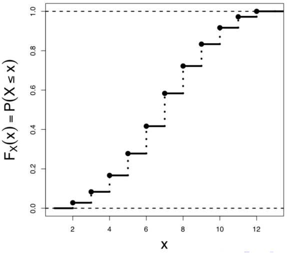
Random variables and distributions



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Recall

- Let $(\Omega, \mathcal{F}, P)$ be a probability space.
  - Discrete: Sample space Ω is countable/listable (finite or infinite)
    - $\Omega = \{\omega_1, \omega_2, \ldots\}, \quad P(\omega_i) = p_i$
    - Probability table

Elementary events ω_i	ω_1	ω_2	$\ldots$	ω_n	$\ldots$	
Probability p_i	p_1	p_2	$\ldots$	p_n	$\ldots$	$\sum_i p_i = 1$

$0 \leqslant p_i \leqslant 1$

    - Probability of an event A

$$P(A) = \sum_{\omega_i \in A} P(\omega_i)$$

  - Continuous: Ω is a subset of the set of real numbers (an interval)
- We now introduce random variables.

Example: A game with two dice

- Random experiment: We roll **two fair** dice and write down the outcomes as (i,j), where i is the result of the first and j is the result of the second die.

  - Probability space $(\Omega, \mathcal{P}(\Omega), P)$: $\quad \Omega = \{(i,j) \,|\, i,j = 1,\ldots 6\}, \quad P(\{(i,j)\}) = \frac{1}{36}$.

  - Game: We win \$500 if the sum is 7, otherwise we lose \$100.

    - We denote this payoff function by

$$D(i,j) = \begin{cases} 500 & \text{if } i+j=7 \\ -100 & \text{if } i+j \neq 7. \end{cases}$$

...This function is an example of a random variable.

A random variable assigns <u>a number to each outcome</u> in the sample space.

Random variables

- A (measurable) function
$$X : \Omega \to \mathbb{R}$$
which assigns a real number $X(\omega) = x$ to every $\omega \in \Omega$
$$\omega \in \Omega \mapsto X(\omega) = x \in \mathbb{R}$$
is called a random variable.
    - The number $x \in \mathbb{R}$ is called the realization of X.
    - The set of values of X is the image space (or feature space)
$$\{x \,|\, X(\omega) = x, \;\; \omega \in \Omega\}$$

Random variables

- Two different types of random variables:
  - Discrete
    - If the image space is finite or countably infinite, the random variable X is discrete.
  - Continuous
    - A random variable X that can take values in one or more intervals, i.e. values that are infinite and uncountable are called continuous.

  - Example:
    - The number of cash registers opened in a grocery store is a discrete random variable.

      The time spent waiting in the queue is a continuous random variable.

Discrete random variables

- The probability mass function (pmf) of a discrete random variable X is the function

$$p(a) = P(X = a).$$

... auf Deutsch: Wahrscheinlichkeitsgewichtungsfunktion (Gewichte)

  - It always holds $0 \leqslant p(a) \leqslant 1$
  - We allow a to be any real number, i.e. $a \in \mathbb{R}$.
    If a is a value that X never takes, then $p(a) = 0$.

- Example
  Let Ω be our earlier sample space for rolling **two fair** dice.
  Let M to be the maximum value of the two dice:

$$M(i,j) = \max(i,j).$$

  - M is a discrete random variable. We describe it by listing its possible values and the probabilities associated with those values.

value	a	1	2	3	4	5	6
pmf	$p(a)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

$$\sum_{a=1}^{6} p(a) = 1$$

Continuous random variables

- $(\Omega, \mathcal{F}, P)$ is a probability space.
  - Continuous range of values $[c, d]$, $[0, 1]$, $[0, +\infty)$, $(-\infty, +\infty)$

- A random variable X is continuous if there exists a function f such that for any $c \leqslant d$ it holds

$$P(c \leqslant X \leqslant d) = \int_c^d f(x)\, dx.$$

  - $f(x)$ is called the probability density function (density function or density)

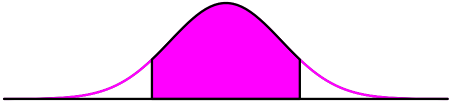
... auf Deutsch: Wahrscheinlichkeitsdichtefunktion (Dichtefunktion oder Dichte)

  - Properties:
    - f is nonnegative: $f(x) \geqslant 0$
    - The area under f is one: $\int_{-\infty}^{+\infty} f(x)\, dx = 1$

Continuous random variables

- Visualization

$$
\begin{aligned}
P(c \leqslant X \leqslant d) &= \int_c^d f(x)\,dx \\
&= \text{Area under } f \text{ between } c \text{ and } d
\end{aligned}
$$



Probability density function and probability

Continuous vs. discrete random variables

- The probability density function $f(x)$ of a continuous random variables (pdf) is the analogue to the probability mass function $p(x)$ of a discrete random variable (pmf).

- Important differences:

  - In contrast to $p(x)$, the probability density function $f(x)$ is not a probability. We have to integrate it to get probabilities.

  - Since $f(x)$ is not a probability, there is no restriction that $f(x)$ is less than or equal to 1.

Cumulative distribution function

- The cumulative distribution function (cdf) of a random variable X is the function F_X defined by

$$\underline{F_X(x) = P(X \leqslant x)}, \quad \text{for all} \quad x \in \mathbb{R}.$$

Auf Deutsch: Verteilungsfunktion

  - $F_X(x) = F(x)$ is defined for **all** real values x.

- A **cumulative distribution function** satisfies the following properties:
  (1) $0 \leqslant F(x) \leqslant 1$ for all $x \in \mathbb{R}$
  (2) F is monotonically increasing, i.e. from $x \leqslant y$ it follows $F(x) \leqslant F(y)$
  (3) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$
  (4) F is right continuous, i.e. $\lim_{h \searrow 0} F(x+h) = F(x)$ for all $x \in \mathbb{R}$.

- **Important properties**:
  - $P(X > x) = 1 - P(X \leqslant x) = 1 - F(x)$ and
  - $P(a \leqslant X \leqslant b) = F(b) - F(a)$.

Example: Maximum of two dice

- Discrete random variables
  - (a) Let Ω be our sample space from the example with two fair dice. Let M be the maximum of the two dice: $M(i,j) = \max(i,j)$. We want to obtain $F_M(a) = F(a) = P(M \leqslant a)$.

From the distribution table

value	a	1	2	3	4	5	6	
pmf	$p(a)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$	1
cdf	$F(a)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{9}{36}$	$\frac{16}{36}$	$\frac{25}{36}$	1	

we get, for example

$F(9) = 1, \quad F(-2) = 0, \quad F(2.5) = \frac{4}{36} \quad \text{and} \quad F(\pi) = \frac{9}{36}.$

Example: Maximum of two dice

- Discrete random variables

  (a) Let Ω be our sample space from the example with two dice.
  Let M be the maximum of the two dice: $M(i,j) = \max(i,j)$.
  We want to obtain $F_M(a) = F(a) = P(M \leqslant a)$

  - We summarize, the cdf of M is of the following form

$$F_M(a) = F(a) = \begin{cases} 0\,, & a < 1 \\ \frac{1}{36}\,, & 1 \leqslant a < 2 \\ \frac{4}{36}\,, & 2 \leqslant a < 3 \\ \frac{9}{36}\,, & 3 \leqslant a < 4 \\ \frac{16}{36}\,, & 4 \leqslant a < 5 \\ \frac{25}{36}\,, & 5 \leqslant a < 6 \\ 1\,, & a \geqslant 6 \end{cases}.$$

F_M is a **step** (piecewise continuous) function

Example: Sum of two dice

(b) Let Ω be our sample space for the same example of rolling two fair dice. Let X be the sum of two dice: $X(i,j) = i + j$.

- e.g. $\{X = 8\} = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$ and $P(X = 8) = \frac{5}{36}$
- In general, the pmf of X is of the form

$$P(X = x) = \frac{6 - |7 - x|}{36}, \quad \text{for} \quad x = 2, 3, \ldots, 12$$

... HW Check!

the distribution table

value	x	2	3	4	5	6	7	8	9	10	11	12	
pmf	$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1

Example: Sum of two dice

(b) Let Ω be our sample space for the same example of rolling two dice.
Let X be the sum of two dice: $X(i,j) = i + j$.

- e.g. $\{X = 8\} = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$ and $P(X = 8) = \frac{5}{36}$
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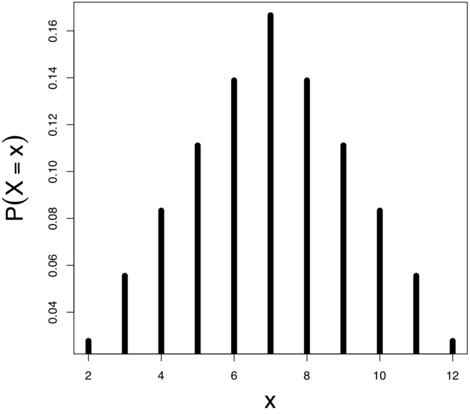
value	x	2	3	4	5	6	7	8	9	10	11	12	
pmf	$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1
cdf	$F(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	1	

- e.g. $F(5.3) = P(X \leqslant 5.3)$
$= P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) = \frac{10}{36}$

Example: Sum of two dice

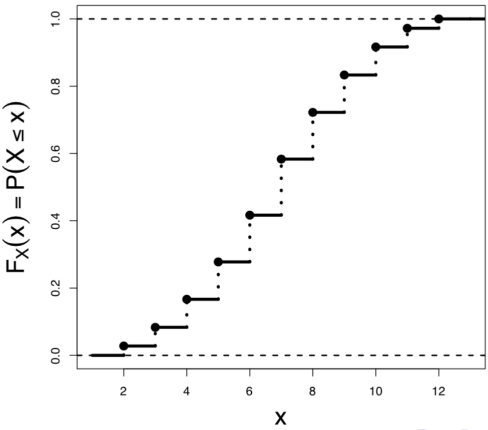
- The probability mass function

$$P(X = x) = \frac{6 - |7 - x|}{36}, \quad \text{for} \quad x = 2, 3, \ldots, 12$$



Example: Sum of two dice

- The graph of the cumulative distribution function is a step function.



Example: Payoff function

HW We roll two dice and write down the outcomes as (i, j), where i is the result of the first and j is the result of the second die. We play a game in which we win \$500 if the sum is 7, otherwise we lose \$100.

- Find the pmf and the cdf of the payoff function D.
- What would be the expected win in this game?

Continuous random variable: Cumulative distribution function

- The cumulative distribution function of a **continuous** random variable X is given by

$$F(x) = P(X \leqslant x) = \int_{-\infty}^{x} f(t)\, dt$$

- $P(a \leqslant X \leqslant b) \;=\; F(b) \;-\; F(a)$
  $= \int\limits_{-\infty}^{b} f(x)dx - \int\limits_{-\infty}^{a} f(x)dx = \int_a^b f(x)\, dx$

- It also holds $F'(x) = f(x)$ for all $x \in \mathbb{R}$

Example

- Let X has the probability density function

$$f(x) = \begin{cases} 3, & x \in [0, \frac{1}{3}] \\ 0, & \text{else} \end{cases}.$$

  (a) Compute $P(0.1 \leqslant X \leqslant 0.2)$ and $P(0.1 \leqslant X \leqslant 1)$.

  (b) Find the cumulative distribution function of X.

- Answer:

  (a) $P(0.1 \leqslant X \leqslant 0.2) = \int_{0.1}^{0.2} f(x)dx = \int_{0.1}^{0.2} 3\,dx = 0.3$

  (or area of rectangle = $3 \cdot 0.1 = 0.3$)

  $P(0.1 \leqslant X \leqslant 1) = \int_{0.1}^{1} f(x)dx = \int_{0.1}^{\frac{1}{3}} f(x)dx + \int_{\frac{1}{3}}^{1} f(x)dx = 3\int_{0.1}^{\frac{1}{3}} dx + 0 = 0.7$

  (b) The cumulative distribution function $F(x) = P(X \leqslant x)$ is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ 3x, & 0 \leqslant x < \frac{1}{3} \\ 1, & x \geqslant \frac{1}{3} \end{cases}.$$

More examples

HW Let X has range $[0, 1]$ and probability density function $f(x) = ax^2$.

(a) What is the value of a?

(b) Compute the cumulative distribution function (cdf) $F_X(x)$.

(c) Compute $P(1 \leqslant X \leqslant 2)$.

HW Let Y has range $[0, b]$ and its cdf is given by

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{y^2}{9}, & 0 \leqslant y < b \\ 1, & y \geqslant b \end{cases}$$

(a) What is the value of b?

(b) Find the probbility density function $f(y)$ of Y.

Generalized inverse and quantile function

- One often needs the inverse function of a cdf F.
- As a cdf is not necessary strictly monotonically increasing, one needs the notion of the generalized inverse of F

$$F^{-1}(p) := \inf\{x \mid F(x) \geqslant p\} \qquad \text{for} \quad p \in (0,1).$$

  - If F is strictly monotonically increasing, then F^{-1} is the (usual) inverse function of F.

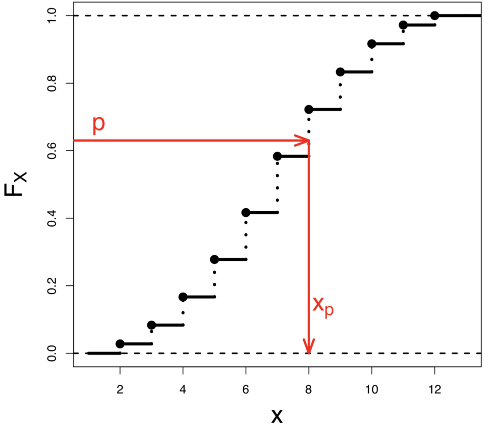
- A function that assigns the value $F^{-1}(p)$ to every $p \in (0,1)$ is called quantile function

$$x_p = F^{-1}(p) \quad \text{for} \quad p \in (0,1) \iff \qquad F(x_p) = p \quad \text{for} \quad p \in (0,1)$$

  - x_p is a p-quantil of F

Example: Sum of two dice

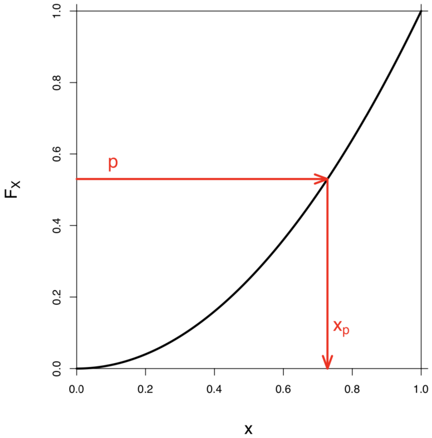
- The graph of the cumulative distribution function is a step function.



Example: continuous random variable

- A p-quantil of F is obtained from

$$F_X(x_p) = P(X \leqslant x_p) = \int_{-\infty}^{x_p} f(t)\,dt = p \quad \Longleftrightarrow \quad x_p = F_X^{-1}(p)$$



Quartiles

- A 50%-quantil $x_{0.5}$ of F is called median.
- A 25%-quantil $x_{0.25}$ of F is called lower quartile
- A 75%-quantil $x_{0.75}$ of F is called upper quartile

Expected value

- The expected value (expectation)
  - For a discrete random variable with the values $x_1, x_2 \ldots$ and pmfs $p(x_i)$, the expected value is defined by:

$$\mathbb{E}(X) = \sum_{i=1}^{\infty} x_i \cdot p(x_i) = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + \ldots$$

  - Let X be a continuous random variable with the density $f(x)$.
    The expected value of X is defined by:

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x \cdot f(x)\, dx$$

It is assumed that the **sum** and the **integral** are absolute convergent!

  - It is a weighted mean (average) of the possible outcomes of X.
  - It is a measure of the central tendency.

Moment of order one of X.

Properties of $\mathbb{E}(X)$

$(e1)$ $\mathbb{E}(aX+b) = a\mathbb{E}(X)+b, \quad a,b\in\mathbb{R}$

$(e2)$ $\mathbb{E}(aX+bY) = a\mathbb{E}(X)+b\mathbb{E}(Y), \quad a,b\in\mathbb{R}$

$(e3)$ For a function h it holds

$$\mathbb{E}(h(X)) = \sum_{i=1}^{\infty} h(x_i)\cdot p(x_i) \quad \text{respectively} \quad \mathbb{E}(h(X)) = \int_{-\infty}^{+\infty} h(x)\cdot f(x)\,dx.$$

Particularly:

- $\mathbb{E}(X^k)$ ... the moment of order k of X
- $\mu_k = \mathbb{E}\big((X-\mathbb{E}(X))^k\big)$ ... the central moment of order k of X
- $\mu_2 = \mathbb{E}\big((X-\mathbb{E}(X))^2\big) = \mathbb{V}ar(X)$ ...is the variance of X
- $\sigma = \sqrt{\mathbb{V}ar(X)}$ ... is the standard deviation of X

$(e4)$ Let X and Y be independent. Then it holds $\mathbb{E}(X\cdot Y) = \mathbb{E}(X)\cdot\mathbb{E}(Y)$.

Properties of $\mathbb{V}ar(X)$

$(v1)$ $\mathbb{V}ar(X) = \mathbb{E}\big((X - \mathbb{E}(X))^2\big) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

$(v2)$ $\mathbb{V}ar(aX + b) = a^2\,\mathbb{V}ar(X), \quad a, b \in \mathbb{R}$

$(v3)$ $\mathbb{V}ar(X + b) = \mathbb{V}ar(X), \quad b \in \mathbb{R}$

$(v4)$ Let X and Y be independent. Then it holds

$$\mathbb{V}ar(X + Y) = \mathbb{V}ar(X) + \mathbb{V}ar(Y).$$

Note: if X and Y are not independent, an additional term arrises in the sum! We will get back to this later!

Example

It is given

X	x	3	4	5	6
pmf	$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$

(a) Compute $\mathbb{E}(X)$ and $\mathbb{E}(3X-2)$.

(b) Compute $\mathbb{E}(X^2)$ and $\mathbb{V}ar(X)$.

Answer:

(a) $\mathbb{E}(X) = 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{2} + 5 \cdot \frac{1}{8} + 6 \cdot \frac{1}{8} = \frac{33}{8}$ and

$\mathbb{E}(3X-2) = 3 \cdot \mathbb{E}(X) - 2 = 3 \cdot \frac{33}{8} - 2 = \frac{83}{8}$

... We have used the properties of the expectation.

II way:

$Y = 3X-2$	y	7	10	13	16
pmf	$p(y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$

$\mathbb{E}(3X-2) = \mathbb{E}(Y) = 7 \cdot \frac{1}{4} + 10 \cdot \frac{1}{2} + 13 \cdot \frac{1}{8} + 16 \cdot \frac{1}{8} = \frac{83}{8}$

Example cont.

(b) $\mathbb{E}(X^2) = \frac{9}{4} + \frac{16}{2} + \frac{25}{8} + \frac{36}{8} = \frac{143}{8}$

$\mathbb{V}ar(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{143}{8} - (\frac{33}{8})^2 = \frac{55}{64}$.

Transformations

- Let X be a continuous random variable, i.e., its density f_X is known.
- What is the distribution of a transformation

$$Y = g(X)$$

with $g : \mathbb{R} \to \mathbb{R}$?

  - Method: Use the cumulative distribution function of X
    - Determine F_X.
    - Determine F_Y for $Y = g(X)$.
    - Find $f_Y(y) = F'_Y(y)$.

Another example

Let X has the density function

$$f(x) = \begin{cases} ax^4, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find a.

(b) Compute $\mathbb{E}(X)$ and $\mathbb{V}ar(X)$.

(c) Find the median value of X.

(d) Let $X_1, X_2, \ldots, X_{25}$ are independentand identically distributed (i.i.d.) copies of X. Let $\bar{X}$ be their average (mean), i.e.

$$\bar{X} = \frac{1}{25} \sum_{i=1}^{25} X_i.$$

Compute the standard deviation of $\bar{X}$.

(e) Let $Y = 2X^3 + 1$. Find the density of Y.

Another example: Solution

(a) From $1 = \int_{\mathbb{R}} f(x)\,dx = \int_0^1 ax^4\,dx$ we obtain $a = 5$.

(b) The expectation and variance are

$$\mu = \mathbb{E}(X) = \int_0^1 5x^5\,dx = \frac{5}{6}$$

$$\mathbb{V}ar(X) = \int_0^1 (x - \frac{5}{6})^2\,5x^4dx = 5\int_0^1 (x^6 - \frac{5}{3}x^5 + \frac{25}{36}x^4)dx = \frac{5}{252} \approx 0.02$$

$$\sigma = \sqrt{\mathbb{V}ar(X)} \approx 0.14$$

(c) First we calculate the cumulative distribution function

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^5, & 0 \leqslant x < 1 \\ 1, & x \geqslant 1 \end{cases}$$

Median $x_{0.5}$ solves the equation

$$F_X(x_{0.5}) = 0.5$$

Then, $F_X(x_{0.5}) = x_{0.5}^5 = 0.5$ and $x_{0.5} = \sqrt[5]{0.5} \approx 0.87$.

Another example: Solution

(d) We compute the expectation $\mathbb{E}(\bar{X})$ and the variance $\mathbb{V}ar(\bar{X})$ of the **sample mean** $\bar{X} = \frac{1}{25}(X_1 + X_2 + \cdots + X_{25})$.

$$\mathbb{E}(\bar{X}) = \frac{1}{25}\mathbb{E}(X_1 + \cdots + X_{25}) = \mathbb{E}(X) = \frac{5}{6}$$

$$\mathbb{V}ar(\bar{X}) = \frac{1}{25^2}\mathbb{V}ar(X_1 + X_2 + \cdots + X_{25}) = \frac{1}{25^2}(\mathbb{V}ar(X_1) + \cdots + \mathbb{V}ar(X_{25}))$$

$$= \frac{1}{25^2} \cdot 25\mathbb{V}ar(X) = \frac{\mathbb{V}ar(X)}{25} < \mathbb{V}ar(X).$$

Also, $\sigma_{\bar{X}} = \frac{\sigma}{5} \approx 0.028$.

Important: $\mathbb{V}ar(\bar{X}) \leqslant \mathbb{V}ar(X)$!

(e) In order to find the pdf of $Y = 2X^3 + 1$, we use the cdf of X

$$F_Y(y) = P(Y \leqslant y) = P(2X^3 + 1 \leqslant y) = P(X \leqslant \sqrt[3]{\frac{y-1}{2}})$$

$$= F_X(\sqrt[3]{\frac{y-1}{2}}) = \begin{cases} 0, & y < 1 \\ (\frac{y-1}{2})^{\frac{5}{3}}, & 1 \leqslant y < 3 \\ 1, & y \geqslant 3 \end{cases}.$$

Also,

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{5}{6} \cdot (\frac{y-1}{2})^{\frac{2}{3}}, & 1 < y < 3 \\ 0, & \text{else} \end{cases}.$$

More examples & some multiple-choice questions

Examples

HW Let X be a random variable that takes on values 0, 2 and 3 with probabilities 0.3, 0.1 and 0.6 respectively. Let $Y = 3(X-1)^2$.

(a) Compute $\mathbb{E}(X)$ and $\mathbb{V}ar(X)$.
(b) Compute $\mathbb{E}(Y)$.
(c) Let $F_Y(y)$ be the cdf of Y. What is $F_Y(7)$?

HW Let X be a random variable with the cdf

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x(2-x) & \text{for } 0 \leqslant x < 1 \\ 1 & \text{for } x > 1 \end{cases}.$$

(a) Compute $P(X \leqslant 0.4)$.
(b) Compute $\mathbb{E}(X)$.
(c) Find the moment of order four of X.
(d) Find the median, upper and lower quartiles.

Examples

HW Let X be a random variable with the range $[0,1]$ and cumulative distribution function

$$F(x) = 2x^2 - x^4, \qquad \text{for } 0 \leqslant x \leqslant 1.$$

(a) Verify that F is a cumulative distribution function.
(b) Compute $P(\frac{1}{4} \leqslant X \leqslant \frac{3}{4})$.
(c) What is the probability density of X?

HW Let X be a random variable with the cumulative distribution function F_X. Let X_1 and X_2 be independent and identically distributed (i.i.d.) copies from X. Let $Y = \max\{X_1, X_2\}$. Find the distribution function F_Y of Y with respect to F_X.

HW An enthusiastic football fan gives away Tototips every week, using the digits 0 (draw), 1 (home win), 2 (victory) with the help of the probability function

$$P(X = k) = \begin{cases} \frac{1}{4} + ak + bk^2 & k = 0, 1, 2 \\ 0 & \text{else} \end{cases}$$

with unknown values a and b. However, it is known that for his tips it holds $P(X = 1) = 1/4$. Determine a and b and the corresponding distribution function.

A few multiple-choice questions

(1) Let X be a random variable with probability density function of the form

$$f(x) = \begin{cases} -2x, & -1 \leqslant x \leqslant 0 \\ 0, & \text{else} \end{cases}.$$

Compute $P(-\frac{3}{4} \leqslant X < -\frac{1}{2})$.

a. $5/16$
b. $1/2$
c. $7/8$
d. $19/64$

(2) Let

$$F(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \leqslant x < 0.5 \\ 1 & x \geqslant 0.5 \end{cases}$$

be the cumulative distribution function of a random variable X and let $Y = 2X + 1$. Then, the expectation $\mathbb{E}(Y)$ equals

(a) 0.25
(b) 0.5
(c) 1.5
(d) 0.75

A few multiple-choice questions

(3) A random variable X has a probability distribution as follows:

X	0	1	2	3
$P(X)$	$2k$	$3k$	$13k$	$2k$

where k is a positive constant. The probability $P(X < 2.0)$ is equal to

a. 0.90
b. 0.25
c. 0.65
d. 0.15

(4) Let X be a random variable that takes values $-2, -1, 0, 1$ and 2, each with probability $1/5$. Let $Y = X^2$. Then,

a. $\mathbb{C}ov(X, Y) > 0$
b. $\mathbb{C}ov(X, Y) < 0$
c. $\mathbb{V}ar Y < 2\mathbb{V}ar X$
d. $\mathbb{V}ar Y = 2\mathbb{V}ar X$

... Note: We will learn soon what is the covariance $\mathbb{C}ov(X, Y)$!

A few multiple-choice questions

(5) Let us consider ballgames which cannot end in a tie. Suppose a bookie will give you \$6 for every \$1 you risk if you pick the winners in three ballgames. Thus, for every \$1 you bet you will either lose \$1 or gain \$5. What is the bookie's expected earnings per dollar wagered?

a. $-\$2/8$
b. $\$34/8$
c. $\$2/8$
d. $\$21/27$

(6) Let X and Y be two independent random variables. Suppose that we know $\mathbb{V}ar(2X - Y) = 6$ and $\mathbb{V}ar(X + 2Y) = 9$. Then, the variance $\mathbb{V}ar(Y)$ equals

a. 1
b. 2
c. 3
d. 5

A few multiple-choice questions

(7) Let

$$F(x) = \begin{cases} 0, & x < 1 \\ 1 - \frac{1}{x^3}, & x \geqslant 1 \end{cases}$$

be the cumulative distribution function of a random variable X. Then, its second central moment is

a. not finite
b. 0.75
c. 6.75
d. 3

(8) Fourty people are invited to a party. Each person accepts the invitation, independently of all others, with probability $1/4$. Let X be the number of accepted invitations. Then, the expectation of $X^2 - 8X + 5$ equals

a. -167.5
b. 1.25
c. 27.5
d. 32.5

A few multiple-choice questions

(9) Anna rolls two fair dice and observe two numbers X and Y.
If $Z = X - Y$, what is the probability $P(Z < 4)$?

a. $\frac{1}{18}$
b. $\frac{17}{18}$
c. $\frac{11}{12}$
d. $\frac{1}{3}$

(10) Let

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.$$

be the probability density function of a random variable X. Then,

a. the expectation $\mathbb{E}X = 0$
b. the variance $\mathbb{V}arX = 1$
c. the variance $\mathbb{V}arX = \sqrt{\pi}$
d. the expectation $\mathbb{E}X$ does not exist

Thank you for your attention!