median of which Uni? D
- The third quartile of the respondants of Uni C was about? 60

Reminder: One-sample situation



- $H_0 : \mu = \mu_0$
- Mean $\bar{x}$ far away from μ_0?
- Answer: Yes! Many standard errors $sem = s/\sqrt{n}$ away (approx. $6 \cdot sem$)
- In the model $X_1, \ldots, X_n$ i.i.d. RVs with $X_1 \sim N(\mu, \sigma^2)$ it holds under H_0
  $T = \frac{\bar{X} - \mu_0}{SEM} \sim t(n-1)$
- The estimated standard deviation of $\bar{X}$ is $1 \cdot SEM$ (not $6 \cdot SEM$)
- Judge the evaluated data $t = \frac{\bar{x} - \mu_0}{sem}$ according to the $t(n-1)$-distribution
- Here: $p < \alpha = 5\%$ resp. $t \in R$, thus reject H_0 on the α level

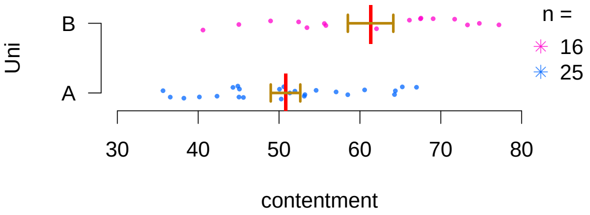
Reminder: One-sample situation



Message:
We judge the discrepancy of $\bar{x}$ and μ_0 in the units 'standard error'

$$\boxed{sem = \frac{s}{\sqrt{n}}}$$

Today: Two-sample situation

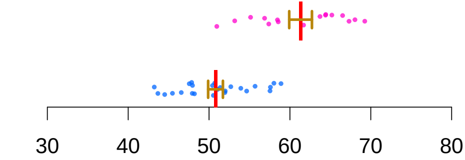


- The satisfaction $(x_1, \ldots, x_{n_1})$ of the students from Uni A ('index 1') has the tendency to be smaller than the satisfaction $(y_1, \ldots, y_{n_2})$ in Uni B ('index 2')
- Can this discrepancy observed in the data be easily explained by chance, if there is actually no difference between the Unis (in the 'populations' of all students of the program)?
- Measure discrepancy through the distance of means $\bar{y}$ and $\bar{x}$
- Question: Is this discrepancy large? Respectively, are the means far apart?
- Answer: This depends on the standard errors sem_y and sem_x

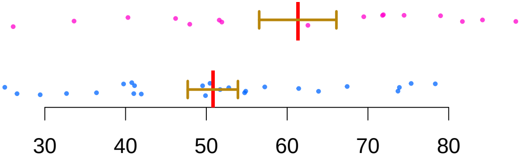
sem_y is the standard error of $\bar{y}$, i.e., based on the data $(y_j)_j$ only, and analogously sem_x is based on the $(x_i)_i$.

Standard errors!

Discrepancy huge



Discrepancy smaller



(while the means in the upper and lower graphic coincide)

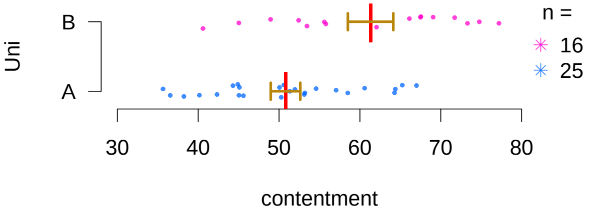
Thus: discrepancy large if the standard errors small

Two-sample t-statistic

The two-sample t-statistic

$$\boxed{t = \frac{\bar{y} - \bar{x}}{\sqrt{sem_y^2 + sem_x^2}}}$$

measures the discrepancy of the means in relation to their standard errors

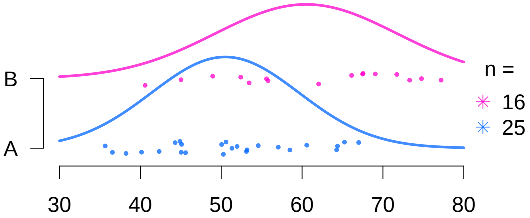


Here: $t \approx 3.1$

$\rightarrow$ Need a model, in order to explicitly judge the value of t

Model

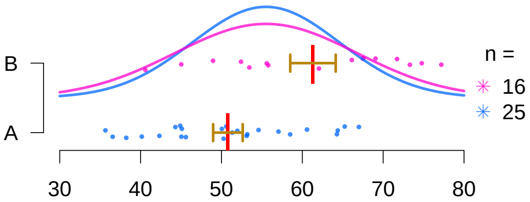
<u>Model</u>: Let $X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}$ be independent RVs with
$X_i \sim N(\mu_1, \sigma_1^2)$ for $i = 1, \ldots, n_1$ and
$Y_j \sim N(\mu_2, \sigma_2^2)$ for $j = 1, \ldots, n_2$ with $(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$



- Independence
- Normal distribution
- Each group has their own (unknown) parameters
  in particular there is a possible 'shift'

Null hypothesis

Let $X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}$ be independent RVs with
$X_i \sim N(\mu_1, \sigma_1^2)$ for $i = 1, \ldots, n_1$ and
$Y_j \sim N(\mu_2, \sigma_2^2)$ for $j = 1, \ldots, n_2$ with $(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$



- Null hypothesis: $H_0 : \mu_1 = \mu_2$
  i.e., distributions are not shifted against each other, 'no difference'
- In the model, how unlikely is the observed discrepancy, resp. $t \approx 3.1$, if H_0 holds true, and hence there is no shift?

Distribution of the t-statistic

Let $X_1, \dots, X_{n_1}, Y_1, \dots, Y_{n_2}$ be independent RVs with
$X_i \sim N(\mu_1, \sigma_1^2)$ for $i = 1, \dots, n_1$ and
$Y_j \sim N(\mu_2, \sigma_2^2)$ for $j = 1, \dots, n_2$ with $(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$

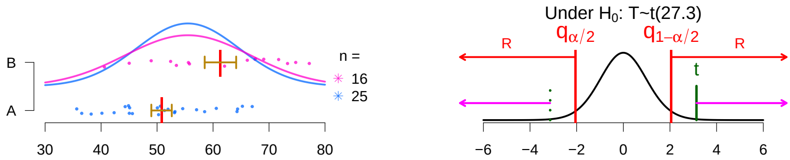
Under $H_0 : \mu_1 = \mu_2$ it holds (approx), that

$$T := \frac{\bar{Y} - \bar{X}}{\sqrt{SEM_y^2 + SEM_x^2}} \sim t(\nu)$$

- $t(\nu)$ is the t-distribution with ν degrees of freedom
- ν depends on the sample sizes n_1 und n_2, as well as on the standard errrors SEM_x and SEM_y $\rightarrow$ sufficient if R knows ν
- Why is this distribution plausible?
  - Due to independence, standardization of the difference $\bar{Y} - \bar{X}$ yields $\frac{(\bar{Y}-\bar{X})-(\mu_2-\mu_1)}{\sqrt{\sigma_2^2/n_2+\sigma_1^2/n_1}} \sim N(0,1)$
  - But under H_0 the difference in expectations $\mu_2 - \mu_1$ vanishes
  - The estimation of σ_1 and σ_2 again yields heavier tails as in $N(0,1)$
- Rough approximation: If H_0 holds true, then $|t| \approx 1$ is a typical value, while $|t| \gtrapprox 3$ barely happens (as T is approx $N(0,1)$ distributed)

Judgement of the discrepancy

How unlikely is the observed discrepancy, resp. $t \approx 3.1$, if H_0 holds true, i.e., if there is no difference in the population means?



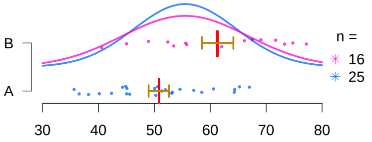
'Business as usual'

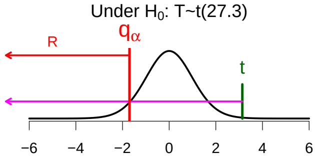
- Under $H_0 : \mu_x = \mu_y$ it is $T \sim t(27.3)$
- From the data we compute $t \approx 3.1$
- p-value: $p = \mathbb{P}_{H_0}(|T| \geqslant |t|) \approx 0.004$
- For $\alpha = 5\%$ (previously set) it is $t \in R$ ($\Leftrightarrow p \leqslant \alpha$), i.e., reject H_0 on the 5%-level
- Interpretation: A discrepancy as extreme as observed in the data appears only in about 4 of 1000 cases, if there is no difference between the Unis. In this sense the data speak against the null hypothesis.

Remark

Uni B: $y_1, \ldots, y_{n_2}$
Uni A: $x_1, \ldots, x_{n_1}$

$$t = \frac{\bar{y} - \bar{x}}{\sqrt{sem_y^2 + sem_x^2}}$$





- We perfomed a two-sided test (null hypothesis: $H_0 : \mu_1 = \mu_2$)
  Alternative $H_A : \mu_1 \neq \mu_2$
  Extreme values of t speak against H_0 (here: H_0 rejected)
- One-sided tests:
  - right-sided: $H_A : \mu_1 < \mu_2$
    Large values of t speak against H_0 (here: H_0 rejected)
  - left-sided: $H_A : \mu_1 > \mu_2$
    Small values of t speak against H_0 (here: H_0 not rejected)

t-test in R (according to 'Welch')

```
# Enter data
x <- c(...)
y <- c(...)
# Perform t-test
t.test(y,x,...)
# Output

        Welch Two Sample t-test

data:  y and x
t = 3.1365, df = 27.311, p-value = 0.004067
alternative hypothesis:
true difference in means is not equal to 0
95 percent confidence interval:
  3.641362 17.396356
sample estimates:
mean of x mean of y
 61.31892  50.80006
```

- The t-test presented is also known as the *Welch*-test
- The degrees of freedom 27.3 can be read from here

Generalization

Let $X_1, \dots, X_{n_1}, Y_1, \dots, Y_{n_2}$ be independent RVs with
$X_i \sim N(\mu_1, \sigma_1^2)$ for $i = 1, \dots, n_1$ and
$Y_j \sim N(\mu_2, \sigma_2^2)$ for $j = 1, \dots, n_2$ with $(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$

Under $H_0 : \mu_1 = \mu_2$ it holds (approx) that

$$T = \frac{\bar{Y} - \bar{X}}{\sqrt{SEM_y^2 + SEM_x^2}} \sim t(\nu)$$

H_0 reformulated: $d := \mu_2 - \mu_1 = 0$ (no difference)

Generalization of the null hypothesis: $H_0 : d = d_0$ (difference is $d_0 \in \mathbb{R}$)

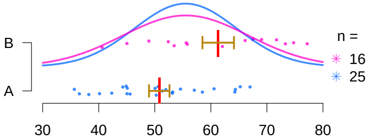
Under $H_0 : d = d_0$ it holds (approx) that

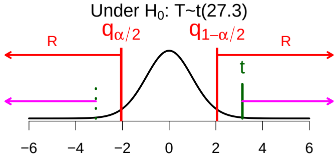
$$T := \frac{(\bar{Y} - \bar{X}) - d_0}{\sqrt{SEM_y^2 + SEM_x^2}} \sim t(\nu)$$

Analogous procedure, the only difference is that d_0 has to be subtracted in the numerator of t (Statistic has the same structure as in the one-sample case: $T = [\spadesuit - \spadesuit]/\spadesuit$)

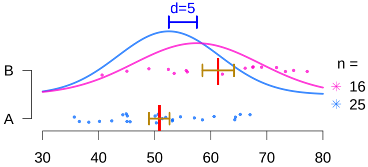
Example

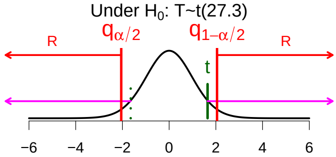
Under $H_0 : d = 0$ ($\Leftrightarrow \mu_1 = \mu_2$) $\rightarrow$ reject H_0





Unter $H_0 : d = 5$ ($\Leftrightarrow \mu_2 = \mu_1 + 5$) $\rightarrow$ do <u>not</u> reject H_0





t-test in R

```
# Enter data
x <- c(...)
y <- c(...)
# perform t-test
t.test(y,x,mu=5,...)
# Output

        Welch Two Sample t-test

        data:  y and x
t = 1.6456, df = 27.311, p-value = 0.1113
alternative hypothesis:
true difference in means is not equal to 5
95 percent confidence interval:
  3.641362 17.396356
sample estimates:
mean of x mean of y
 61.31892  50.80006
```

- Where does the confidence interval come from?

Two-sample t-test and confidence interval

- Let $X_1, \dots, X_{n_1}, Y_1, \dots, Y_{n_2}$ be independent RVs with
  $X_i \sim N(\mu_1, \sigma_1^2)$ for $i = 1, \dots, n_1$ and
  $Y_j \sim N(\mu_2, \sigma_2^2)$ for $j = 1, \dots, n_2$ with $(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$
- $d = \mu_2 - \mu_1$
- Let $q_{1-\alpha/2}$ be the $(1-\alpha/2)$-quantile of the $t(\nu)$-distribution (R knows ν)

Under $H_0 : d = d_0$ it holds (approx)

$$T := \frac{(\bar{Y} - \bar{X}) - d_0}{\sqrt{SEM_y^2 + SEM_x^2}} \sim t(\nu)$$

and equivalently: The confidence interval

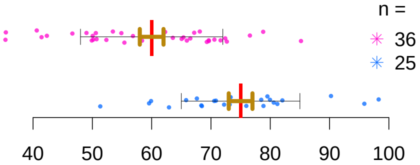
$$I := \left((\bar{Y} - \bar{X}) - q_{1-\alpha/2} \cdot \sqrt{SEM_y^2 + SEM_x^2},\ (\bar{Y} - \bar{X}) + q_{1-\alpha/2} \cdot \sqrt{SEM_y^2 + SEM_x^2} \right)$$

overlaps the parameter d_0 with probability (approx) $1 - \alpha$

- C.I. has 'structure' as in the one-sample case: $I = (\spadesuit - q \cdot \spadesuit,\ \spadesuit + q \cdot \spadesuit)$
- Derivation analogously: $\alpha = \mathbb{P}_{H_0}(T \in R) = \cdots = \mathbb{P}_{H_0}(I \not\ni d_0)$
- For n_1, n_2 large: possibly use normal approximation, i.e., replace $t(\nu)$ with $N(0,1)$. In this case the normality assumption of the RVs can be dropped

Question

Can $H_0 : \mu_2 = \mu_1$ be rejected on the 5%-level?



Here t-test naively:

- $\bar{x} \approx 75$ and $\bar{y} \approx 60$ (balances in equilibrium)
- $s_x \approx 10$ and $s_y \approx 12$ (approx bell-shaped distributions, $\approx 2/3$ of the data captured)
- $sem_x \approx 10/5 = 2$ and $sem_y \approx 12/6 = 2 \rightarrow$ means are far apart
- $$t \approx \frac{60 - 75}{\sqrt{2^2 + 2^2}} = \frac{-15}{\sqrt{8}} \approx \frac{-15}{\sqrt{9}} = -5 \qquad \text{(extreme value!)}$$
- $R \approx (-\infty, -2] \cup [2, \infty)$ (as 97.5%-quantile of $N(0,1)$ is $q \approx 1.96 \approx 2$)
- $t \in R$, thus reject H_0 … or equivalently
- C.I. $\approx (-15 - 2 \cdot 3, -15 + 2 \cdot 3) = (-21, -9)$ does not overlap $d = 0$

Again: Clear, only naive estimations. But $t \approx -5$ is extreme, regardless of our rough estimation. What does R say?

t-test in R

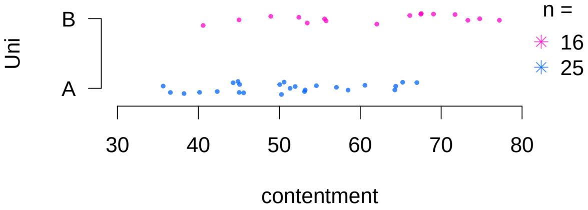
```
# Enter data
x <- c(...)
y <- c(...)
# Perform t-Test
t.test(y,x,...)
# Output

        Welch Two Sample t-test

data:  y and x
t = -4.9829, df = 56.052, p-value = 6.358e-06
alternative hypothesis:
true difference in means is not equal to 0
95 percent confidence interval:
 -21.097942  -8.998583
sample estimates:
mean of x mean of y
 59.27828  74.32654
```

Our naive estimations were very precise (absurd how well it fits). But again: Even if the estimations in the picture were biased and we had obtained e.g., $t \approx -4$, then still this would have been an extreme value!

Welch vs. Student



Model 'Welch': Let $X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}$ be independent RVs with
$X_i \sim N(\mu_1, \sigma_1^2)$ for $i = 1, \ldots, n_1$ and
$Y_j \sim N(\mu_2, \sigma_2^2)$ for $j = 1, \ldots, n_2$ with $(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$

A different approach, also known as Student's t-test, assumes the variances to be equal in both groups

Model 'Student': Let $X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}$ be independent RVs with
$X_i \sim N(\mu_1, \sigma^2)$ for $i = 1, \ldots, n_1$ and
$Y_j \sim N(\mu_2, \sigma^2)$ for $j = 1, \ldots, n_2$ with $(\mu_1, \mu_2, \sigma^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$

In our data, the spread in both groups is similar, so Student's additional assumption is plausible. (In general, Welch's version is applicable to a wider range of data, as it allows for different variances in both groups)

Welch vs. Student

Model 'Welch': Let $X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}$ be independent RVs with
$X_i \sim N(\mu_1, \sigma_1^2)$ for $i = 1, \ldots, n_1$ and
$Y_j \sim N(\mu_2, \sigma_2^2)$ for $j = 1, \ldots, n_2$ with $(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$

Under $H_0 : \mu_1 = \mu_2$ it holds (approx), that

$$T := \frac{\bar{Y} - \bar{X}}{\sqrt{SEM_y^2 + SEM_x^2}} \sim t(\nu)$$

Model 'Student': Let $X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}$ be independent RVs with
$X_i \sim N(\mu_1, \sigma^2)$ for $i = 1, \ldots, n_1$ and
$Y_j \sim N(\mu_2, \sigma^2)$ for $j = 1, \ldots, n_2$ with $(\mu_1, \mu_2, \sigma^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$

Under $H_0 : \mu_1 = \mu_2$ it holds (exactly), that

$$T := \frac{\bar{Y} - \bar{X}}{\sqrt{n_2^{-1} + n_1^{-1}} \cdot S_p} \sim t(n_2 + n_1 - 2)$$

It is $S_p^2 := \frac{(n_2-1)S_y^2 + (n_1-1)S_x^2}{n_2+n_1-2}$, which is often called the *pooled empirical variance*

- downside Student: need spread to be similar in both groups
- upside Student: the test is exact and the degrees of freedom have a simple form

For a derivation of Student's t-test see e.g., Messer, M. and Schneider, G. *Statistik: Theorie und Praxis im Dialog*,

t-test in R (according to 'Student')

```
# Enter data
x <- c(...)
y <- c(...)
# Perform t-test
t.test(y,x,var.equal=TRUE,...)
# Output

        Two Sample t-test

data:  y and x
t = 3.2846, df = 39, p-value = 0.002163
alternative hypothesis:
true difference in means is not equal to 0
95 percent confidence interval:
  4.041178 16.996539
sample estimates:
mean of x mean of y
 61.31892  50.80006
```

- The argument `var.equal` allows to assume equal variances ('Student')
- The degrees of freedom are $n_2 + n_1 - 2 = 16 + 25 - 2 = 39$

Multiple-choice questions

(1) Two features of a novel operating system are compared using a two-sample t-test. The statistics for the first feature are $\bar{x} = 23$, $s_x^2 = 66$ and $n_x = 6$ and those for the second feature are $\bar{y} = 29$, $s_y = 10$ and $n_y = 4$. The rejection region is given through $R = (-\infty, -q] \cup [q, \infty)$. Then, it holds

a. we reject for $q = 0.4$ but not for $q = 1.2$
b. we reject for both $q = 0.4$ and $q = 1.2$
c. we do not reject for $q = 0.4$ but for $q = 1.2$
d. we do neither reject for $q = 0.4$ nor for $q = 1.2$

Multiple-choice questions

(2) The statement "If there is sufficient evidence to reject a null hypothesis at the 10significance level, then there is sufficient evidence to reject it at the 5% significance level" is __________ .

Please select the best answer of those provided below.

a. always true

b. never true

c. sometimes true. The p-value for the statistical test needs to be provided for a conclusion.

d. not enough information; this would depend on the type of statistical test used.

Multiple-choice questions

(3) A sociologist focusing on popular culture and media believes that the average number of hours per week (hrs/week) spent using social media is greater for women than for men. Examining two independent simple random samples of 100 individuals each, the researcher calculates sample standard deviations of 2.3 hrs/week and 2.5 hrs/week for women and men respectively. He performed a one-sided two-sample t-test testing $H_0 : \mu_W = \mu_M$ against $H_0 : \mu_W > \mu_M$. If the average number of hrs/week spent using social media for the sample of women is 1 hour greater than that for the sample of men, what conclusion can be made from a hypothesis test?

a. The observed difference in average number of hrs/week spent using social media is not significant.
b. The observed difference in average number of hrs/week spent using social media is significant.
c. A conclusion is not possible without knowing the average number of hrs/week spent using social media in each sample.
d. A conclusion is not possible without knowing the population sizes.

Multiple-choice questions

(4) In hypothesis testing, a β-error occurs when

a. the null hypothesis is not rejected when the null hypothesis is true.
b. the null hypothesis is rejected when the null hypothesis is true.
c. the null hypothesis is not rejected when the alternative hypothesis is true.
d. the null hypothesis is rejected when the alternative hypothesis is true.

Multiple-choice questions

(5) Anna performed a two-sample t-test for testing the equality of sample means of two given data sets x and y. When she applied `t.test()` in `R`, she obtained the follwing output

```
        Welch Two Sample t-test

data:  x and y
t = -1.8479, df = 12.441, p-value = 0.08852
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -4.6198400  0.3706765
sample estimates:
 mean of x  mean of y
-0.2745817  1.8500000
```

Please select the correct statement.

a. Anna tests $H_0 : \mu_x = \mu_y$ against $H_A : \mu_x < \mu_y$.

b. Anna should reject H_0 on the level of significance 5%.

c. Anna should not reject H_0 on the level of significance 5%.

d. Since the test statistics t belongs to the 95% confidence interval, Anna can conclude that H_0 is correct.

Multiple-choice questions

(6) A prospective observational study on the relationship between sleep deprivation and heart disease was done by researchers at the Medical University. Women who slept at most 5 hours a night were compared to women who slept for 8 hours a night (reference group). After adjusting for potential confounding variables like smoking, a 95% confidence interval for the relative risk of heart disease was $(1.10, 1.92)$. Based on this confidence interval, a consistent conclusion would be

a. Sleep deprivation is associated with a modestly increased risk of heart disease.
b. Sleep deprivation is associated with a modestly decreased risk of heart disease.
c. There was no evidence of an association between sleep deprivation and heart disease.
d. Lack of sleep causes the risk of heart disease to increase by 10% to 92%.

Multiple-choice questions

(7) A random sample of 25 college stidents was obtained and each was asked to report their actual height and what they wished as their ideal height. A 95% confidence interval for the average difference between their ideal and actual heights μ_d (in centimeters) was (2.0, 5.6). Based on this interval, which one of the null hypotheses below (versus a two-sided alternative) can be rejected?

a. $H_0 : \mu_d = 0.5$
b. $H_0 : \mu_d = 1.0$
c. $H_0 : \mu_d = 1.5$
d. $H_0 : \mu_d = 2.0$

Multiple-choice questions

(8) As the degrees of freedom for the t-distribution increase, the distribution approaches

a. value of zero for the mean.
b. the $t(1)$-distribution.
c. the normal distribution.
d. the binomial distribution.

Multiple-choice questions

(9) Anna considers moving to Innsbruck, and is concerned aif the average one-way commute time excees 25 minutes? She took a random sample of 50 Innsbruck residents and finds an average commute time of 29 minutes with a standard deviation of 7 minutes. What are the null and alternative hypotheses for her test?

a. $H_0 : \mu = 25$ vs. $H_1 : \mu > 25$

b. $H_0 : \mu = 25$ vs. $H_1 : \mu < 25$

c. $H_0 : \mu = 29$ vs. $H_1 : \mu > 29$

d. $H_0 : \mu = 29$ vs. $H_1 : \mu < 29$

Multiple-choice questions

(10) It is known that for right-handed people, the dominant (right) hand tends to be stronger. For left-handed people who live in a world designed for right-handed people, the same may not be true. To test this, muscle strength was measured on the right and left hands of a random sample of 15 left-handed men and the difference (left - right) was found. The alternative hypothesis is one-sided (left hand stronger). Assuming the conditions are met, based on the t-statistic of 1.80 the appropriate conclusion for this test using $\alpha = 5\%$ is

a. $df = 14$, so p-value< 0.05 and the null hypothesis can be rejected.

b. $df = 14$, so p-value> 0.05 and the null hypothesis cannot be rejected.

c. $df = 28$, so p-value< 0.05 and the null hypothesis can be rejected.

d. $df = 28$, so p-value> 0.05 and the null hypothesis cannot be rejected.

Thank you for your attention!