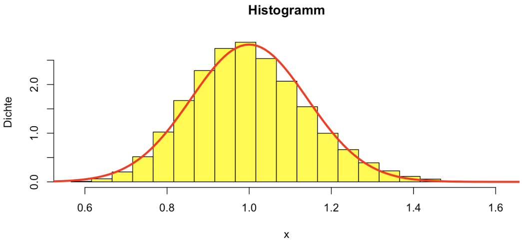
TECHNISCHE UNIVERSITÄT WIEN

Central limit theorem and Law of large numbers



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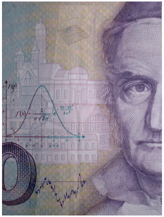
Reminder

- $X \sim \mathcal{N}(\mu, \sigma^2)$
  - One of the most important distributions in statistics and probability theory.



Normal distribution on the front of the German 10 Mark banknote from 1990s

  - X has a Normal distribution (or Gauss distribution) with the parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, if its pdf is given by

$$f(x) = \frac{1}{\sqrt{2\pi\,\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for} \quad x \in \mathbb{R}$$

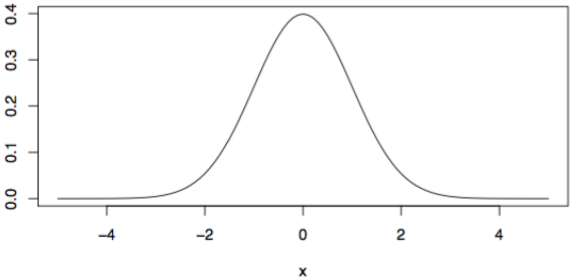
    - $\mathbb{E}(X) = \mu, \quad \mathbb{V}ar(X) = \sigma^2$

$\underline{Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)}$

We recall

- $Z \sim \mathcal{N}(0, 1)$ is the Standard normal random variable

... $\mathbb{E}(Z) = 0, \quad \mathbb{V}ar(Z) = 1$

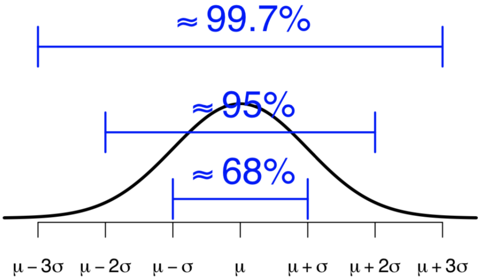


- Its cdf is given by $\Phi(z) = P(Z \leqslant z)$
- The following hold

$$P(-1 \leqslant Z \leqslant 1) \approx 0.68$$
$$P(-2 \leqslant Z \leqslant 2) \approx 0.95$$
$$P(-3 \leqslant Z \leqslant 3) \approx 0.997$$

We recall

- Let $X \sim \mathcal{N}(\mu, \sigma^2)$.
    - Standardization
$$Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$$
    - For $X \sim \mathcal{N}(\mu, \sigma^2)$ the 68-95-99.7-Rule holds



Sample mean

- Let $X_1, X_2, \ldots, X_n$ be **independent identically distributed** (i.i.d.) random variables with $\mathbb{E}(X_1) = \mu$ and $\mathbb{V}ar(X_1) = \sigma^2$.

... $X_1, X_2, \ldots, X_n$ is allso called a **random sample of size n**

- For the sample mean $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ it holds

$$\mathbb{E}(\bar{X}) = \mu \qquad \text{and} \qquad \mathbb{V}ar(\bar{X}) = \frac{\sigma^2}{n}$$

- $$\mathbb{E}(\bar{X}) = \mathbb{E}\Big(\frac{1}{n}\sum_{i=1}^{n} X_i\Big) \overset{\text{linearity}}{=} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_i) \overset{\text{ident. distr.}}{=} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_1) = \frac{1}{n} \cdot n\,\mathbb{E}(X_1) = \mathbb{E}(X_1) = \mu$$

- $$\mathbb{V}ar(\bar{X}) = \mathbb{V}ar\Big(\frac{1}{n}\sum_{i=1}^{n} X_i\Big) = \frac{1}{n^2}\mathbb{V}ar\Big(\sum_{i=1}^{n} X_i\Big) \overset{\text{independ.}}{=} \frac{1}{n^2}\sum_{i=1}^{n} \mathbb{V}ar(X_i) \overset{\text{ident. distr.}}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(X_1) = \frac{1}{n^2} \cdot n\,\mathbb{V}ar(X_1) = \frac{\mathbb{V}ar(X_1)}{n} = \frac{\sigma^2}{n}$$

Sample mean

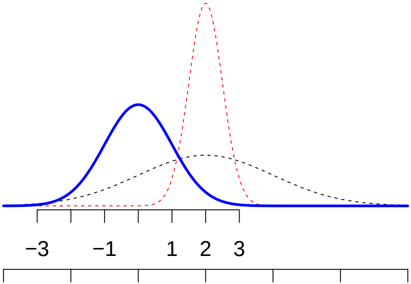
- Let $X_1, X_2, \ldots, X_n$ be independent identically distributed (i.i.d.) random variables with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$.

... $X_1, X_2, \ldots, X_n$ is allso called a **random sample of size n**

- For the sample mean $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ it holds

$$E(\bar{X}) = \mu \qquad \text{and} \qquad Var(\bar{X}) = \frac{\sigma^2}{n}$$

  - Especially if $X_1, \ldots X_n$ are normally distributed, the sample mean is also normally distributed $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$



Convergence

- Two important theorems of probability theory:
  - Law of Large Numbers (LLN)
    - in German: Gesetz der großen Zahlen (GGZ)
  - Central Limit Theorem (CLT)
    - in German: Zentraler Grenzwertsatz (ZGWS)

... We state the theorems for

independent and identically distributed random variables

Theorems

- Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables with $\mathbb{E}(X_1) = \mu$ and finite $\mathbb{V}ar(X_1) = \sigma^2 < \infty$.

... n i.i.d. random variables $X_1, \ldots, X_n$ are called a random sample of size n

  - Note that $X_1, \ldots, X_n$ are **not** necessarily normally distributed.

- We consider the sample mean

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

  - The Law of large numbers and the Central limit theorem provide information about the value and the distribution of $\bar{X}_n$.

    - LLN: As n grows, the probability of $\bar{X}_n$ being in the neighborhood of μ tends to 1.

    - CLT: For large n, the distribution of $\bar{X}_n$ is approximately $\mathcal{N}(\mu, \frac{\sigma^2}{n})$.

Law of large numbers

- The Law of large numbers says that the sample mean $\bar{X}_n$ will be with high probability very close to the expectation μ of the underlying distribution

- Formal statement:

  Suppose $X_1, X_2, \ldots, X_n \ldots$ are i.i.d. random variables with expectation μ and finite variance σ^2. For each n, let $\bar{X}_n$ be the mean of the first n variables. Then for any $a > 0$, we have

$$\lim_{n \to \infty} P(|\bar{X}_n - \mu| < a) = 1$$

Example

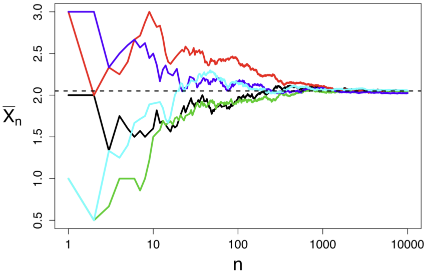
- We consider a discrete random variable with the pmf

x	0	1	2	3	4
$p(x)$	0.1	0.2	0.3	0.35	0.05

  - Then, $\mathbb{E}(X) = \sum_{x=0}^{4} x\,p(x) = 2.05$ and $\mathbb{V}ar(X) = \sum_{x=0}^{4} (x - 2.05)^2\,p(x) = 1.1475$

- Since the variance of X exists, the Law of Large Numbers says

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \;\to\; \mathbb{E}(X) = 2.05 \qquad \text{for large } n$$



Example 2

- The mean values of the Bernoulli random variables
  - Let $X_1 ..., X_n$ denote n independent flips of a fair coin
    - $X_i \sim ber(0.5)$ and $\mathbb{E}(X_i) = \mu = 0.5$
    - $S_n = X_1 + \cdots + X_n \sim B(n, 0.5)$
  - Then $\bar{X}_n = \frac{1}{n} S_n$ is the average number of heads obtained in n flips.
  - We expect that this number for large n is close to $\mu = 0.5$.
    - LLN: The sample mean $\bar{X}_n$ is very likely to be close to the expected value 0.5 for large n.
    - For example, let us consider the probability of being within 0.1 from the expected value $\mu = 0.5$, i.e.

      $P(|\bar{X}_n - \mu| \leqslant 0.1)$ or equivalently $P(0.4 \leqslant \bar{X}_n \leqslant 0.6) = P(0.4n \leqslant S_n \leqslant 0.6n)$
    - in R:

$n = 10$:	`diff(pbinom(c(3,6),10,0.5))`	= 0.65625
$n = 50$:	`diff(pbinom(c(19,30), 50, 0.5))`	= 0.8810795
$n = 100$:	`diff(pbinom(c(39,60), 100, 0.5)`	= 0.9647998
$n = 500$:	`diff(pbinom(c(199,300), 500, 0.5))`	= 0.9999941
$n = 1000$:	`diff(pbinom(c(399,600), 1000, 0.5))`	= 1

HW Repeat the calculations and determine the probability

$$P(|\bar{X}_n - \mu| \leqslant 0.01).$$

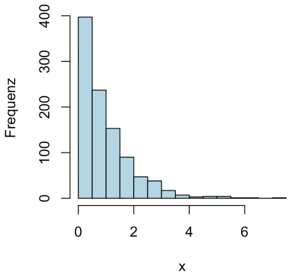
Histograms

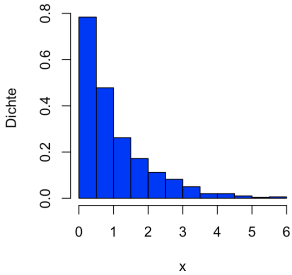
- Histograms (frequency and density histogram)

  ... A graphical representation of a frequency distribution based on a previous classification of the data.

  - Made by *binning* data ... data binning is a way to group a number of continuous values into a smaller number of *bins*
  - **Frequency:** height of bar over bin = number of data points in bin.
  - **Density:** area of bar is the fraction of all data points that lie in the bin.

    ... Thus, total area is 1.

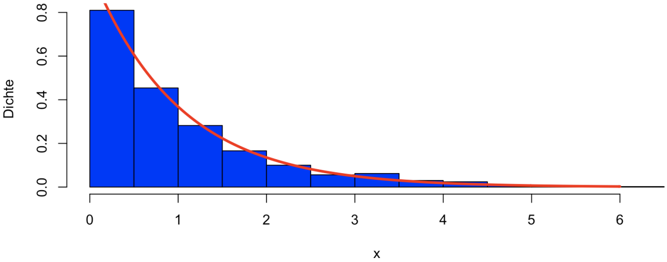




in R: `hist(x,freq=TRUE)` `hist(x,freq=FALSE)` or `hist(x,prob=TRUE)`

Law of Large Numbers and histograms

- The Law of Large Numbers implies that density histogram converges to probability density function.



The histogram with bin width 0.5 showing 1000 draws from an exponential $\exp(1)$ distribution. The pdf of $\exp(1)$ is given in red.

Central Limit Theorem

- The Central Limit Theorem states that the sum, resp. mean, of many independent copies of a random variable is approximately a normal random variable.
- Let $X_1, X_2, \dots$ be a sequence of i.i.d. with expectation μ and standard deviation σ.
  - For each n:
    - the sum $S_n = X_1 + X_2 + \cdots + X_n$
    - the sample mean $\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$
  - The Central Limit Theorem says that for large n it holds

$$S_n \approx \mathcal{N}(n\mu, n\sigma^2)$$

$$\bar{X}_n \approx \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

    - Thus, the standardized $\bar{X}_n$ and S_n have approximately $\mathcal{N}(0,1)$, i.e.

$$\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \approx \mathcal{N}(0,1) \quad \text{and} \quad \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \approx \mathcal{N}(0,1)$$

Example

- Simulation for CLT
  - Let $X_1, X_2, \dots$ be a sequence of i.i.d. $\exp(1)$ random variables.
    - Then, $\mathbb{E}(X_i) = 1$ and $\mathbb{V}ar(X_i) = 1 < \infty$.
  - We consider sample means $\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$.
  - Central Limit Theorem:
    - $\bar{X}_n$: $\quad \mathbb{E}(\bar{X}_n) = 1 \quad$ and $\quad \mathbb{V}ar(\bar{X}_n) = \frac{1}{n}$

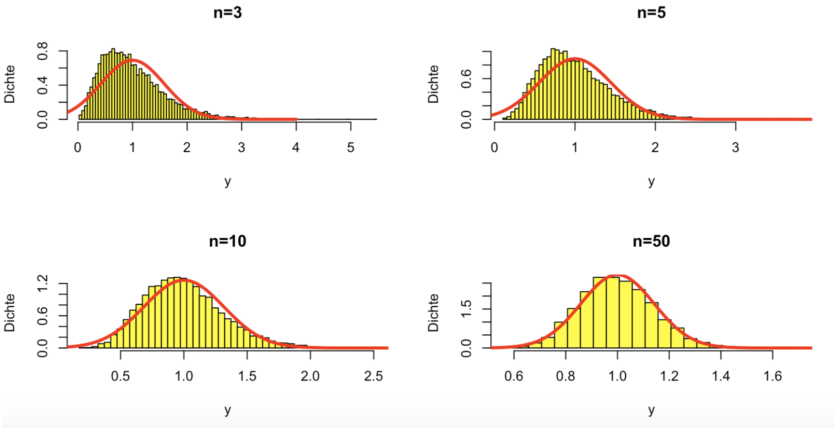
$$\bar{X}_n \approx \mathcal{N}(1, \frac{1}{n})$$

    - The standardized means Y_n

$$Y_n = \frac{\bar{X}_n - \mathbb{E}(\bar{X}_n)}{\sqrt{\mathbb{V}ar(\bar{X}_n)}} = \frac{\bar{X}_n - 1}{\frac{1}{\sqrt{n}}} \approx \mathcal{N}(0, 1)$$

Example

- The following plots show the results for respectively $n = 3, 5, 10, 50$ based on 10000 simulated values for $\bar{X}_n$. The density of the $\mathcal{N}(1, \frac{1}{n})$ distribution is drawn in red.



Normal approximation of $B(n,p)$ distribution

- The **sum** $\underline{S_n = X_1 + \cdots + X_n}$ of independent $ber(p)$ random variables $X_1, \ldots, X_n$ has $\underline{B(n,p)}$ distribution. $\mathbb{E}(S_n) = np$ and $\mathbb{V}ar(S_n) = np(1-p)$

- Using the continuity correction it holds for $a \leqslant b$, where $a, b \in \{0, 1, \ldots, n\}$:

$$P(a \leqslant S_n \leqslant b) \approx \Phi\left(\frac{b + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{a - \frac{1}{2} - np}{\sqrt{np(1-p)}}\right)$$

  - Let T be a continuous random variable such that $\mathbb{E}(T) = \mathbb{E}(S_n) = np$ and $\mathbb{V}ar\,(T) = \mathbb{V}ar\,(S_n) = np(1-p)$. Then,

$$\begin{aligned}
P(a \leqslant S_n \leqslant b) &= P\left(a - \frac{1}{2} \leqslant S_n < b + \frac{1}{2}\right) \approx P\left(a - \frac{1}{2} \leqslant T < b + \frac{1}{2}\right) \\
&= P\left(\frac{a - \frac{1}{2} - \mathbb{E}(T)}{\sqrt{Var\,(T)}} \leqslant \frac{T - \mathbb{E}(T)}{\sqrt{\mathbb{V}ar\,(T)}} < \frac{b + \frac{1}{2} - \mathbb{E}(T)}{\sqrt{\mathbb{V}ar\,(T)}}\right) \\
&\overset{\text{CLT}}{\approx} P\left(\frac{a - \frac{1}{2} - np}{\sqrt{np(1-p)}} \leqslant Z < \frac{b + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) \\
&= \Phi\left(\frac{b + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{a - \frac{1}{2} - np}{\sqrt{np(1-p)}}\right)
\end{aligned}$$

Normal approximation of $B(n,p)$ distribution

- The sum $S_n = X_1 + \cdots + X_n$ of independent $ber(p)$ random variables $X_1, \ldots, X_n$ has $B(n,p)$ distribution.
  - $\mathbb{E}(S_n) = np$ and $\mathbb{V}ar(S_n) = np(1-p)$
- Using the continuity correction it holds for $a \leqslant b$, where $a, b \in \{0, 1, \ldots, n\}$:

$$P(a \leqslant S_n \leqslant b) \approx \Phi\left(\frac{b + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{a - \frac{1}{2} - np}{\sqrt{np(1-p)}}\right)$$

- Approximation of the cdf of S_n:

$$P(S_n \leqslant x) \approx \Phi\left(\frac{x + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right)$$

Φ is the cdf of $\mathcal{N}(0,1)$

- Rule of thumb: The approximation is considered reasonable, when $\min\{np, np(1-p)\} \geqslant 10$

Normal approximation of $\mathcal{P}(\lambda)$ distribution

- Let $X \sim \mathcal{P}(\lambda)$. Then, by providing the continuity correction for $a \leqslant b$, with $a, b \in \mathbb{N}_0$

$$P(a \leqslant X \leqslant b) \approx \Phi\left(\frac{b + \frac{1}{2} - \lambda}{\sqrt{\lambda}}\right) - \Phi\left(\frac{a - \frac{1}{2} - \lambda}{\sqrt{\lambda}}\right),$$

- Approximation of the cdf of X:

$$P(X \leqslant x) \approx \Phi\left(\frac{x + \frac{1}{2} - \lambda}{\sqrt{\lambda}}\right)$$

- Rule of thumb: The approximation is considered to be reasonable when $\lambda > 15$.

Example

- An accountant rounds to the nearest euro. We assume the error in rounding follows uniform distribution on $(-0.5, 0.5)$. Estimate the probability that the total error in 300 entries is more than 5 euro.
  - Let X_j be the rounding error of the jth entry. Then, $X_j \sim U(-0.5, 0.5)$
    $\mathbb{E}(X_j) = 0$ and $\mathbb{V}ar\,(X_j) = \frac{1}{12}$
  - The overall error is $S = X_1 + \cdots + X_{300}$.
    $\mathbb{E}(S) = 0$, $\mathbb{V}ar\,(S) = \frac{300}{12} = 25$ and $\sigma_S = 5$.
  - CLT: for the standardized S it holds

$$Z = \frac{S - \mathbb{E}(S)}{\sqrt{\mathbb{V}ar(S)}} = \frac{S}{5} \sim \mathcal{N}(0,1)$$

  - Then,

$$P(|S| > 5) = 1 - P(|S| \leqslant 5) \approx 1 - P(|Z| \leqslant 1) \approx 0.32$$

Computations: The pdf is of the form $f(x) = 1$ if $x \in (-0.5, 0.5)$ and $f(x) = 0$ otherwise. Then, $\mathbb{E}(X_j) = \int_{-0.5}^{0.5} x dx = 0$ and $\mathbb{V}ar(X_j) = \mathbb{E}(X_j^2) = \int_{-0.5}^{0.5} x^2 dx = 2\int_0^{0.5} x^2 dx = \frac{2}{3} x^3 |_0^{0.5} = \frac{1}{12}$. Thus, $\mathbb{E}S = 300 \cdot \mathbb{E}X_1 = 0$ and $\mathbb{V}arS = 300 \cdot \mathbb{V}ar(X_1) = 300 \cdot \frac{1}{12} = 25$.

Examples

HW Let $X_1, X_2, \ldots, X_{25}$ be independent and identically distributed (i.i.d.) random variables and $X_1 \sim \mathcal{N}(1, 4)$. Find the probability $P(X_1 + X_2 + \ldots\ldots X_{25} \geqslant 26)$.

HW Transportation officials tell us that 60% of the population wear their seatbelts while driving. A random sample of 1000 drivers has been taken. What is the probability that between 580 and 630 of the drivers were wearing their seatbelts?

HW Let $X_1, X_2, \ldots X_{121}$ be i.i.d. with the expectation $\mu = 35$ and variance $\sigma^2 = 25$. Denote by

$$\bar{X}_{121} = \frac{1}{121}(X_1 + \cdots + X_{121})$$

the sample mean. Approximate the probability $P(\bar{X}_{121} > 35.2)$ using the Central limit theorem.

HW We toss a fair coin 100 times.
What is the probability of obtaining 60 or more heads?

A few multiple-choice questions

(1) Let $X_1, X_2, \ldots X_{81}$ be i.i.d. sample from a population with population mean $\mu = 5$ and population variance $\sigma^2 = 4$ and let $S = X_1 + X_2 + \ldots X_{81}$. Approximate the probability $P(S \notin [387, 423])$ using the Central limit theorem.

a. 68%
b. 78%
c. 45%
d. 32%

(2) Assume that X is a binomial random variable with $n = 100$ and $p = 0.1$. Use the normal probability distribution to compute $P(X \leqslant 15)$.

a. 0.5336
b. 0.9664
c. 0.0336
d. 0.4664

A few multiple-choice questions

(3) Transportation officials tell us that 60% of the population wear their seatbelts while driving. A random sample of 1000 drivers has been taken. What is the probability that between 220 and 550 of the drivers were wearing their seatbelts?

a. 1.0
b. 0.4015
c. 0.9066
d. 0

(4) Consider a sequence of independent tosses of a coin that is biased so that it comes up heads with probability 0.7 and tails with probability 0.3. Let X_i be 1 if the ith toss comes up heads and 0 otherwise. Let Y be the number of heads in the first 4900 tosses of the biased coin, i.e. $Y = \sum_{i=1}^{4900} X_i$. Use a normal random variable to approximate the probability that $Y \leqslant 3450$. Approximate the result in terms of the cdf $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} \, dx$.

a. $\Phi(0.62)$
b. $\Phi(0.02)$
c. $1 - \Phi(1.41)$
d. $\Phi(1.92)$

A few multiple-choice questions

(5) Let $X_1, \ldots, X_{64}$ be a random sample from a distribution with the expectation -1.2 and variance 4. Let

$$\bar{X} = \frac{1}{64} \sum_{i=1}^{64} X_i$$

be the sample mean. Determine the approximate value of $P(\bar{X} > -0.9)$ using the Central limit theorem and express it in terms of a suitable `R`-function.

a. `pnorm(1.2)`
b. `pnorm(-1.2)`
c. `pnorm(-0.9, 1.2, 0.5)`
d. `pnorm(-0.9, -1.2, 0.25)`

(6) The central limit theorem is important in statistics because

a. For large n, it says that the population is approximately normal
b. For any population, it says the sampling distribution of the sample mean is approximately normal, regardless of the sample size.
c. For large n, it says the sampling distribution of the sample mean is approximately normal, regardless of the population.
d. For any size sample, it says that the sampling distribution of the sample mean is approximately normal.

A few multiple-choice questions

(7) Consider a sequence of independent tosses of a coin that is biased so that it comes up heads with probability $3/4$ and tails with probability $1/4$. Let X be the number of heads in the first 4800 tosses of the biased coin. Use a normal random variable to approximate the probability

$$P(X > 3590).$$

In the computations include the continuity correction.

a. 0.3632
b. 0.6293
c. 0.6368
d. 0.3707

(8) If the heights of women are normally distributed with expectation 163 centimeters, which of the following is the smallest?

a. The probability of randomly choosing one woman and finding her height is between 160 and 166 centimeters.
b. The probability of randomly choosing 15 women and finding that their mean height is between 160 and 166 centimeters.
c. The probability of randomly choosing 100 women and finding that their mean height is between 160 and 166 centimeters.
d. All of the above have the same probability.

A few multiple-choice questions

(9) Assume that a baseball team has an average pitcher, that is, one whose probability of winning any decision is 0.5. If this pitcher has 30 decisions in a season, what is the probability that he will win at least 23 games?

a. less than 0.5%
b. between 0.5% and 1%
c. between 1% and 5%
d. more than 5%

(10) Assume the given distribution is normal. Pumpkins grown on a certain farm have weights with a mean of 10 kilograms. What is the standard deviation (in kilograms) of the weight if 37% of the pumpkins weigh more than 13 kilograms?

a. 1.22
b. 2.97
c. 4.76
d. 8.82

Thank you for your attention!