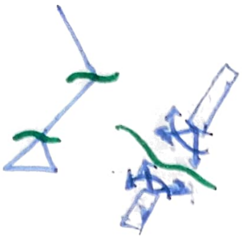
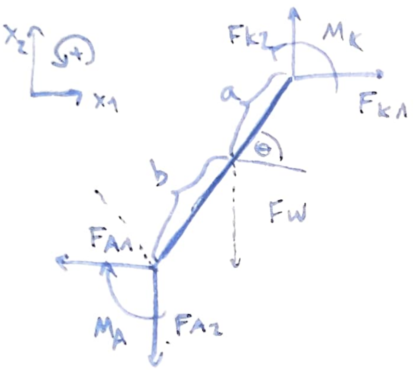
$\boxed{1}$



ankle reaction force: $\vec{F}_A = \begin{pmatrix} 21{,}6\,N \\ -453{,}5\,N \end{pmatrix} = \begin{pmatrix} F_{A1} \\ F_{A2} \end{pmatrix}$

shank segment



$m = 3{,}255\ kg\ ,\quad \vec{a} = \begin{pmatrix} 1{,}21\,m/s^2 \\ -0{,}95\,m/s^2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$\boxed{\Sigma F = m \cdot a}$

$x_1: \quad -F_{A1} + F_{K1} = m \cdot a_1$

$F_{K1} = m \cdot a_1 + F_{A1} = 3{,}26\,kg \cdot 1{,}21\,m/s^2 + 21{,}6\,N \approx \underline{25{,}5\ N}$

$x_2: \quad -F_{A2} - m \cdot g + F_{K2} = m \cdot a_2$

$F_{K2} = m \cdot a_2 + m \cdot g + F_{A2}$

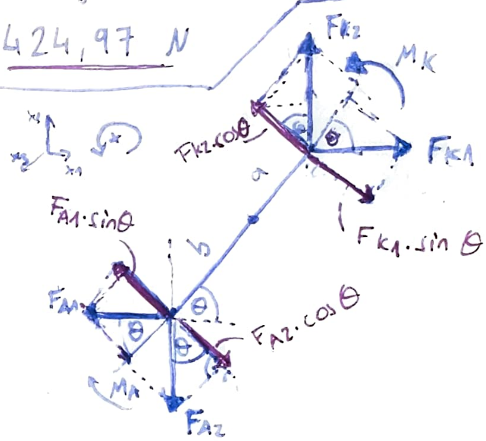
$= 3{,}26\,kg \cdot (-0{,}95\,m/s^2) + 3{,}26\,kg \cdot 9{,}81\,m/s^2 + (-453{,}5\,N)$

$= \underline{-424{,}97\ N}$

$a = 18{,}62\,cm,\ b = 24{,}38\,cm$

$I = m \cdot r_g^2 = 0{,}055\ kgm^2$

$\alpha = -0{,}22\ rad/s^2,\ \theta = 78{,}7^\circ$



$\boxed{\Sigma M = I \cdot \alpha}$

– calculate moments around COM –

$I\alpha_1 = -M_A - b \cdot F_{A1} \cdot \sin\theta + b \cdot F_{A2} \cdot \cos\theta$

$+ a \cdot F_{K2} \cdot \cos\theta - a \cdot F_{K1} \cdot \sin\theta + M_K$

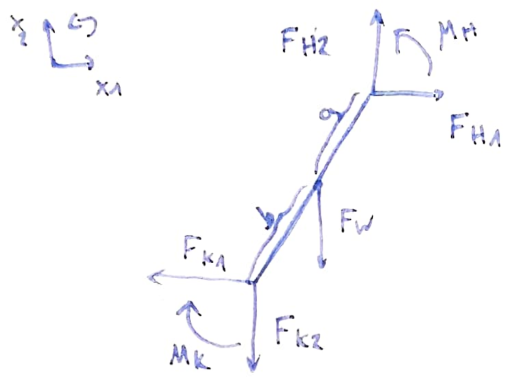
$M_K = I \cdot \alpha_1 + M_A + \sin\theta \cdot (b \cdot F_{A1} + a \cdot F_{K1}) - \cos\theta (b \cdot F_{A2} + a \cdot F_{K2})$

$= \underline{16{,}7\ N}$

(Note: result may vary dep. on how much you rounded in the steps before ...)

(1 cont.)

tigh segment $\quad m = 7\ kg \quad \ddot{a} = \begin{pmatrix} 1,15 \\ -1,46 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$



$\Sigma F = m \cdot a$

$x_1: \quad -F_{K1} + F_{H1} = m \cdot a_1 \Rightarrow F_{H1} \approx \underline{\underline{33,6\ N}}$

$x_2: \quad -F_{K2} - F_W + F_{H2} = m \cdot a_2$

$\Rightarrow F_{H2} = 7\,kg \cdot (-1,46\ m/s^2) + (-424,97\ N) + 7\,kg \cdot 9,81\,m/s$

$\approx \underline{\underline{-366,2\ N}}$

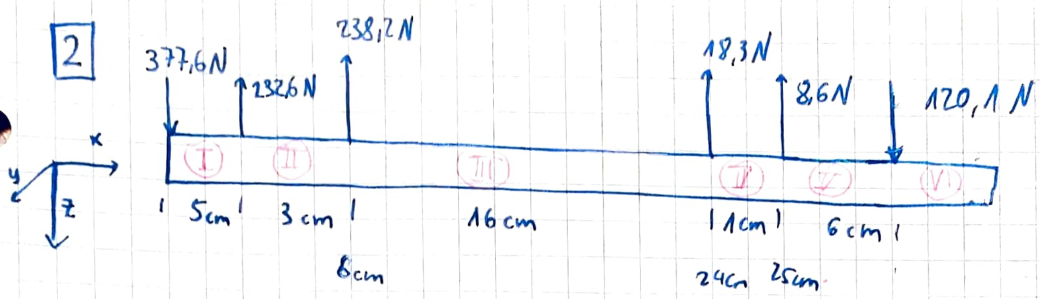
$I = 0,075\ kgm^2, \quad a = 13,8\,cm, \quad b = 18,14\,cm, \quad \Theta = 88,7°, \quad \alpha = 0,01\ rad/s^2$

$I\alpha = -M_K - b \cdot \sin\theta\, F_{K1} + b \cdot \cos\Theta \cdot F_{K2} - a \cdot \sin\theta \cdot F_{H1} + a \cdot \cos\Theta \cdot F_{H2} + M_H$

$\Rightarrow M_H = I\alpha + M_K + \sin\theta\,(b \cdot F_{K1} + a \cdot F_{H1}) - \cos\Theta\,(b \cdot F_{K2} + a \cdot F_{H2})$

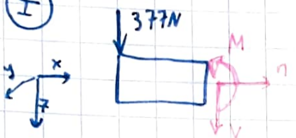
$= \underline{\underline{28,78\ N}}$

2



i) Bending moments

I

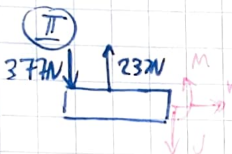


$M + 377{,}6\,N \cdot x = 0$

$M = -377{,}6\,N \cdot 5\,cm$

$M(5\,cm) = -1888\,Ncm$

II



$M + 377{,}6\,N \cdot x - 232{,}6\,N \cdot (x - 5\,cm) = 0$

$M = -377{,}6\,N \cdot x + 233{,}6\,N\,(x - 5\,cm)$

$M(8\,cm) = \underline{-2323\,Ncm}$ ⇒ this is the max. bending moment

N.B: be careful when you are not using SI - UNITS!

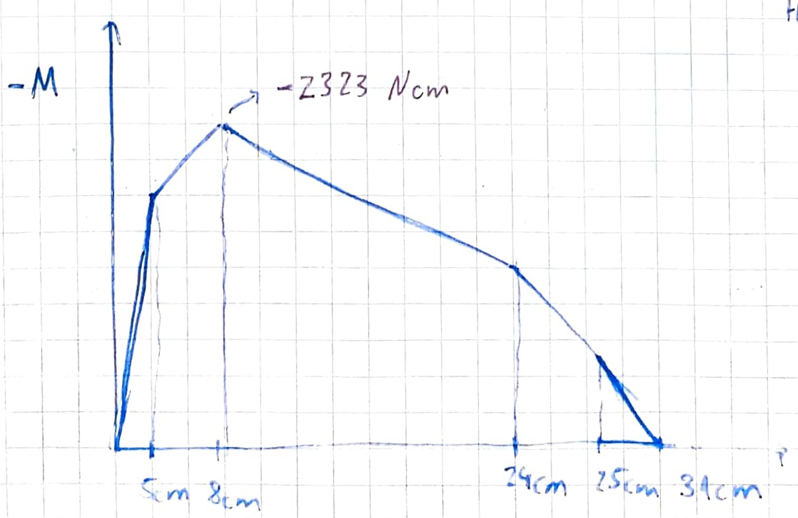
III $M = -377{,}6\,N \cdot x + 232{,}6\,N\,(x - 5\,cm) + 238{,}2\,N \cdot (x - 8\,cm)$

$M(24\,cm) = -831{,}8\,N\,Ncm$

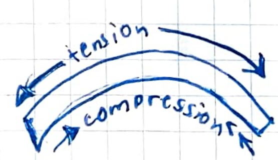
IV $M = -377{,}6\,N \cdot x + 232{,}6\,N\,(x - 5\,cm) + 238{,}2\,N\,(x - 8\,cm) + 18{,}3\,N \cdot (x - 24\,cm)$

$M(25\,cm) = -720{,}3\,Ncm$

V $M = -377{,}6\,N \cdot x + 232{,}6\,N\,(x - 5\,cm) + 238{,}2\,N\,(x - 8\,cm) + 18{,}3\,N\,(x - 24\,cm) + 8{,}6\,N\,(x - 25\,cm)$



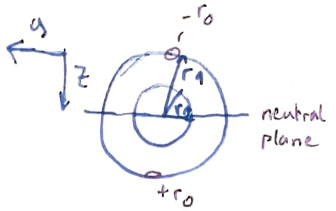
the forearm bends like this:



$\boxed{2}$ cont.

ii) $\boxed{\frac{M}{I} = \frac{\sigma}{\Delta z}}$

For one of the 2 bones : $M_u = \frac{1}{2} M_{MAX} = -1161{,}5\ Ncm$



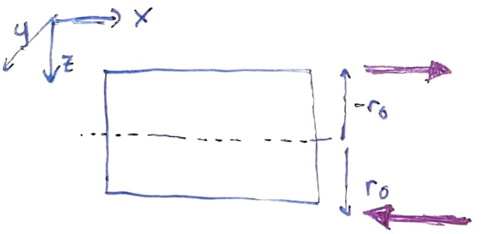
$I = \int_{\circledcirc} z^2 \cdot dydz = \frac{\pi}{4}\left(r_o^{\,4} - r_i^{\,4}\right) \approx 0{,}177\ cm^4$

see lecture

$\Rightarrow$ maximum stresses at $z = \pm r_o$

$$\sigma(z = r_o) = \frac{M_u}{I} r_o = \frac{-1161{,}5\ Ncm}{0{,}17...\ cm^4} \cdot 0{,}7\,cm = -4599\ Ncm^2$$

$$\sigma(z = -r_o) = \frac{M_u}{I} \cdot (-r_o) \approx 4599\ N\,cm^2$$



at $-r_o$: σ is positive $\Rightarrow$ tension

at r_o : σ is negative $\rightarrow$ compression

iii) add a compressive force of 50 N - to both bones together $\rightarrow$ 25 N to one bone

$$\sigma_c = \frac{F}{A} = \frac{-25\,N}{1{,}15\,cm^2} = -21{,}74\ N/cm^2$$

$$\rightarrow\ \tilde{\sigma}(z = r_o) = \sigma(z = r_o) + \sigma_c = -4599\ N/cm^2 - 21{,}76\ N/cm^2 = -4620{,}76\ N/cm^2$$

$$\tilde{\sigma}(z = -r_o) = \sigma(z = r_o) + \sigma_c = 4577{,}24\ N/cm^2$$

iv) $\sigma = \varepsilon \cdot E$ $\qquad E = 17 \cdot 10^9\ Pa = 17 \cdot 10^9\ N/m^2 = 17 \cdot 10^5\ N/cm^2$

w. σ from ii) : $\varepsilon = \dfrac{-4599\ N/cm^2}{17 \cdot 10^5\ N/cm^2} = -0{,}0027052.. = -0{,}2705\,\%$

w. $\tilde{\sigma}$ from iii) $\varepsilon = \dfrac{-4620\ N/cm^2}{17 \cdot 10^5\ N/cm^2} = -0{,}002718 = -0{,}2718\,\%$

$\boxed{3}$ i) $\underline{\underline{\varepsilon}}' = R^T \underline{\underline{\varepsilon}} R$

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \varepsilon_{11}c + \varepsilon_{12}s & -\varepsilon_{11}s + \varepsilon_{12}c \\ \varepsilon_{12}c + \varepsilon_{22}s & -\varepsilon_{12}s + \varepsilon_{22}c \end{pmatrix} = \underline{\underline{\varepsilon}} R$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \varepsilon_{11}c + \varepsilon_{12}s & -\varepsilon_{11}s + \varepsilon_{12}c \\ \varepsilon_{12}c + \varepsilon_{22}s & -\varepsilon_{12}s + \varepsilon_{22}c \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_{11}c^2 + \varepsilon_{12}sc + \varepsilon_{12}sc + \varepsilon_{22}s^2 & -\varepsilon_{11}sc + \varepsilon_{12}c^2 - \varepsilon_{12}s^2 + \varepsilon_{22}sc \\ -\varepsilon_{11}sc - \varepsilon_{12}s^2 + \varepsilon_{12}c^2 + \varepsilon_{22}sc & +\varepsilon_{11}s^2 - \varepsilon_{12}sc - \varepsilon_{12}sc + \varepsilon_{22}c^2 \end{pmatrix}$$

$$\varepsilon_{11}' = \varepsilon_{11}c^2 + \varepsilon_{22}s^2 + 2 \cdot \varepsilon_{12}sc$$

$$\varepsilon_{12}' = \varepsilon_{22}sc - \varepsilon_{11}sc + \varepsilon_{12}c^2 - \varepsilon_{12}s^2$$

$$\varepsilon_{22}' = \varepsilon_{11}s^2 + \varepsilon_{22}c^2 - 2 \cdot \varepsilon_{12}sc$$

using $\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta \Rightarrow \sin^2\theta = \cos^2\theta - \cos(2\theta)$

$\cos^2\theta = 1 - \sin^2\theta = 1 - \cos^2\theta + \cos(2\theta) \Rightarrow 2\cos^2\theta = 1 + \cos(2\theta)$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\varepsilon_{11}' = \left(\frac{1 + \cos(2\theta)}{2}\right)\varepsilon_{11} + \left(\frac{1 - \cos(2\theta)}{2}\right)\varepsilon_{22} + \varepsilon_{12}\sin(2\theta)$$

$$= \frac{\varepsilon_{11} + \varepsilon_{22}}{2} + \frac{\varepsilon_{11} - \varepsilon_{22}}{2} \cdot \cos(2\theta) + \varepsilon_{12} \cdot \sin(2\theta)$$

$$\varepsilon_{12}' = \frac{1}{2}(\varepsilon_{22} - \varepsilon_{11}) \cdot \sin(2\theta) + \varepsilon_{12} \cdot \left(\frac{1 + \cos(2\theta)}{2}\right) - \varepsilon_{12} \cdot \left(\frac{1 - \cos(2\theta)}{2}\right)$$

$$= \frac{\varepsilon_{22} - \varepsilon_{11}}{2}\sin(2\theta) + \varepsilon_{12} \cdot \cos(2\theta)$$

$$\varepsilon_{22}' = \varepsilon_{11}\left(\frac{1 - \cos(2\theta)}{2}\right) + \varepsilon_{22}\left(\frac{1 + \cos(2\theta)}{2}\right) - \varepsilon_{12} \cdot \sin(2\theta)$$

$$= \frac{\varepsilon_{11} + \varepsilon_{12}}{2} + \frac{\varepsilon_{22} - \varepsilon_{11}}{2} \cdot \cos(2\theta) - \varepsilon_{12} \cdot \sin(2\theta)$$

$\boxed{3}$ ii) option A, insert in

$$\varepsilon_{11}' = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} + \frac{\varepsilon_{11}-\varepsilon_{22}}{2}\cdot\cos(2\theta) + \varepsilon_{12}\sin(2\theta)$$

$$\varepsilon_{12}' = \frac{\varepsilon_{22}-\varepsilon_{11}}{2}\sin(2\theta) + \varepsilon_{12}\cos(2\theta)$$

$$\varepsilon_{22}' = \frac{\varepsilon_{22}+\varepsilon_{11}}{2} + \frac{\varepsilon_{22}-\varepsilon_{11}}{2}\cos(2\theta) - \varepsilon_{12}\sin(2\theta)$$

with $\boxed{\theta = 45° \rightarrow \cos(90°) = 0 \quad , \sin(90°) = 1}$

$$\varepsilon_{11}' = \varepsilon_{12}$$

$$\varepsilon_{12}' = 0$$

$$\varepsilon_{22}' = -\varepsilon_{12}.$$

alternatively: $\cos(45°) = \sin(45°) = \frac{1}{\sqrt{2}}$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_{12} \\ \varepsilon_{12} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{\varepsilon_{12}}{\sqrt{2}} & \frac{\varepsilon_{12}}{\sqrt{2}} \\ \frac{\varepsilon_{12}}{\sqrt{2}} & -\frac{\varepsilon_{12}}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\varepsilon_{12}}{2} + \frac{\varepsilon_{12}}{2} & \frac{\varepsilon_{12}}{2} - \frac{\varepsilon_{12}}{2} \\ -\frac{\varepsilon_{12}}{2} + \frac{\varepsilon_{12}}{2} & -\frac{\varepsilon_{12}}{2} - \frac{\varepsilon_{12}}{2} \end{pmatrix} = \begin{pmatrix} \varepsilon_{12} & 0 \\ 0 & -\varepsilon_{12} \end{pmatrix}$$

4 We only consider e_1, e_2 plane, because our stresses are only in this plane, and we want to find ε_{12}. $\Rightarrow \underline{\underline{\sigma}} = \begin{pmatrix} 0 & \sigma_{12} \\ \sigma_{12} & 0 \end{pmatrix}$

rotate by $45°$ $\Rightarrow \underline{\underline{\sigma}}' = \begin{pmatrix} \sigma_{12} & 0 \\ 0 & -\sigma_{12} \end{pmatrix}$

$$\underbrace{\text{direction 1} \quad \varepsilon_1 = \sigma_1 \cdot \frac{1}{E}}_{\text{Hooke's law}} \qquad \underbrace{\nu = \frac{-\varepsilon_t}{\varepsilon_a} = \frac{-\varepsilon_2}{\varepsilon_1}}_{\text{Poisson effect}} \Rightarrow \varepsilon_2 = -\varepsilon_1 \cdot \nu \Rightarrow \varepsilon_2 = -\frac{\nu}{E}\sigma_1$$

$$\text{direction 2} \quad \varepsilon_2 = \sigma_2 \cdot \frac{1}{E} \qquad \nu = -\frac{\varepsilon_t}{\varepsilon_a} = \frac{-\varepsilon_1}{\varepsilon_2} \Rightarrow \varepsilon_1 = -\nu \cdot \varepsilon_2$$

$$\Rightarrow \varepsilon_1 = -\frac{\nu}{E}\sigma_2$$

$\Rightarrow$ So in each direction, there are 2 displacements,

$$\varepsilon_1 = \sigma_1 \cdot \frac{1}{E} - \frac{\nu}{E} \cdot \sigma_2$$

$$= \sigma_{12} \cdot \frac{1}{E} + \frac{\nu}{E} \cdot \sigma_{12} = \frac{1}{E}(1+\nu)\,\sigma_{12}$$

$$\varepsilon_2 = \sigma_2 \cdot \frac{1}{E} - \frac{\nu}{E}\sigma_2 = -\frac{1}{E}(1+\nu)\,\sigma_{12}$$

$$\boldsymbol{\varepsilon}' = \frac{1}{E}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \qquad \text{with } a = \frac{1}{E}(1+\nu)\,\sigma_{12}$$

rotate back $-45°$

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\boldsymbol{\varepsilon} = R^T \boldsymbol{\varepsilon}' R$$

$$[\dots]\ \varepsilon = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} = \frac{1}{E}\begin{pmatrix} 0 & \sigma_{12} + \nu\sigma_{12} \\ \sigma_{12}+\nu\sigma_{12} & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\varepsilon_{12} = \frac{1}{E}(1+\nu)\cdot\sigma_{12}}$$

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix} = \frac{1}{E}\begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{pmatrix} \cdot \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$$

Note: we could do the same w. 3×3 Matrices and the generalized Hook's law in 3D.

$\boxed{5}$

$\varepsilon_{n_1} = 0{,}004$

$\varepsilon_{n_2} = 0{,}002$

$\varepsilon_{n_3} = -0{,}001$

project $\underline{\underline{\varepsilon}}$ in the direction of a vector $\underline{n}$: $\underline{\varepsilon_n} = \underline{n}^T \underline{\underline{\varepsilon}}\, \underline{n}$

$$n = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \underline{n_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \underline{n_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \underline{n_3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\varepsilon_{n_1} = (1 \quad 0) \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 \quad 0) \cdot \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \end{pmatrix} = \varepsilon_{11} \Rightarrow 0{,}004 = \varepsilon_{11}$$

$$\varepsilon_{n3} = (0 \ 1) \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{11} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0 \ 1) \begin{pmatrix} \varepsilon_{12} \\ \varepsilon_{22} \end{pmatrix} = \varepsilon_{22} \Rightarrow -0{,}001 = \varepsilon_{22}$$

$$\varepsilon_{n2} = \left(\tfrac{1}{\sqrt{2}} \quad \tfrac{1}{\sqrt{2}}\right) \cdot \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \left(\tfrac{1}{\sqrt{2}} \quad \tfrac{1}{\sqrt{2}}\right) \begin{pmatrix} \frac{1}{\sqrt{2}}(\varepsilon_{11} + \varepsilon_{12}) \\ \frac{1}{\sqrt{2}}(\varepsilon_{12} + \varepsilon_{22}) \end{pmatrix}$$

$$= \frac{1}{2} \cdot (\varepsilon_{11} + \varepsilon_{12}) + \frac{1}{2} \cdot (\varepsilon_{12} + \varepsilon_{22})$$

$$\Rightarrow \varepsilon_{12} = \varepsilon_{n2} - \frac{1}{2} \cdot \varepsilon_{11} - \frac{1}{2} \cdot \varepsilon_{22} = 0{,}002 - 0{,}002 + 0{,}0005$$

$$= 0{,}0005$$

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} 0{,}004 & 0{,}0005 \\ 0{,}0005 & -0{,}001 \end{pmatrix}$$