Exercise 6

Discrete Mathematics

November 19, 2020

https://aofa.cs.princeton.edu/30gf/

https://www.youtube.com/watch?v=-drdeNMoe8w

https://www.math.upenn.edu/~wilf/gfology2.pdf

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https://math.stackexchange.com/questions/1540225/closed-form-expression-for-sum-k-0nk2-3k-2

https://math.stackexchange.com/questions/3437722/finding-a-closed-form-expression-for-sum-k-

We know from the lecture that

$$\sum_{n\geq 0}\binom{n+k-1}{k-1}z^n = \frac{1}{(1-z)^k} \tag{1}$$

and

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \tag{2}$$

and (Cauchy product)

$$\sum_{n\geq 0} a_n z^n \cdot \sum_{n\geq 0} b_n z^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n \tag{3}$$

and

$$1 + z + z^2 + \dots = \frac{1}{1-z} \tag{4}$$

Let $a_n = (k^2 + 3k + 2)$ and $b_n = 1$. Note that $k^2 + 3k + 2 = (k+2)(k+1)$.

We apply 3 to get

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(k+1)(k+2)\right)x^n = \sum_{n\geq 0}(k+1)(k+2)x^n \cdot \sum_{n\geq 0} x^n \tag{5}$$

By 2 we get $(k+2)(k+1) = 2\binom{k+2}{2}$. We can then apply 1 to get

$$\sum_{n\geq 0}(k+1)(k+2)x^n = 2\sum_{n\geq 0}\binom{k+2}{2}x^n = \frac{2}{(1-x)^3}.$$

Additionally, we get by 4 that

$$\sum_{n\geq 0} x^n = \frac{1}{1-z}$$

So both factors from 5 give

$$\frac{2}{(1-x)^k}\cdot\frac{1}{1-x} = \frac{2}{(1-x)^4}$$

which we can again plug into 1 to get

$$\frac{2}{(1-x)^4} = 2\sum_{n\geq 0}\binom{n+3}{3}x^n$$

so $a_n = 2\binom{n+3}{3}$ which we can now plug into 2 to finally get

$$\sum_{k=0}^{n}(k^2+3k+2) = \frac{(n+3)(n+2)(n+1)}{3}$$

Exercise 52

https://math.stackexchange.com/questions/340124/binomial-coefficients-1-2-choose-k

https://en.wikipedia.org/wiki/Factorial#Factorial_of_non-integer_values

https://en.wikipedia.org/wiki/Generating_function#Rational_functions

https://math.stackexchange.com/questions/69270/show-sum-limits-n-0-infty2n-choose-nxn-1-4x-1

https://math.stackexchange.com/questions/205898/how-to-show-that-1-over-sqrt1-4x-generates-s

https://math.stackexchange.com/questions/379249/proving-frac1-sqrt1-4x-sum-n-geq02n-choose-n

We know from the lecture that

$$\sum_{n\geq 0}\binom{\alpha}{n}z^n = (1+z)^{\alpha}. \tag{6}$$

Substituting $u := -4z$ we calculate

$$\frac{1}{\sqrt{1-4z}} = (1+u)^{-0.5} = \sum_{n\geq 0} \binom{-0.5}{n} (-4z)^n = \sum_{n\geq 0} \binom{-0.5}{n} (-1)^n 4^n z^n$$

It is known that

$$\binom{x}{y} = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$$

so we get

$$\binom{-0.5}{n} = \frac{\Gamma(0.5)}{\Gamma(n+1)\Gamma(0.5-n)}$$

Coming from the other side we see

$$\binom{2n}{n} = \frac{\Gamma(2n+1)}{\Gamma(n+1)\Gamma(2n-n+1)}$$

By the recursion formula $\Gamma(s+1) = s\Gamma(s)$ for complex s we get

$$\binom{2n}{n} = \frac{2n\Gamma(2n)}{n\Gamma(n) \cdot n\Gamma(n)}$$

and by the duplication formula $\Gamma(2z) = \pi^{-0.5} 2^{2z-1} \Gamma(z)\Gamma(z+0.5)$ we get

$$\binom{2n}{n} = \frac{2n\pi^{-0.5} 2^{2n-1} \Gamma(n)\Gamma(n+0.5)}{n\Gamma(n) \cdot n\Gamma(n)} = \frac{4^n \Gamma(n+0.5)}{\Gamma(n+1)\sqrt{\pi}}$$

Adding the factor $(-1)^n 4^n$ again, we now show the identity

$$\frac{4^n \Gamma(n+0.5)}{\Gamma(n+1)\sqrt{\pi}} = \frac{\Gamma(0.5)}{\Gamma(n+1)\Gamma(0.5-n)} (-1)^n 4^n \tag{7}$$

Some multiplications and divisions lead to

$$\Gamma(n+0.5)\Gamma(0.5-n) = \Gamma(0.5)(-1)^n \sqrt{\pi}$$

We can now apply the reflection rule $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$

$$\frac{\pi}{\sin(\pi(n+0.5))} = \Gamma(0.5)(-1)^n \sqrt{\pi}$$

We observe that for integer values $\sin(\pi(n+0.5))$ alternates between -1 and 1, to be precise $\sin(\pi(n+0.5)) = (-1)^n$ for integers n and as additionally $1/(-1)^n = (-1)^n$ we get

$$\pi(-1)^n = \Gamma(0.5)(-1)^n \sqrt{\pi}$$

We can now first divide the equation by $(-1)^n$. We consider the reflection rule $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ with $z = 0.5$ to get $\Gamma^2(0.5) = \pi$. The identity

$$\pi = \pi$$

concludes the proof for equation 7. Therefore we have

$$\binom{2n}{n} = \binom{-0.5}{n}(-1)^n 4^n$$

and finally we get the formula that concludes the proof

$$\frac{1}{\sqrt{1-4z}} = \sum_{n\geq 0} \binom{2n}{n} z^n$$

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WolframAlpha

MathStackexchange

Taylor's theorem: If $A(x)$ is the generating function for a sequence $a_0, a_1, \ldots$ then $a_n = A^{(n)}(0)/n!$ where $A^{(n)}$ is the nth derivate of A and $0! = 1$.

The Taylor series of function $f(x)$ is defined as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

The Taylor series of $\frac{2+3z^2}{\sqrt{1-5z}}$ with center 0 (=Maclaurin) is

$$2 + 5z + \frac{87}{4}z^2 + \frac{685}{8}z^3 + \frac{23675}{64}z^4 + \ldots$$

$[z^n]\,\frac{2+3z^2}{\sqrt{1-5z}}$ can be read directly from that series, for example

$$[z^3]\,\frac{2+3z^2}{\sqrt{1-5z}} = \frac{685}{8}$$

To get an explicit formula, we split the term into two summands

$$\frac{2+3z^2}{\sqrt{1-5z}} = \frac{2}{\sqrt{1-5z}} + \frac{3z^2}{\sqrt{1-5z}}$$

Correction: Missed the factor 5. Using the lemma $\sum_{n\geq 0}\binom{n+k-1}{k-1}z^n = \frac{1}{(1-z)^k}$ from the lecture we get $1/\sqrt{1-5z} = \sum_{n\geq 0}\binom{n+0.5-1}{0.5-1}$. Therefore we get

$$\frac{2+3z^2}{\sqrt{1-5z}} = 2\sum_{n\geq 0}\binom{n-0.5}{-0.5}z^n + 3z^2\sum_{n\geq 0}\binom{n-0.5}{-0.5}z$$

We can now right shift $zA(z) = \sum_{n\geq 1} a_{n-1}z^n$ or in other words $[z^n]A(z) = A_n \rightarrow [z^n]zA(z) = A_{n-1}$ twice which leads to

$$\frac{2+3z^2}{\sqrt{1-5z}} = 2\sum_{n\geq 0}\binom{n-0.5}{-0.5}z^n + 3\sum_{n\geq 0}\binom{n-2.5}{-0.5}z^n$$

and therefore

$$[z^n]\frac{2+3z^2}{\sqrt{1-5z}} = 2\binom{n-0.5}{-0.5} + 3\binom{n-2.5}{-0.5}$$

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Leaves

Claim: A full t-ary tree with n internal nodes has $tn+1$ nodes total.

Proof. There are two types of nodes: Nodes with and nodes without parents. A tree has exactly one node with no parent. We can count the nodes with a parent by taking the number of parents in the tree n and multiplying by the branching factor t. This concludes the proof.

By our claim, the number of leaves in a full t-ary tree with n internal nodes is $(tn+1) - n = (t-1)n + 1$

Functional equation

`https://math.stackexchange.com/questions/3179040/generalizing-a-formula-for-enumerating-root`

Claim: The number a_n of t-ary trees with n internal nodes is given by

$$a_n = \frac{1}{(t-1)n+1}\binom{tn}{n}. \tag{8}$$

Proof. t-ary trees $\mathcal{A}_t$ can be formally described by

$$\mathcal{A}_t = \square + \circ \times \mathcal{A}_t^t.$$

where $\square$ symbolise external and $\circ$ internal nodes. Thus, the generating function

$$A(z) = \sum_{n\geq 0} a_n z^n$$

satisfies the relation

$$A(z) = 1 + zA(z)^t.$$

Note from presentations: We're actually done here.

Setting $\tilde{A}_t(z) = A(z) - 1$ we get

$$\tilde{A}(z) = z(1+\tilde{A}(z))^t$$

Theorem (Lagrange's inversion formula): Let $\phi(x)$ be a power series with $\phi(0) \neq 0$ and $y(x)$ the (unique) power series solution of the equation $y(x) = x\phi(y(x))$. Then $y(x)$ is invertible and the n-th coefficient of $g(y(x))$ (where $g(x)$ is an arbitrary power series) is given by

$$[x^n]g(y(x)) = \frac{1}{n}[u^{n-1}]g'(u)\phi(u)^n \tag{9}$$

By using 9 with $\phi(z) = (1+z)^2$, $y = \tilde{A}$ (for $n \geq 1$)

$$\begin{aligned} a_n = [z^n]\,\tilde{A}(z) &= \frac{1}{n}\left[u^{n-1}\right](1+u)^{tn} \\ &= \frac{1}{n}\binom{tn}{n-1} = \frac{1}{(t-1)n+1}\binom{tn}{n}. \end{aligned}$$

Note that $[u^{n-1}]$ just selects the $(n-1)$th coefficient of $(1+u)^{tn}$. By symmetry the "starting side" is irrelevant. Start counting at 0. This is equal to $\binom{tn}{n-1}$. Example for $n = t = 3$: $\binom{3\cdot 3}{3-1} = 36$

$$(1+u)^{3\cdot 3} = u^9 + 9u^8 + \mathbf{36}u^7 + 84u^6 + 126u^5 + 126u^4 + 84u^3 + \mathbf{36}u^2 + 9u + 1$$

This concludes the proof. Thus, we get the functional equation

$$A(z) = \sum_{n\geq 0} \frac{1}{(t-1)n+1}\binom{tn}{n} z^n$$

Example: For $t = 4$ we can calculate $a_0 = 1$, $a_1 = 1$, $a_2 = 4$, $a_3 = 22$ leading to

$$A(z) = 1z^0 + 1z^1 + 4z^2 + 22z^3 \ldots$$

Figure 1: One single external (= no children) node, 0 internal nodes (=$1z^0$)

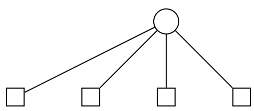


Figure 2: 1 internal, 4 external (= no children) nodes (=$1z^1$)

Exercise 55

https://math.stackexchange.com/q/3025656/844881

By the structure of the equation we get

$$T(z) = z^4 + z^2 * T(z)^2 \implies 0 = z^2 T(z)^2 - T(z) + z^4$$

which we can solve using $(-b \pm \sqrt{b^2 - 4ac})/(2a)$ to get

$$T(z) = \frac{1 \pm \sqrt{1 - 4z^2 z^4}}{2z^2} \tag{10}$$

Note that the + of the $\pm$ doesn't lead to a power series, so we take the -. After applying the identity $\sqrt{1-4z^6} = (1-4z^6)^{1/2}$ we can use the binomial formula 6 to get

$$\sqrt{1-4z^6} = \sum_{n \geq 0} \binom{1/2}{n} (-4)^n z^{6n} \tag{11}$$

$$\begin{aligned}
T(z) &= \frac{1}{2z^2} \left(1 - \sum_{n \geq 0} \binom{0.5}{n} (-4z^6)^n \right) \\
&= \frac{1}{2z^2} \left(1 - \left(1 - \frac{0.5}{1!} 4z^6 + \frac{0.5(0.5-1)}{2!} 4^2 z^{12} - \frac{0.5(0.5-1)(0.5-2)}{3!} 4^3 z^{18} + \dots \right) \right) \\
&= \frac{1}{2z^2} \left(\frac{1}{2} 4z^6 + \frac{1}{4} \cdot \frac{1}{2!} 4^2 z^{12} + \frac{1}{3!} \cdot \frac{3}{8} 4^3 z^{18} + \frac{1}{16} \cdot \frac{5 \cdot 3 \cdot 1}{4!} 4^4 z^{24} + \dots \right) \\
&= z^4 + \frac{1}{2} \cdot \frac{2!}{1!1!} z^{10} + \frac{1}{3} \cdot \frac{4!}{2!2!} z^{16} + \frac{1}{4} \cdot \frac{6!}{3!3!} z^{22} + \dots \\
&= \sum_{n \geq 0} \frac{1}{1+n} \binom{2n}{n} z^{6n+4}
\end{aligned}$$

Now replace n by $(n-4)/6$ to get

$$t_n = \frac{6}{n+2} \binom{(n-4)/3}{(n-4)/6}$$

This does not give the correct result

... but in the lecture an example was done in a similar fashion, so it might make sense to do it better than I did.

We saw in the lecture that $\binom{1/2}{n} = (-1)^{n-1} \frac{1}{2^n} \frac{1 \cdot 3 \cdot 5 \dots (2n-3)}{n!}$, so there are only odd numbers in the numerator, so we can add the even numbers to get

$$\binom{1/2}{n} = \frac{(-1)^{n-1}}{2^n} \frac{1 \cdot 2 \cdot 3 \cdot 4 \dots (2n-3)(2n-2)}{n! \cdot 2^{n-1} 1 \cdot 2 \dots (n-1)} = \frac{(-1)^{n-1}}{2^{2n-1}} \frac{(2n-2)!}{n!(n-1)!} \tag{12}$$

So all in all we get

$$T(z) = \frac{1 - \sum_{n\geq 0} \frac{(-1)^{n-1}}{2^{2n-1}} \frac{(2n-2)!}{n!(n-1)!}(-4)^n z^{6n}}{2z^2}$$

With left shifting we get (as the first coefficient in the taylor series of $\sqrt{1-4z^6}$ is 1)

$$\begin{aligned} T(z) &= \frac{1}{2z^2} - \frac{1 + \sum_{n\geq 0} \frac{(-1)^n}{2^{2n}} \frac{(2n)!}{(n+1)!(n)!}(-4)^{n+1} z^{6n+1}}{2z^2} \\ &= \frac{1}{2z^2} - \frac{1}{2z^2} + \frac{\sum_{n\geq 0} \frac{(-1)^n}{2^{2n}} \frac{(2n)!}{(n+1)!(n)!}(-4)^{n+1} z^{6n}}{2z} \end{aligned}$$

Considering that $(-1)^n \cdot (-4)^n / 2^{2n} = 1$ we can simplify this to

$$T(z) = -2 \cdot \frac{\sum_{n\geq 0} \frac{(2n)!}{(n+1)!(n)!} z^{6n}}{z}$$

Note that in this sum the z with the least exponent is z^6 for $n = 0$. Therefore, we don't have to make a complete left shift (including extracting the first coefficient from the sum) but can divide directly

$$T(z) = -2 \cdot \sum_{n\geq 0} \frac{(2n)!}{(n+1)!(n)!} z^{6n-1}$$

which is

$$T(z) = -2 \cdot \sum_{n\geq 0} \binom{2n}{n} \frac{1}{n+1} z^{6n-1}$$

Because of the exponent of z we then replace n with $\frac{n+1}{6}$

$$T(z) = -2 \cdot \sum_{n\geq 0} \binom{\frac{n+1}{3}}{\frac{n+1}{6}} \frac{6}{n+7}$$

The series expansion of $T(z)$ is

$$z^4 + z^{10} + 2z^{16} + 5z^{22} + 14z^{28} + 42z^{34} + 132z^{40} \ldots ...$$

Exercise 56

https://www.whitman.edu/mathematics/cgt_online/book/section03.05.html

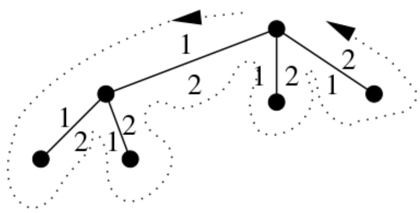
A bijection is not the preferred approach. It would be better to use sequence constructions.

Plane trees are also called ordered trees.

A Dyck word is a string of i 1's and i 2's such that in every prefix, the number of 1's is at least as high as the number of 2's. Let W_n be the set of all Dyck words on i 1's and i 2's. It is known that for the Catalan numbers $C_n = |W_n|$.

We now show a bijection from ordered trees to W_n.

$w = 1121221212$



Given Dyck word w, form an ordered tree as follows:

- Draw the root.
- Read w from left to right.

  For 1, add a new rightmost child to the current vertex and move to it. For 2, go up to the parent of the current vertex.

For any prefix of w with a 1s and b 2s, the depth of the vertex you reach is $a - b \geq 0$, so you do not go above the root. At the end, $a = b = m$ and the depth is $a - b = 0$ (the root).

Conversely, trace an ordered tree counterclockwise from the root. Label each edge 1 going down its left side, and 2 going up its right.

Thus, W_n is in bijection with ordered trees on m edges (hence $m + 1$ vertices), so the Catalan number C_m counts these too.

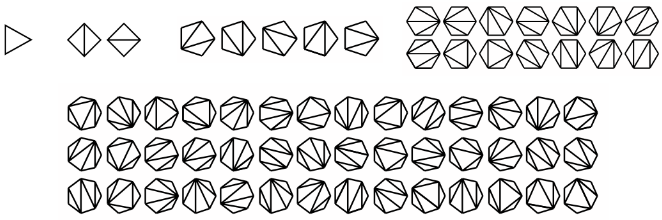
As a consequence, the number of plane rooted trees with $n = m - 1$ nodes is

$$C_{m-1} = \frac{1}{m-1+1}\binom{2(m-1)}{m-1} = \frac{1}{m}\binom{2m-2}{m-1}$$

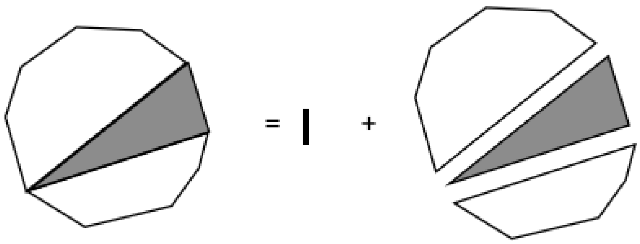
Exercise 57

Presentations: The fixed edge on the "border" is important to avoid double counting. Can also be solved by bijection to binary trees. Convex polygons are sufficient.

The collection $\mathcal{T}$ of all triangulations of regular polygons, with size defined as the number of triangles, is a combinatorial class, whose counting sequence starts as $T_0 = 1, T_1 = 1, T_2 = 2, T_3 = 5, T_4 = 14, T_5 = 42$.



Fix $n + 2$ points arranged in anticlockwise order on a circle and conventionally numbered from 0 to $n + 1$ (for instance the $(n + 2)$th roots of unity). A triangulation is defined as a (maximal) decomposition of the convex $(n + 2)$-gon defined by the points into n triangles (figure in the beginning). The size of the triangulation is the number of triangles; that is, n. Given a triangulation, we define its root as a triangle chosen in some conventional and unambiguous manner (e.g., at the start, the triangle that contains the two smallest labels). Then, a triangulation decomposes into its root triangle and two subtriangulations (that may well be empty) appearing on the left and right sides of the root triangle; the decomposition is illustrated by the following diagram:



The class $\mathcal{T}$ of all triangulations can be specified recursively as

$$\mathcal{T} = \{\epsilon\} + (\mathcal{T} \times \triangle \times \mathcal{T})$$

provided that we agree to consider a 2-gon (a segment) as giving rise to an empty triangulation of size 0. (The subtriangulations are topologically and combinatorially equivalent to standard ones, with vertices regularly spaced on a circle.) Consequently, the OGF $T(z)$ satisfies the equation $T(z) = 1 + zT(z)^2$, so that $T(z) = \frac{1}{2z}(1 - \sqrt{1 - 4z})$. This is the same OGF as the OGF of the Catalan numbers. Therefore, we get that the triangulations are enumerated by Catalan numbers:

$$T_n = C_n \equiv \frac{1}{n+1}\binom{2n}{n}$$