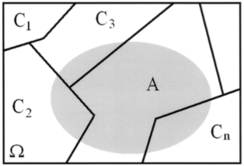
Probability

Conditional probability and the Bayes theorem



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Last week

- Introduction to probability theory
  - In order to **compute probabilities** we deal with **sets**.
  - **Goal:** learn techniques for **counting** the number of elements in a set.

... so far, only for finite sets!

- Permutations
  Lining things up = order is important

  The number of permutations of a set of k elements equals
  $k! = k \cdot (k-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$

... here $k \in \mathbb{N}_0$ and by definition $0! = 1$!

- Combinations
  Selection of a subset - order is not important

  The number of all combinations of k from a set of n elements is
  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

... here $k, n \in \mathbb{N}_0$ and $0 \leqslant k \leqslant n$

- **Permutations and combinations**

  The number of permutations of k out of a set of n, with $\mathbf{0} \leqslant \mathbf{k} \leqslant \mathbf{n}$ is
  $\binom{\mathbf{n}}{\mathbf{k}} \cdot \mathbf{k}! = \frac{n!}{(n-k)!} = n \cdot (n-1) \cdot (n-k+1)$

Terminology

- Random experiment ... a repeatable procedure with well-defined possible outcomes
- Sample space Ω ... the set of all possible outcomes

  $\omega \in \Omega$ are elementary events

  - Discrete sample space - countable (listable) and it can be finite or infinite
    - Example: $\{H, T\}$, $\{1, 3, 5, \dots\}$ are discrete sample spaces
      The interval $0 \leqslant x \leqslant 1$ is not discrete, but continuous
    - Describe (list) all possible outcomes
    - Describe the probability of the outcomes

  ... Probability tables

- Event A ...a subset of the sample space
  - Events are sets that can be described in: words, notation, with Venn diagrams ... $\emptyset$=impossible event, Ω= sure event, $A^c = \Omega \setminus A$= opposite event, $A \cup B$, $A \cap B$, $A^c \cup (B \cap C)$ are events ...
- Probability mass function ...a function that gives the probability of each event
  $P\colon \mathcal{F} \to [0, 1]$

  <u>Axioms</u>: $P(\Omega) = 1$; $P(A) \geqslant 0$ for all $A \in \mathcal{F}$; and
  $P(A \cup B) = P(A) + P(B)$ for all disjoint $A, B \in \mathcal{F}$
- $(\Omega, \mathcal{F}, P)$ is called a probability space

Discrete probability space

- Let $\Omega = \{\omega_1, \omega_2, \ldots\}$ be **countable**. Let p_i be the probability $p_i = P(\omega_i)$.
  - The probability (mass function) P is uniquely determined by the probabilities of the elementary events
    - Probability table

elementary events ω_i	ω_1	ω_2	$\ldots$	ω_n	$\ldots$
probability p_i	p_1	p_2	$\ldots$	p_n	$\ldots$

$$\boxed{0 \leqslant p_i \leqslant 1 \quad \text{and} \quad \sum_i p_i = 1}$$

    - Probability of an event A

$$\boxed{P(A) = \sum_{\omega_i \in A} p_i}$$

(1) Let us toss a **fair** coin **twice**.

- Random experiment: toss a fair coin twice, list the outcomes
- Sample space: $\Omega = \{HH, HT, TH, TT\} = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $|\Omega| = \mathbf{4}$
- $\mathcal{F} = \mathcal{P}(\Omega)$ the power set and $|\mathcal{F}| = 2^4 = 16$

  $\mathcal{F} = \{\emptyset, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\},$
  $\{HT, TT\}, \{TH, TT\}, \{HH, HT, TH\}, \{HH, HT, TT\}, \{HT, TH, TT\}, \Omega\}$

  $|\mathcal{F}| = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = (1+1)^4 = 16$
- Probability: the outcomes ω_1, ω_2, ω_3 and ω_4 are **equally likely** (fair coin), they appear with probability $\frac{1}{4}$

ω_i	ω_1 HH	ω_2 HT	ω_3 TH	ω_4 TT	
p_i	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1

- Event A: head appeared once

  $A = \{HT, TH\} = \{\omega_2, \omega_3\}$ and $P(A) = p_2 + p_3 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
- Event B: head appeared at least once

  $B = \{HH, HT, TH\} = \Omega \setminus \{\omega_4\}$ and $P(B) = 1 - \frac{1}{4} = \frac{3}{4}$
- Event C: head appeared three times

  $C = \emptyset$ and $P(C) = 0$

(2) Let us toss a **fair** coin **until the first head**.

- Random experiment: toss a fair coin until the first head
- Sample space: $\Omega = \{H, TH, TTH, TTTH, TTTTH \ldots\}$
- Probability:

ω_i	H	TH	TTH	TTTH	TTTTH	...
p_i	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$	...

Is the sum of all probabilities 1?
We use geometric series:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{1}{2} \cdot (1 + \frac{1}{2} + \frac{1}{2^2} + \ldots) = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = 1$$

- Event A: head was obtained in a maximum of 4 tosses

$$A = \{H, TH, TTH, TTTH\} \quad \text{and} \quad P(A) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = 0.9375$$

- Event B: head appears in at least 3 throws

$$B = \Omega \setminus \{H, TH\} \quad \text{and} \quad P(B) = 1 - (\frac{1}{2} + \frac{1}{2^2}) = 0.25$$

(3) **Taxis**

- Random experiment: count the **number of taxis** that pass through Wiedner Hauptstrasse during our class
- Sample space: $\Omega = \{0, 1, 2, \dots\}$.
- Probability: this is often modelled with the probability mass function

$$p_k = P(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \quad k = 0, 1, \dots$$

known as the Poisson distribution, where λ is the average number of taxis.

- We can put the outcomes in a table:

ω_i	0	1	$\dots$	n	$\dots$
p_i	$e^{-\lambda}$	$e^{-\lambda} \cdot \lambda$	$\dots$	$e^{-\lambda} \cdot \frac{\lambda^n}{n!}$	$\dots$

Is the sum of the probabilities of all possible outcomes 1? Yes!

Using the Taylor series $e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$, we obtain

$$\sum_{k=0}^{\infty} p_k = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

HW What is the probability that 2 taxis will pass by? 5? at least 8?

(4) **Two fair dice**

- Random experiment (a): roll two fair dice, note the pair of numbers showing on the dice (first, second)
- Sample space: the product of the result sets for each die

$$\Omega = \{(1,1),(2,1),\dots(6,5),(6,6)\}, \ \ |\Omega| = 36$$

- The probability of each outcome is $\frac{1}{36}$.

		die 2					
		1	2	3	4	5	6
	1	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	2	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
die 1	3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	4	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$
	6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

two dimensional probability table (Table 1)

- A: one gets the same number on both dice
  - $A = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$ and $P(A) = \frac{6}{36} = \frac{1}{6}$

(4) **Two fair dice**

- Random experiment (b): roll two dice and note the sum of the numbers appeared on the dice
- Sample space: $\Omega = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}, \quad |\Omega| = 11$
- probability: these 11 outcomes are *not* all equally likely.

ω_i	2	3	4	5	6	7	8	9	10	11	12	
p_i	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1

Table 2

- A: the sum is smaller than 6
  - $A = \{2, 3, 4, 5\}$ and $P(A) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} = \frac{5}{18}$

HW
  - B: The total is even
  - C: the sum is at most 10
  - D: the sum is 1
  - Are A and C **disjoint**? Compute $A \cap B$, $B \cup C$ and $B^c \cap D$.

- Question: What is the relationship between the two probability tables above?

Simple examples

HW A statistical experiment

- Experiment: random selection of k objects from a set of n distinguishable objects $\mathcal{M} = \{1, 2, \ldots, n\}$, $k \leqslant n$, assuming an object that has been selected cannot be selected a second time.

  Describe the result sets and determine their size when selected a second time.

  - the order of selection does not matter
  - the order of selection is important.

(5) Measure the **lifetime** of a transistor

- Random experiment: measure the lifetime (in operating hours) of a transistor and report the result
- Sample space: $\Omega = [0, \infty)$, i.e. we can obtain any nonnegative real value
- Probability: Since there is a continuum of possible outcomes, we have to use a probability density function.

We will talk about it next week.

Some rules of probability

From now on, we assume $(\Omega, \mathcal{F}, P)$ is a probability space. Recall, the space $\mathcal{F}$ containing all events is closed under unions, intersections and complements.

- Let A, B and C be three events.

  Then, A^c, $B \cup C$ are $A \cap B$ are also events and the following hold:

  (1) $P(A^c) = 1 - P(A)$

  (2) $P(B \cup C) = P(B) + P(C) - P(B \cap C)$

  (3) $A \subseteq B \implies P(A) \leqslant P(B)$

  (4) $A \cap B = \emptyset \implies P(A \cap B) = 0$
  A and B are <u>disjoint</u>

  We can visualize these rules using Venn diagrams.

Examples: Rules of probability

- For the following examples suppose we have an experiment that produces a random integer between 1 and 20. The probabilities are not necessarily the same for each outcome.

  (a) If the probability of an odd number is 0.65 what is the probability of an even number?

  - If A is an event of being an odd random number, then A^c is an event of being an even random number. Then, $P(A^c) = 1 - P(A) = 0.35$.

  (b) Consider two events, A is an integer divisible by two and B is odd and less than 10. Suppose $P(A) = 0.6$ and $P(B) = 0.25$. What is $A \cap B$? What is the probability $P(A \cup B)$?

  - A and B are disjoint, since all numbers in A are even and all numbers in B are odd, i.e. $A \cap B = \emptyset$.
  - $P(A \cup B) = P(A) + P(B) = 0.85$

  (c) Consider three events, A is a random number and is a multiple of 2, B is a multiple of 3 and C is a multiple of 6. Suppose $P(A) = 0.6$ and $P(B) = 0.3$ and $P(C) = 0.2$. What is the probability of the event A or B?

  - Since an integer is divisible by 6 if and only if it is divisible by both 2 and 3, we have $C = A \cap B$. Then,
    $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(C) = 0.6 + 0.3 - 0.2 = 0.7.$

HW Examples

HW If $P(A^c) = \frac{1}{4}$ and $P(B) = \frac{1}{3}$, can A and B be disjoint? Justify your answer.

HW The 24h express delivery has increased significantly in the last ten years. Customer service has consistently been shown to be the greatest influence on a company's success. A study was conducted to examine customer satisfaction with an express delivery service. In addition to their satisfaction, the customers were asked how often they had used the overnight shipping option. The results are shown below

Frequency of Use	**Satisfaction**	**level**		**Total**
	High	**Medium**	**Low**	
<2 per month	250	140	10	400
2-5 per month	140	55	5	200
>5 per month	70	25	5	100
Total	460	220	20	700

- What is the probability that a randomly selected person will use the company two to five times a month or have moderate satisfaction?

Another example

(1) Playing **poker**
52 cards: 13 ranks $2, \ldots 9, 10$, J, Q, K, A and 4 suits $\heartsuit, \spadesuit, \diamondsuit, \clubsuit$
Poker hands consist of five cards

- Random experiment: playing poker with cards
- Sample space: Ω contains all poker hands ω
  In total $|\Omega| = \binom{52}{5}$ poker hands. Also, $\mathcal{F} = \mathcal{P}(\Omega)$ and $|\mathcal{F}| = 2^{52}$.
- Probability: all $\omega \in \Omega$ are **equally probable**

(a) *A*: we got **exactly one pair**

Question: What is the probability of getting exactly one pair?

(a) less than 5%
(b) between 5% and 10%
(c) between 10% and 20%
(d) between 20% and 40%
(e) bigger than 40%.

Another example

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52 cards: 13 ranks $2, \ldots 9, 10$, J, Q, K, A and 4 suits $\heartsuit, \spadesuit, \diamondsuit, \clubsuit$
Poker hands consist of 5 cards

- Random experiment: playing poker with cards
- Sample space: Ω contains **all poker hands**; in total $\binom{52}{5}$ poker hands.
- Probability: all poker hands are **equally probable**

(a) A: we got **exactly one pair**
What is the probability of getting exactly one pair?

Answer: The number of hands with exactly one pair is

$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3 = 1098240$$

- 13 ... number of ways to select the pair
- $\binom{4}{2}$ ... number of ways to choose the two cards (suits) from the pair
- $\binom{12}{3}$ ... number of ways to specify the other three cards
- 4^3 ... number of ways to choose the other three cards (suits) from these values

$$P(A) = P(\textbf{exactly one pair}) = \frac{13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3}{\binom{52}{5}} = \frac{1098240}{2598960} \approx 42.3\%$$

Another example

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(e) bigger than 40%

Answer: (e) bigger than 40%

Another example

(b) B: we get **four kings**

- How many different hands are there with four kings?
  What is the probability of getting four kings?
  - If we specify that four of the cards are kings, there are **48 different ways** to determine the fifth card. Thus,

$$P(B) = P(\textbf{four kings}) = \frac{48}{\binom{52}{5}} = \frac{1}{54145} < \frac{1}{50000}.$$

- We can also calculate the probability $P(B)$ through an "updating" argument, by calculating **conditional probabilities**.
  - The probability that the first card is a king is $\frac{4}{52}$. Assuming the first card is a king, the probability that the second card is a king is $\frac{3}{51}$. If we continue this argument, we get that the probability of a poker hand with four kings equals

$$P(B) = P(\textbf{four kings}) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49} \cdot \frac{48}{48} \cdot \binom{5}{1} = \frac{1}{54145}.$$

This is called the conditional probability because additional conditions are taken into account.

Conditional probability

- The conditional probability answers the question of how the probability of an event changes when we have additional information.
  - The conditional probability of A given B is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \qquad \text{with } P(B) > 0.$$

  - If B becomes the sample space we have $P(B|B) = 1$
  - For disjoint events A and B it holds $P(A|B) = P(B|A) = 0$.

HW Draw two cards from a deck of cards. Let the events be D_1 = "first card is a diamond" and D_2 = "second card is a diamond". What are $P(D_2|D_1)$, $P(D_1|D_2)$ und $P(D_2|D_1^c)$?

Multiplication theorem and the Bayes theorem

- We rewrite the definition of conditional probability and obtain

$$P(A \cap B) = P(A|B) \cdot P(B) \quad \text{for } P(B) > 0 \qquad \text{and}$$
$$P(A \cap B) = P(B|A) \cdot P(A) \quad \text{for } P(A) > 0.$$

... these equations are called Multiplication theorem ...

- We equate the two expressions and obtain

$$\boxed{P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} \qquad \text{and} \qquad P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}}$$

which gives the formula for **reversing** ("turning around") conditional probabilities. The previous equation is called the Bayes theorem (for two events).

Independence

- If for two events A und B it holds

$$P(A|B) = P(A)$$

then we say that A and B are independent.

  - Then,

$$\underline{P(A \cap B)} = P(A|B) \cdot P(B) = \underline{P(A) \cdot P(B)}$$

Thus, two events A and B are independent if it holds

$$P(A \cap B) = P(A) \cdot P(B)$$

HW Let A and B be independent. Show that

- A and B^c ... Prove $P(A \cap B^c) = P(A) \cdot P(B^c)$
- A^c and B^c ... Prove $P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$

are also independent.

HW Let A and B be independent. Knowning that $P(A|B) = 0.2$ and $P(B|A) = 0.5$, compute the probability $P(A^c \cap B)$.

Example

- We toss a fair coin four times. Consider the events
  $A =$ " at least three heads" and
  $B =$ "first toss is tail"
  What are $P(A|B)$ and $P(B|A)$? Are A and B independent?

  Answer:

  - $\Omega = \{HHHH, HHHT, \ldots, TTTT\}, \quad |\Omega| = 2^4 = 16, \quad |\mathcal{F}| = |\mathcal{P}(\Omega)| = 2^{16}$
  - $A = \{HHHT, HHTH, HTHH, THHH, HHHH\}, \quad |A| = 5, \quad P(A) = \frac{5}{16}$
  - $B = \{Tijk : \quad i, j, k \in \{T, H\}\}, \quad |B| = 2^3 = 8, \quad P(B) = \frac{1}{2}$
  - $A \cap B = \{THHH\}, \quad |A \cap B| = 1, \quad P(A \cap B) = \frac{1}{16}$
  - The conditional probabilities are

  $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{16}}{\frac{1}{2}} = \frac{1}{8} \quad \text{and} \quad P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{16}}{\frac{5}{16}} = \frac{1}{5}.$$

  - Since $P(A|B) = \frac{1}{8} \neq \frac{5}{16} = P(A)$, we conclude that A and B are not independent.

Independence

- Three events A, B and C are independent if the following hold

$$P(A \cap B) = P(A) \cdot P(B)$$
$$P(A \cap C) = P(A) \cdot P(C)$$
$$P(B \cap C) = P(B) \cdot P(C)$$
$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

HW Roll two dice and consider the following events

$A =$ "first die is 3"
$B =$ "the sum is 6"
$C =$ "the sum is 7"

A is independent of :

(a) B and C (b) only B (c) only C (d) neither B nor C

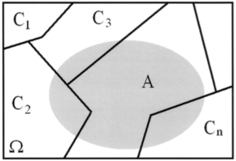
Law of total probability

- If only the conditional probabilities and the probabilities of the conditional event are known, the **total probability** of A results from

$$P(A) = P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c),$$

  where B^c denotes the opposite event to B.

- Let $C_1, C_2, \dots$ with $P(C_j) > 0$ be a partition of the sample space Ω, i.e. these events are pairwise disjoint $C_i \cap C_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{\infty} C_i = \Omega$.



- The **total probability** of an event A is given by

$$P(A) = \sum_{i=1}^{\infty} P(A \cap C_i) = \sum_{i=1}^{\infty} P(A|C_i) \cdot P(C_i)$$

The Bayes theorem

- The Bayes theorem describes the probability of an event based on **prior knowledge of the conditions** that might be related to the event.

  - Let $C_1, C_2, \ldots$ be a partition of Ω and $P(C_i) > 0$ for all $i \geqslant 1$. Then, for an event A with $P(A) > 0$ it holds:

$$\boxed{P(C_i|A) = \frac{P(A \cap C_i)}{P(A)} = \frac{P(A|C_i) \cdot P(C_i)}{\sum\limits_{j=1}^{\infty} P(A|C_j) \cdot P(C_j)}}$$

The Bayes theorem
(for two and more events)

- The partition events $C_1, C_2, \ldots$ are also called hypotheses
- $P(C_i)$ is called the a priori probability of C_i, i.e. the probability **before** the entry of A
- $P(C_i|A)$ is called the posterior probability, i.e. the probability **after** the entry of A

Example

- An urn contains 5 red and 2 green marbles. A random marble is selected and replaced with a marble of the other color, then a second marble is drawn.
  (1) What is the probability the second marble is red?
  (2) What is the probability the first marble was red given the second marble was red?

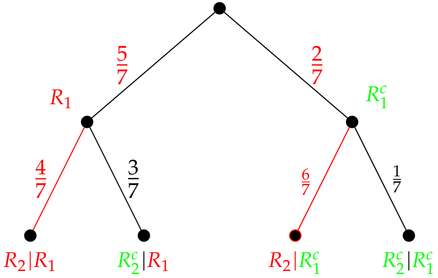
  Use the probability tree to solve the problems.

Answer:

- Denote by R_1 = "the first marble is red" and R_2= "the second marble is red".
- R_1^c="the first marble is green" and R_2^c="the second marble is green"
- $P(R_1) = \frac{5}{7}$ and $P(R_1^c) = 1 - \frac{5}{7} = \frac{2}{7}$
- $P(R_2|R_1) = \frac{4}{7}$ and $P(R_2^c|R_1) = \frac{3}{7}$
- $P(R_2|R_1^c) = \frac{6}{7}$ and $P(R_2^c|R_1^c) = \frac{1}{7}$

- We draw the probability tree

Urn = 5 & 2 marbles



(1) From the **Law of total probability** we obtain

$$P(R_2) = P(R_2|R_1) \cdot P(R_1) + P(R_2|R_1^c) \cdot P(R_1^c) = \frac{5}{7} \cdot \frac{4}{7} + \frac{2}{7} \cdot \frac{6}{7} = \frac{32}{49}$$

(2) By the Bayes theorem we obtain

$$P(R_1|R_2) = \frac{P(R_1 \cap R_2)}{P(R_2)} = \frac{P(R_2|R_1) \cdot P(R_1)}{P(R_2)} = \frac{20/49}{32/49} = \frac{20}{32}$$

- A container contains three types of batteries Type 1, Type 2 and Type 3 in the ratio of 20 : 30 : 50. A battery is good if it lasts more than 100 hours. The probability that a Type 1 battery is good is 0.7, the probability that a Type 2 battery is good is 0.4 and the probability that a Type 3 battery is good is 0.3. A battery is randomly taken from the container.
  (1) What is the probability the chosen battery is good?
  (2) What is the probability the battery is of Type 3 given it was not good?

Answer:

- Let $G =$ 'battery is good", $H_i =$ "battery is of Type i", where $i = 1, 2, 3$.

(1) Then,

$$P(G) = P(G|H_1) \cdot P(H_1) + P(G|H_2) \cdot P(H_2) + P(G|H_3) \cdot P(H_3)$$
$$= 0.7 \cdot 0.2 + 0.4 \cdot 0.3 + 0.3 \cdot 0.5 = 0.41$$

(2)

$$P(H_3|G^c) = \frac{P(G^c|H_3) \cdot P(H_3)}{P(G^c)} = \frac{P(G^c|H_3) \cdot P(H_3)}{1 - P(G)} = \frac{0.7 \cdot 0.5}{0.59} = \frac{35}{59} = 0.593.$$

More examples & some multiple-choice questions

More examples

HW A blood test reacts 95% positive if there is a disease. It also reacts to 1% "false positive". It is assumed that 0.5% of the population are ill.

(1) What is the probability a randomly selected person is actually sick?
(2) What is the probability of being sick if the blood test is positive?

HW Let B be the set of all natural numbers less or equal 100. Define the events B_i, for any natural number i to be the set of numbers in B that are divisible by i. For example, $B_3 = \{3, 6, 9, \ldots, 96, 99\}$. Compute $|B_2 \cup B_3 \cup B_5|$.

HW The Randomizer holds the 6-sided die in one fist and the 8-sided die in the other. The Roller selects one of the Randomizer's fists and takes the die face down. The Roller rolls the die in secret and reports the result to the table.

Given the reporeted number, what is the probability that the 6-sided die was selected? (Find the probability for each possible number reported.)

HW Show that if $P(A) > 0$ then $P(A \cap B | A) \geqslant P(A \cap B | A \cup B)$.

HW A professor thinks students who live on campus are more likely to get the best grade ("sehr gut") in the Statistics and Probability Theory course. To check this, the professor combines the data from the past few years: 600 students have taken the course, 120 students have gotten "sehr gut", 200 students lived on campus, 80 students who did not live on the campus and got "sehr gut". Does this data suggest that *getting the grade "sehr gut"* and *living on the campus* are independent events?

HW There are 15 people in a party, including Anna and Mia. We divide the 15 people into 3 groups, where each group has 5 people. What is the probability that Anna and Mia are in the same group?

HW There are two bags of marbles left on the table. Bag 1 contains 10 blue marbles, while Bag 2 contains 15 blue marbles. Sandra picks one of the bags at random, and throws 6 red marbles in it. Then she shakes the bag and chooses 5 marbles (without replacement) at random from the bag. If there are exactly 2 red marbles among the 5 chosen marbles, what is the probability that Sandra has chosen Bag 2?

More examples

HW 36 members of one club play tennis, 28 play basketball and 18 swim. In addition, 22 play tennis and basketball, 12 play tennis and swim, 9 play basketball and swim and 4 play all three sports.

(1) How many members are there in the club?
(2) How many members of the club play only one sport?
(3) How many members of the club play at least one of the three sports?

HW Five cards are dealt from a shuffled deck of cards.
What is the probability that the dealt hand contains exactly two kings, given that we know it contains at least one king?

HW We choose a point (X, Y) uniformly at random in the unit square

$$S = \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}.$$

Let A be the event $A = \{(x, y) \in S : |x - y| \leqslant \frac{1}{2}\}$ and B be the event $B = \{(x, y) \in S : y \geqslant x\}$. Are the events A and B independent?

A few multiple-choice questions

(1) A medical treatment has a success rate of 0.8. Two patients will be treated with this treatment. Assuming the results are independent for the two patients, what is the probability that at least one of them will be successfully cured?

(a) 0.96
(b) 0.32
(c) 0.64
(d) 0.04

(2) In a list of 15 households, 9 own homes and 6 do not own homes. Four households are randomly selected from these 15 households. Find the probability that the number of households in these four who own homes is at most one.

(a) 0.1536
(b) 0.1792
(c) 0.3456
(d) 0.4752

A few multiple-choice questions

(3) Suppose that there are 4 women and 8 men. How many 5 person committees can be formed with exactly 2 women and 3 men?

(a) $\binom{12}{5}$
(b) $\binom{4}{2} \cdot \binom{8}{2}$
(c) $\binom{4}{2} \cdot \binom{8}{3}$
(d) $\binom{12}{5} - \binom{4}{1} \cdot \binom{8}{4}$

(4) Suppose box A contains 4 red and 5 blue coins and box B contains 6 red and 3 blue coins. A coin is chosen at random from the box A and placed in box B. Finally, a coin is chosen at random from among those now in box B. What is the probability a blue coin was transferred from box A to box B given that the coin chosen from box B is red?

(a) $\frac{15}{29}$
(b) $\frac{14}{29}$
(c) $\frac{1}{2}$
(d) $\frac{7}{10}$

A few multiple-choice questions

(5) A fair die is rolled. Find the probability of getting an even number or a number bigger than 2.

(a) $\frac{1}{3}$

(b) $\frac{7}{12}$

(c) $\frac{2}{3}$

(d) $\frac{5}{6}$

(6) Anna has an unfair coin. She knows that when fliping this coin, head is obtained twice as often as a tail. Anna rolls this coin until she obtains a tail. What is the probability that she rolled the coin at most three times?

(a) $\frac{19}{27}$

(b) $\frac{26}{27}$

(c) $\frac{5}{9}$

(d) $\frac{4}{27}$

A few multiple-choice questions

(7) Let A and B be two independent events.If we additionally know that $P(A|B) = 0.6$ and $P(B|A) = 0.3$, compute the probability of the event "at most one of A or B".

(a) 0.82
(b) 0.54
(c) 0.42
(d) 0.72

(8) A multiple choice exam has 4 choices for each question. A student has studied enough so that the probability they will know the answer to a question is 0.5, the probability that they will be able to eliminate one choice is 0.25, otherwise all 4 choices seem equally plausible. If they know the answer they will get the question right. If not they have to guess from the 4 or 3 choices. As the examiner you want the test to measure what the student knows. If the student answers a question correctly what is the probability they knew the answer?

(a) 0.129
(b) 0.443
(c) 0.774
(d) 0.097

A few multiple-choice questions

(9) Consider two independent (nontrivial) events A and B. Which one of the following statements is **true**?

(a) $P(A \cap B) = P(A) + P(B)$

(b) $P(A \cap B) = 0$

(c) $P(A \cap B^c) = P(A) - P(A)P(B)$

(d) $P(A \cap B^c) = 1 - P(A)P(B)$

(10) Tom enters a chess tournament where his probability of winning a game is 0.3 against half of the players (category I), 0.4 against a quarter of the players (category II), and 0.5 against the remaining quarter of the players (category III). He plays a game against a randomly chosen opponent. What is the probability that he wins the game?

(a) 0.375

(b) 0.150

(c) 0.850

(d) 0.625

Thank you for your attention!