Problem Set 4

Problem 4.1 Consider binary phase shift keying transmission of a symbol $\mathsf{s} \in \{-a, a\}$, $a > 0$, with $\mathrm{P}\{\mathsf{s} = -a\} = 1/2$. The symbol is transmitted over a fading channel with coefficient $\mathsf{h} \sim \mathcal{N}(\mu_\mathsf{h}, \sigma_\mathsf{h}^2)$, $\mu_\mathsf{h} > 0$. Let $\mathsf{r} = \mathsf{h} \cdot \mathsf{s}$ denote the received symbol. The channel coefficient is statistically independent of the source symbol.

a) Find a such that $\sigma_\mathsf{s}^2 = 2$.

b) Specify and sketch both conditional pdfs of r given the source symbol.

c) Specify and sketch the unconditional pdf of the received symbol.

d) Find the conditional probability that $\mathsf{r} > 0$ given $\mathsf{s} = -a$ and sketch it in your figure. *Hint: use the Q-function to express the probability.*

e) The detected symbol is $\hat{\mathsf{s}} = a \cdot \operatorname{sign}(\mathsf{r})$. Calculate the probability of a symbol error, i.e., $\hat{\mathsf{s}} \neq \mathsf{s}$, for $\mu_\mathsf{h} = \sqrt{2}$ and $\sigma_\mathsf{h}^2 = 1/4$.

f) Assume a block of 400 independent symbols is transmitted. Calculate the probability that the entire block is received correctly.

g) Is data detection possible if $\mu_\mathsf{h} < 0$? How about $\mu_\mathsf{h} = 0$? Justify your answers.

Problem 4.2 Consider the joint pdf

$$f_{\mathsf{x},\mathsf{y}}(x, y) = \begin{cases} xe^{-x(y+1)}, & x \geq 0 \wedge y \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

a) Find the marginal pdfs $f_\mathsf{x}(x)$ and $f_\mathsf{y}(y)$.

b) Find the conditional pdfs $f_{\mathsf{x}|\mathsf{y}}(x|y)$ and $f_{\mathsf{y}|\mathsf{x}}(y|x)$.

c) Are x and y statistically independent? Justify your answer.

Problem 4.3 Let x and y be two statistically independent random variables, distributed according to the following pdfs:

$$f_\mathsf{x}(x) = \frac{1}{3}e^{-x/3}u(x), \quad f_\mathsf{y}(y) = \frac{1}{2}e^{-y/2}u(y), \quad \text{where} \quad u(z) = \begin{cases} 1, & z \geq 0 \\ 0, & z < 0 \end{cases}.$$

a) Calculate the probability $\mathrm{P}\{\mathsf{x} > 5 \wedge \mathsf{y} > 10\}$.

b) Let $\mathsf{z} = \mathsf{x} + \mathsf{y}$. Find the mean μ_z of the random variable z.

c) Calculate the characteristic function $\Phi_\mathsf{z}(\omega)$.

Problem 4.4 Consider two statistically independent random variables x and y with

$$f_{\mathsf{x}}(x) = \begin{cases} \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \qquad f_{\mathsf{y}}(y) = \begin{cases} \frac{1}{\pi\sqrt{1-y^2}}, & |y| < 1 \\ 0, & |y| \geq 1 \end{cases}.$$

a) Show that $\mathsf{z} = \mathsf{xy}$ is Gaussian distributed.

b) Specify the mean and variance of z.

Hint: $\displaystyle\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}},\ a > 0.$