Makroökonomische Vertiefung

WS22

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1 Introduction

Macroeconomists...

- empirically **describe** the aggregate economy.
- theoretically **explain** the determination of production, prices, employment, exchange rates, etc.
- quantitatively **evaluate** economic policies.

1.1 Trend versus Cycle

Macroeconomic time series are often decomposed into two components:

- **trend component**: long-term growth
- **cyclical component**: fluctuations around trend (business cycles)

1.2 HP-Filter

$$y_t = g_t + c_t$$

$$\min_{(g_t)_{t=1}^T} \quad \underbrace{\sum_{t=1}^{T}(y_t - g_t)^2}_{\text{cyclical fluctuation}} + \lambda \cdot \underbrace{\sum_{t=2}^{T-1}\big((g_{t+1} - g_t) - (g_t - g_{t-1})\big)^2}_{\text{change in growth trend}}$$

The parameter λ smoothes the trend:

- $\lambda = 0 \Rightarrow g_t = y_t$ (no cyclical fluctuations)
- $\lambda \to \infty \Rightarrow g_{t+1} - g_t = g_t - g_{t-1}$ (linear trend)

2 One-Period Real Macroeconomic Model

2.1 Assumptions

representative rousehold

- Life-time utility function $u(C, l)$ describes the preferences over consumption C and leisure l:

$$\begin{aligned} u_C(\cdot) > 0 \quad & u_{CC}(\cdot) < 0 \\ u_l(\cdot) > 0 \quad & u_{ll}(\cdot) < 0 \end{aligned}$$

- Time constraint: $l + N^S = h$, where N^S is labour supply and h is the available time.
- Flow budget constraint: $C \leq wN^S + rK^S$, where N^S and K^S are labour and capital supplied by the household. The household takes the wage w and the interest rate r as given.

representative firm

- The final good Y at given *total factor productivity* A is produced with $Y = AF\big(K^d, N^d\big)$, where $F(\cdot)$ is a *Neoclassical production function* combining capital K^d and N^d.

2.2 Representative Household: utility maximation

$$\begin{aligned} \max_{C,l,K^S} \quad & u(C,l) \\ s.t. \quad & C \leq w(h-l) + rK^S \\ & C \geq 0,\ l \in [0,h],\ K^S \in [0, K_0] \end{aligned}$$

$$\Rightarrow \quad \mathcal{L}\Big(\lambda, C, l, K^S\Big) = u(C,l) + \lambda \cdot \Big(w(h-l) + rK^S - C\Big)$$

The relevant FOCs for an optimum are:

I. $\frac{\partial \mathcal{L}(\cdot)}{\partial C} \overset{!}{=} 0 \quad \Rightarrow \quad u_C(\cdot) = \lambda$

II. $\frac{\partial \mathcal{L}(\cdot)}{\partial l} \overset{!}{=} 0 \quad \Rightarrow \quad u_l(\cdot) = \lambda w$

III. $\frac{\partial \mathcal{L}(\cdot)}{\partial \lambda} \overset{!}{=} 0 \quad \Rightarrow \quad C = w(h-l) + rK^S$

Solutions can be characterised by:

$$\text{MRS}_{l,C} \equiv \frac{u_l(\cdot)}{u_C(\cdot)} = w$$

$$C = w(h-l) + rK^S$$

For given (w, r) and endowment K_0, **2** equations can be solved for **2** unknowns, (C, l).

2.3 Representative Firm: profit maximisation

$$\max_{N^d, K^d} \Pi^F\left(N^d, K^d\right) = AF\left(N^d, K^d\right) - wN^d - rK^d$$

The FOCs for an optimum are:

I. $\frac{\partial \Pi^F(\cdot)}{\partial K^d} \overset{!}{=} 0 \quad \Rightarrow \quad AF_K(\cdot) = r$

II. $\frac{\partial \Pi^F(\cdot)}{\partial N^d} \overset{!}{=} 0 \quad \Rightarrow \quad AF_N(\cdot) = w$

2.4 Competitive Equilibrium

- Every market participant is a price-taker.
- Households **maximise utility** and firms **maximise profit**. $\Rightarrow$ The decision of households and firms are consistent with each other, therefore all markets are clear:
  - Good market clearing: $C = Y$
  - Labour market clearing: $N^d = N^S$
  - Capital market clearing: $K^d = K^S$

The system can be reduced to:

$$\begin{aligned} \frac{u_l(C,l)}{u_C(C,l)} &= AF_N(K, h-l) \\ C &= AF_N(K, h-l)(h-l) + AF_K(K, h-l)K \\ C &= AF(K, h-l) \end{aligned}$$

3 T-Period Real Macroeconomic Model

3.1 Assumptions

representative household

- Life-time utility function $U\Big((C_t, l_t)_{t=0}^T\Big)$ describes the preferences over consumption C_t and leisure l_t and is time-separable:

$$U(\cdot) = \sum_{t=0}^{T} \beta^t u(C_t, l_t)$$

- $\beta \in (0,1)$ is the houshold's discount factor.
- Time constraint: $l_t + N_t = h, \quad \forall t$
- Household's initial capital endowment is $K_0 > 0$.
- Net investment in the capital stock via savings, $S_t = I_t$:
  $K_{t+1} - K_t = I_t - \delta K_t, \quad \forall t$, where $\delta \in [0,1]$ is the *rate of depreciation* and I_t is *gross investment*.
- Flow budget constraint: $C_t + I_t \leq w_t N_t + r_t K_t, \quad \forall t$

representative firm

- The final good Y at given *total factor productivity* A is produced with
  $Y = AF\big(K^d, N^d\big)$, where $F(\cdot)$ is a *Neoclassical production function* combining capital K^d and N^d.

3.2 Representative Household: utility maximisation

$$\begin{aligned}
\max_{(C_t, l_t, I_t)_{t=0}^T} \quad & \sum_{t=0}^{T} \beta^t u(C_t, l_t) \\
s.t. \quad & C_t + I_t \leq w_t(h - l_t) + r_t K_t \\
& K_{t+1} = (1-\delta)K_t + I_t \\
& C_t \geq 0,\ l_t \in [0,h],\ K_{t+1} \geq 0
\end{aligned}$$

$$\Rightarrow \mathcal{L} = \sum_{t+1}^{T} \beta^t u(C_t, l_t) + \lambda_t\big(w_t(h - l_t) + (1 - \delta + r_t)K_t - C_t - K_{t+1}\big)$$

The relevant FCOs are:

I. $\frac{\partial \mathcal{L}(\cdot)}{\partial C_t} \overset{!}{=} 0 \quad \Rightarrow \quad \beta^t u_C(C_t, l_t) = \lambda_t, \quad \forall t$

II. $\frac{\partial \mathcal{L}(\cdot)}{\partial l_t} \stackrel{!}{=} 0 \quad \Rightarrow \quad \beta^t u_l(C_t, l_t) = \lambda_t w, \quad \forall t$

III. $\frac{\partial \mathcal{L}(\cdot)}{\partial K_{t+1}} \stackrel{!}{=} 0 \quad \Rightarrow \quad (1 - \delta + r_{t+1})\lambda_{t+1} = \lambda_t, \quad \forall t \in [0, T-1]$

IV. $\frac{\partial \mathcal{L}(\cdot)}{\partial \lambda_t} \stackrel{!}{=} 0 \quad \Rightarrow \quad K_{t+1} = w_t(h - l_t) - C_t + (1 - \delta + r_t)K_t, \quad \forall t$

Solutions can be characterised by:

$$\begin{aligned}
\mathrm{MRS}_{l_t,C_t} &\equiv \frac{u_l(C_t, l_t)}{u_C(C_t, l_t)} = w_t && \forall t \\
\mathrm{MRS}_{C_t,C_{t+1}} &\equiv \frac{u_C(C_t, l_t)}{\beta u_C(C_{t+1}, l_{t+1})} = 1 - \delta + r_{t+1} && \forall t \in [0, T-1] \\
K_{t+1} &= w_t(h - l_t) + (1 - \delta + r_t)K_t - C_t && \forall t
\end{aligned}$$

3.3 Representative Firm: profit maximisation

$$\max_{N_t, K_t} \Pi^F(N_t, K_t) = AF(N_t, K_t) - w_t N_t - r_t K_t$$

The FOCs for an optimum are:

I. $\frac{\partial \Pi^F(\cdot)}{\partial K_t} \stackrel{!}{=} 0 \quad \Rightarrow \quad AF_K(\cdot) = r_t$

II. $\frac{\partial \Pi^F(\cdot)}{\partial N_t} \stackrel{!}{=} 0 \quad \Rightarrow \quad AF_N(\cdot) = w_t$

3.4 Competitive Equilibrium

- Every market participant is a price-taker.
- Households **maximise utility** and firms **maximise profit**. ⇒ The decision of households and firms are consistent with each other, therefore all markets are clear.

The system can be reduced to a system of $\mathbf{3T + 2}$ nonlinear equations and $\mathbf{3T + 2}$ unknowns (endogenous variables):

$$\begin{aligned}
\frac{u_l(C_t, l_t)}{u_C(C_t, l_t)} &= A_t F_N(K_t, h - l_t) \\
\frac{u_C(C_t, l_t)}{\beta u_C(C_{t+1}, l_{t+1})} &= 1 - \delta + AF_K(K_{t+1}, h - l_{t+1}) \\
K_{t+1} &= AF(K_t, h - l_t) - C_t + (1 - \delta)K_t
\end{aligned}$$

4 Ramsey-Cass-Koopmans Model

4.1 Assumptions

representative household

- The representative household grows at rate $n \geq 0$:
  $N_{t+1} = (1+n)N_t, \quad \forall t, \quad N_0 = 1$
- Life-time utility function $U\big((c_t)_{t=0}^{\infty}\big)$ describes preferences over consumption c_t:

$$U(\cdot) = N_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t u(c_t) \qquad c_t \equiv \frac{C_t}{N_t}$$

- $T \to \infty$ can be justified since the representative household is a family, where altruistic parents care about their offspring (**dynasty**).
- The *instantenious utility function* is isoelastic:

$$u(c_t) = \begin{cases} \frac{c_t^{1-\sigma}-1}{1-\sigma} & \sigma \neq 1 \\ \ln(c_t) & \text{otherwise} \end{cases}$$

- σ measures *relative risk aversion*: $\sigma(c) \equiv -\frac{u_{cc} \times c}{u_c(c)}$
- Household's initial capital endowment is $K_0 > 0$.
- Net investment in the capital stock via savings, $S_t = I_t$:
  $K_{t+1} - K_t = I_t - \delta K_t, \quad \forall t$
- Flow budget constraint: $C_t + I_t \leq w_t N_t + r_t K_t, \quad \forall t$

representative firm

- A_t is interpreted as an exogenous labour-augmenting technology, $g \geq 0$ being the *rate technological progress*: $A_{t+1} = (1+g)A_t, \quad \forall t, \quad A_0 = 1$

$$\begin{aligned} y_t &= F(k_t, A_t) & y_t &\equiv Y_t/N_t \\ \tilde{y}_t &= F\left(\tilde{k}_t, 1\right) = f\left(\tilde{k}_t\right) & \tilde{y}_t &\equiv Y_t/(A_t N_t) \end{aligned}$$

$$\Rightarrow F_K(K_t, A_t N_t) = F_k(k_t, A_t) = f_{\tilde{k}}\left(\tilde{k}_t\right)$$

4.2 Representative Houshold: utility maximisation

$$\max_{(c_t,i_t)_{t=0}^{\infty}} \quad N_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t u(c_t)$$
$$s.t. \quad c_t + i_t \leq w_t + r_t k_t$$
$$(1+n)k_{t+1} = (1-\delta)k_t + i_t$$

$$\Rightarrow \mathcal{L} = \sum_{t=0}^{\infty} \beta^t (1+n)^t u(c_t) + \lambda_t \big(w_t + (1-\delta+r_t)k_t - c_t - (1+n)k_{t+1} \big)$$

The relevant FCOs are:

I. $\frac{\partial \mathcal{L}(\cdot)}{\partial c_t} \overset{!}{=} 0 \quad \Rightarrow \quad \beta^t (1+n)^t u_C(C_t, l_t) = \lambda_t, \quad \forall t$

II. $\frac{\partial \mathcal{L}(\cdot)}{\partial k_{t+1}} \overset{!}{=} 0 \quad \Rightarrow \quad (1-\delta+r_{t+1})\lambda_{t+1} = (1+n)\lambda_t, \quad \forall t \in [0, T-1]$

III. $\frac{\partial \mathcal{L}(\cdot)}{\partial \lambda_t} \overset{!}{=} 0 \quad \Rightarrow \quad (1+n)k_{t+1} = w_t - c_t + (1-\delta+r_t)k_t, \quad \forall t$

Optimal behaviour requires the *transversality condition* to hold:

$$\lim_{T \to \infty} \beta^T (1+n)^{T+1} \lambda_T k_{T+1} = 0$$

Solutions can be characterised by:

$$\text{MRS}_{c_t, c_{t+1}} \equiv \frac{u_c(c_t)}{\beta u_c(c_{t+1})} = 1 - \delta + r_{t+1}$$
$$k_{t+1}(1+n) = w_t + (1-\delta+r_t)k_t - c_t$$

4.3 Representative Firm: profit maximisation

$$\max_{N_t, K_t} \Pi^F(N_t, K_t) = F(K_t, A_t N_t) - w_t N_t - r_t K_t$$

The FOCs for an optimum are:

I. $\frac{\partial \Pi^F(\cdot)}{\partial K_t} \overset{!}{=} 0 \quad \Rightarrow \quad F_K(\cdot) = r_t$

II. $\frac{\partial \Pi^F(\cdot)}{\partial N_t} \overset{!}{=} 0 \quad \Rightarrow \quad F_N(\cdot) = w_t$

4.4 Competitive Equilibrium

- Every market participant is a price-taker.
- Households **maximise utility** and firms **maximise profit**. $\Rightarrow$ The decision of households and firms are consistent with each other, therefore all markets are clear.

The system can be reduced to:

$$\frac{u_c(c_t)}{\beta u_c(c_{t+1})} = 1 - \delta + F_k(k_{t+1}, A_{t+1})$$
$$k_{t+1}(1+n) = F(k_t, A_t) + (1 - \delta + r_t)k_t - c_t$$

It is helpful, to define the composite parameter $(1+z) \equiv (1+g)(1+n)$. With the isolelastic utility function and by normalising by A_t:

$$\frac{\tilde{c}_{t+1}}{\tilde{c}_t} = \frac{\beta^{1/\sigma}\left((1-\delta) + f_{\tilde{k}}\left(\tilde{k}_{t+1}\right)\right)^{1/\sigma}}{1+g}$$
$$\tilde{k}_{t+1} - \tilde{k}_t = \frac{f\left(\tilde{k}_t\right) - \tilde{c}_t}{1+z} - \frac{(z+\delta)\tilde{k}_t}{1+z}$$

4.5 Steady State

Steady-state quilibrium with population growth and technological progress is an equilibrium path with $\tilde{k}_t = \tilde{k}^{ss}$, therefore $\Delta\tilde{k}_t = 0$, $\Delta\tilde{c}_t = 0$ and $\Delta\tilde{y}_t = 0$.

- A steady state equilibrium is a fixed point of a dynamic system.
- No growth in per effective labour variables implies sustained growth in per capita and aggregate variable if $g > 0$.

$\Delta\tilde{c}_t = 0$ determines $\tilde{k}^{ss}$:

$$f_{\tilde{k}}\left(\tilde{k}^{ss}\right) - \delta = \beta^{-1}(1+g)^{\sigma} - 1$$

$\Delta\tilde{k}_t = 0$ and $\tilde{k}^{ss}$ determine $\tilde{c}^{ss}$:

$$\tilde{c}_t = f\left(\tilde{k}^{ss}\right) - (z+\delta)\tilde{k}^{ss}$$

4.6 Log-linearised Equilibrium Conditions

4.7 Equilibrium Dynamics