Hybrid Systems
Modeling, Analysis and Control

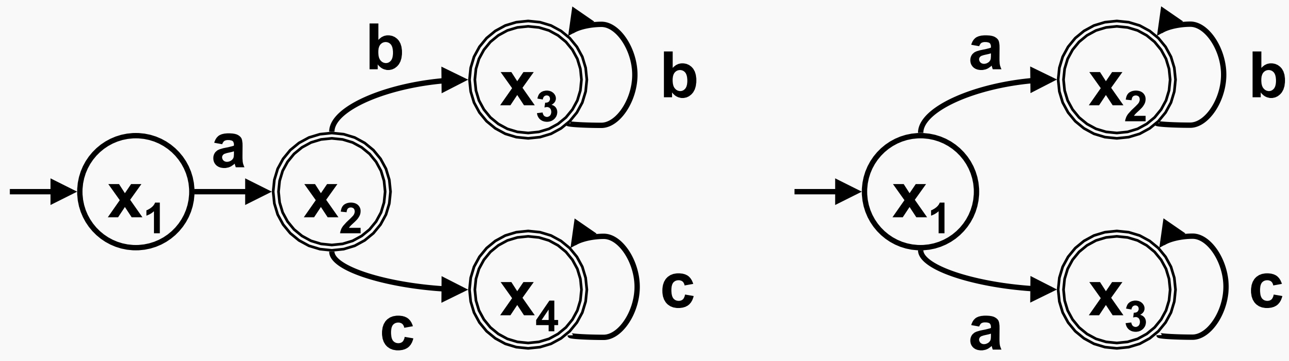
Radu Grosu
Vienna University of Technology

Lecture 5

Finite Automata as Linear Systems
Observability, Reachability and More

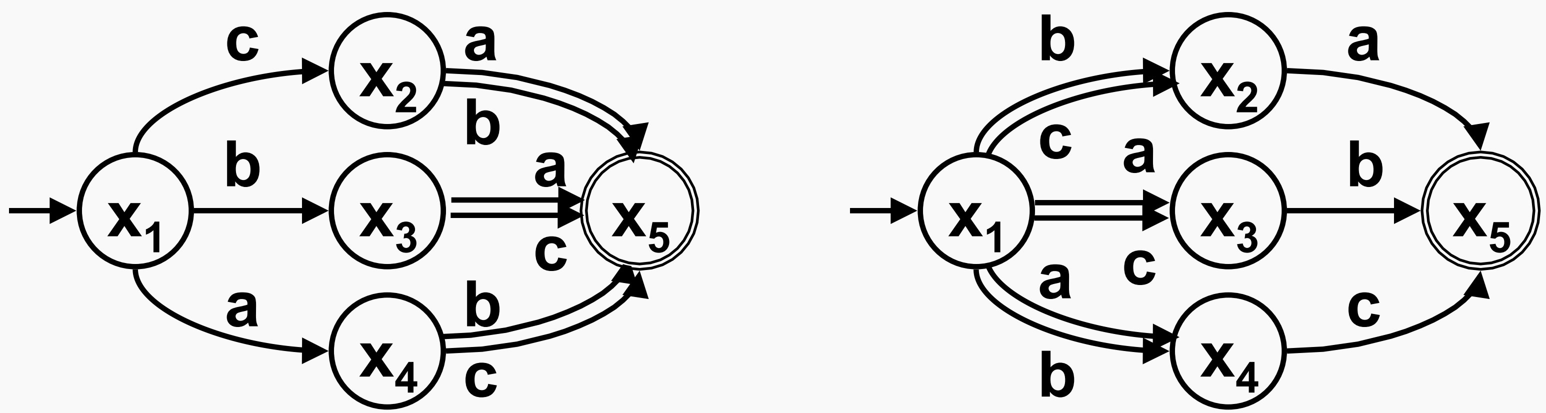
Minimal DFA are Not Minimal NFA

(Arnold, Dicky and Nivat's Example)



$L = a\,(b^* + c^*)$

Minimal NFA: How are they Related?
(Arnold, Dicky and Nivat's Example)

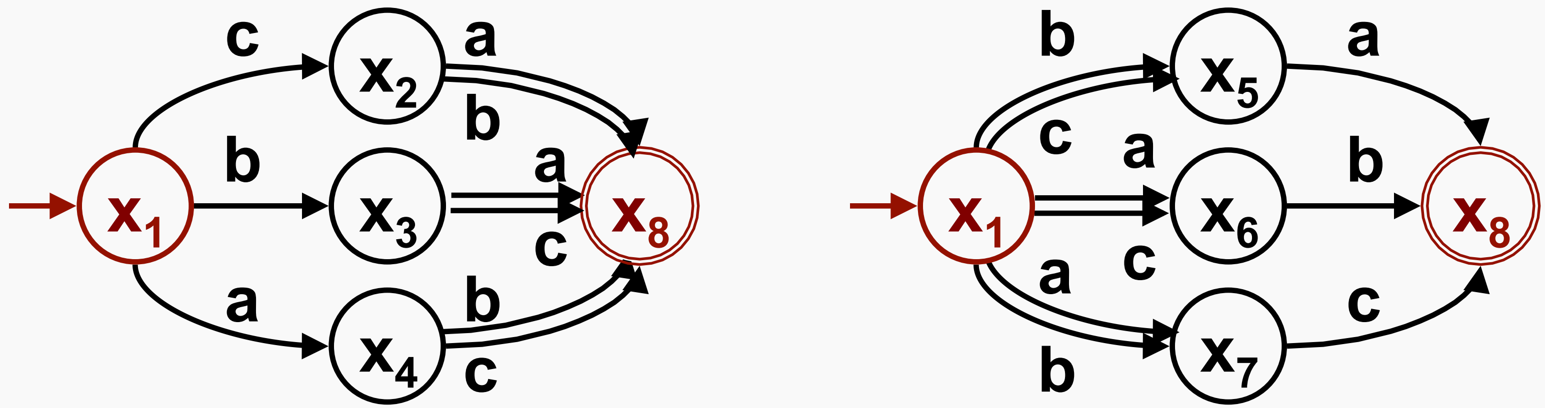


L = ab+ac + ba+bc + ca+cb

No homomorphism of either automaton onto the other.

Minimal NFA: How are they Related?

(Arnold, Dicky and Nivat's Example)



Carrez's solution: Take both in a **terminal NFA.**

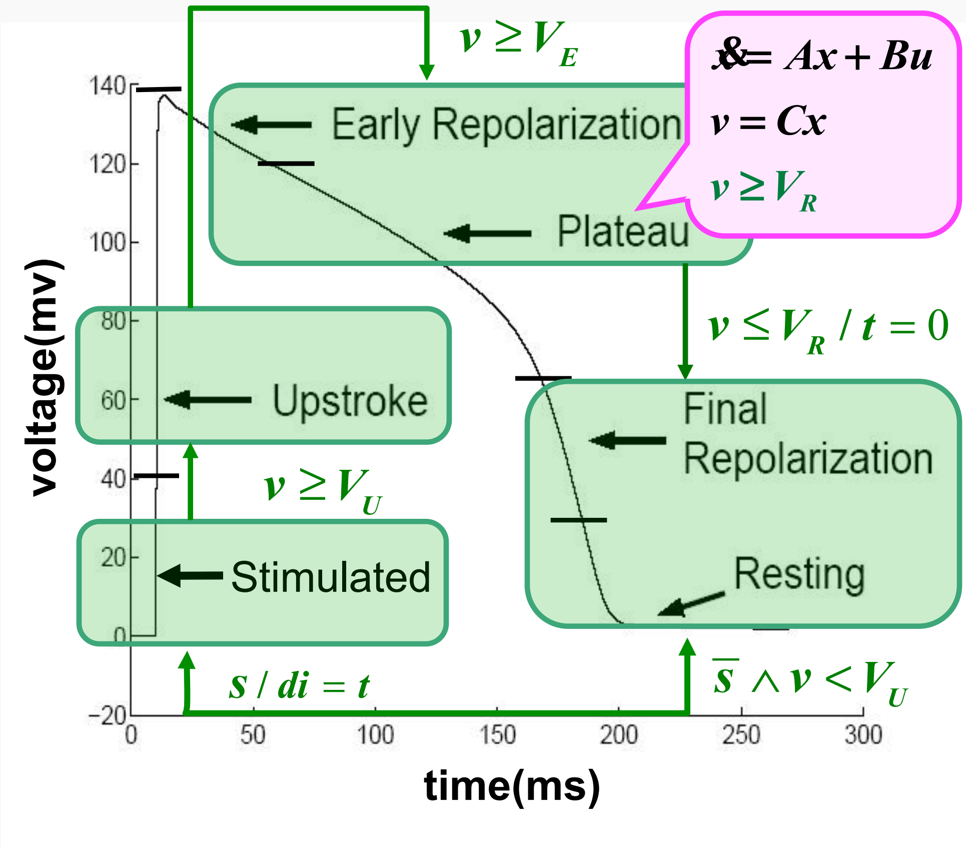
Is this the best one can do?

No! One can use use linear (similarity) transformations.

Convergence of Theories

- **Hybrid Systems Computation and Control:**
  - convergence between control and automata theory.
- **Hybrid Automata:** an outcome of this convergence
  - modeling formalism for systems exhibiting both discrete and continuous behavior,
  - successfully used to model and analyze embedded and biological systems.

Lack of Common Foundation for HA



- **Mode dynamics:**
  - Linear system (LS)
- **Mode switching:**
  - Finite automaton (FA)
- **Different techniques:**
  - LS reduction
  - FA minimization

- **LS & FA taught separately:** No common foundation!

Main Conjecture of this Talk

- Finite automata can be conveniently regarded as time invariant linear systems over semimodules:

  - linear systems techniques generalize to automata

- Examples of such techniques include:

  - linear transformations of automata,

  - minimization and determinization of automata as observability and reachability reductions

  - Z-transform of automata to compute associated regular expression through Gaussian elimination.

Finite Automata as Linear Systems

- Consider a finite automaton $M = (X,\Sigma,\delta,S,F)$ with:
  - finite set of states X, finite input alphabet Σ,
  - transition relation $\delta \subseteq X \times \Sigma \times X$,
  - starting and final sets of states $S,F \subseteq X$

Finite Automata as Linear Systems

- Consider a finite automaton $M = (X,\Sigma,\delta,S,F)$ with:
  - finite set of states X, finite input alphabet Σ,
  - transition relation $\delta \subseteq X \times \Sigma \times X$,
  - starting and final sets of states $S,F \subseteq X$
- For each input letter $a \in \Sigma$:
  - represent $\delta(a) \subseteq X \times X$ as a boolean matrix $A(a)$,
  - write $A = \sum_{a \in \Sigma} A(a)\, a$ where $a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}$

Finite Automata as Linear Systems

- Now define the linear system $L_M = [S,A,C]$:

$$x(n+1) = x(n)A, \quad x_0 = S(\varepsilon)\varepsilon$$

$$y(n) = x(n)C, \quad C = F(\varepsilon)\varepsilon$$

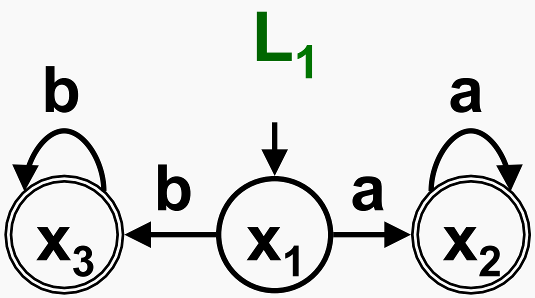
x and y are row vectors

Finite Automata as Linear Systems

- Now define the linear system $L_M = [S,A,C]$:

$$x(n+1) = x(n)A, \quad x_0 = S(\varepsilon)\varepsilon$$
$$y(n) = x(n)C, \quad C = F(\varepsilon)\varepsilon$$

- Example: consider following automaton:



$$A(a) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad x_0(\varepsilon)' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A(b) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C(\varepsilon) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Polynomials and their Operations

- A, C, x(n) and y(n) are polynomials with:
  - powers: strings in Σ^* (the input strings)
  - coefficients: matrices and vectors over B

Polynomials and their Operations

- A, C, x(n) and y(n) are polynomials with:
  - powers: strings in Σ^* (the input strings)
  - coefficients: matrices and vectors over B
- **Addition and multiplication:** done over polynomials

$$(A(a)a + A(b)b)^2 =$$

$$A(a)A(a)aa + A(a)A(b)ab + A(b)A(a)ba + A(b)A(b)bb \mathrel{\hat{=}}$$

$$A(aa)aa \quad + A(ab)ab \quad + A(ba)ba \quad + A(bb)bb$$

Boolean Semimodules

- B is a doubly idempotent, commutative semiring:
  - $(B,+,0)$ is a commutative idempotent monoid (or),
  - $(B,\times,1)$ is a commutative idempotent monoid (and),
  - multiplication distributes over addition,
  - 0 is an annihilator: $0 \times a = 0$

Boolean Semimodules

- B is a doubly idempotent, commutative semiring:
  - (B,+,0) is a commutative idempotent monoid (or),
  - (B,$\times$,1) is a commutative idempotent monoid (and),
  - multiplication distributes over addition,
  - 0 is an annihilator: $0 \times a = 0$
- B^n is a semimodule over scalars in B:
  - $r(x+y) = rx + ry$, $(r+s)x = rx + sx$, $(rs)x = r(sx)$,
  - $1x = x$, $0x = 0$

Boolean Semimodules

- B is a doubly idempotent, commutative semiring:
  - (B,+,0) is a commutative idempotent monoid (or),
  - (B,×,1) is a commutative idempotent monoid (and),
  - multiplication distributes over addition,
  - 0 is an annihilator: $0 \times a = 0$
- B^n is a semimodule over scalars in B:
  - $r(x+y) = rx + ry$, $(r+s)x = rx + sx$, $(rs)x = r(sx)$,
  - $1x = x$, $0x = 0$
- **Note:** No additive and multiplicative inverses!

Divergence of Classic/Discrete Math

- **Canonical partial order in semirings:**

$$a \leq_{+} b \quad \text{iff} \quad \exists! c.\ a + c = b$$

$$a \leq_{\times} b \quad \text{iff} \quad \exists! c.\ a \times c = b$$

Divergence of Classic/Discrete Math

- Canonical partial order in semirings:

$$a \leq_{+} b \quad \text{iff} \quad \exists! c.\ a + c = b$$

$$a \leq_{\times} b \quad \text{iff} \quad \exists! c.\ a \times c = b$$

- Example of canonical PO for Natural numbers:

$$1 \leq_{+} 5 \quad \text{iff} \quad \exists! 4.\ 1 + 4 = 5$$

Divergence of Classic/Discrete Math

- Canonical partial order in semirings:

  $a \leq_{+} b$ iff $\exists! c.\ a + c = b$

  $a \leq_{\times} b$ iff $\exists! c.\ a \times c = b$

- Example: Canonical PO for Natural numbers:

  $1 \leq_{+} 5$ iff $\exists! 4.\ 1 + 4 = 5$

- Example: Canonical PO for Integer numbers:

  $5 \leq_{+} -1$ iff $\exists! (-6).\ 5 + (-6) = 1$

Semiring: Either inverses or partial order!

Observability

- Let L = [S,A,C] be an n-state automaton. It's output:

$$[y(0)\ y(1)\ \dots\ y(n-1)] = x_0[C\ AC\ \dots\ A^{n-1}C] = x_0 O \qquad (1)$$

L is observable if x_0 is uniquely determined by (1).

Observability

- Let L = [S,A,C] be an n-state automaton. It's output:

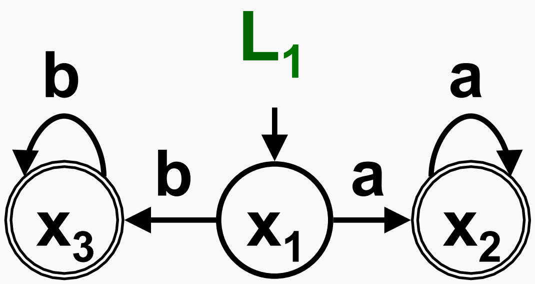
$$[y(0)\ y(1)\ \dots\ y(n\text{-}1)] = x_0^t[C\ AC\ \dots\ A^{n-1}C] = x_0^tO \qquad (1)$$

L is observable if x_0 is uniquely determined by (1).

- Example: the observability matrix O of L_1 is:

O =

A^nC	ε	a	b	aa	ab	ba	bb
x_1	0	1	1	1	0	0	1
x_2	1	1	0	1	0	0	0
x_3	1	0	1	0	0	0	1



Linear Dependence

- **Initial vector x_0 selects a sum of rows from O. Hence:**
  - **if L is deterministic and therefore has a single initial state, x_0 is uniquely determined if all rows O_i in O are distinct**

Linear Dependence

- Initial vector x_0 selects a sum of rows from O. Hence:
  - if L is deterministic and therefore has a single initial state, x_0 is uniquely determined if all rows O_i in O are distinct
  - if L is nondeterministic and has several initial states, x_0 is not uniquely determined if there are boolean a_i, b_i and:

$$\exists I,J \subset [1..n].\ \ I \cap J = \varnothing \ \wedge \ \sum_{i \in I} a_i O_i = \sum_{i \in J} b_i O_i \qquad (2)$$

Linear Dependence

- **Initial vector x_0 selects a sum of rows from O. Hence:**
  - if L is deterministic and therefore has a single initial state, x_0 is uniquely determined if all rows O_i in O are distinct
  - if L is nondeterministic and has several initial states, x_0 is not uniquely determined if there are boolean a_i ,b_i and:

$$\exists I,J \subset [1..n].\quad I \cap J = \varnothing \;\wedge\; \sum_{i\in I} a_i O_i = \sum_{i\in J} b_i O_i \qquad (2)$$

- **Linear dependence:** **(2) for finite I,J and any vector set.**
  - Def (2) generalizes linear dependence in vector spaces

Linear Dependence

- Initial vector x_0 selects a sum of rows from O. Hence:
  - if L is deterministic and therefore has a single initial state, x_0 is uniquely determined if all rows O_i in O are distinct
  - if L is nondeterministic and has several initial states, x_0 is not uniquely determined if there are boolean a_i, b_i and:

$$\exists I,J \subset [1..n].\ \ I \cap J = \varnothing \ \wedge\ \sum_{i \in I} a_i O_i = \sum_{i \in J} b_i O_i \qquad (2)$$

- Linear dependence: **(2) for finite I,J and any vector set.**
  - Def (2) generalizes linear dependence in vector spaces
  - Linear independence **is consequently:**

$$\forall I,J \subset [1..n].\ \ I \cap J = \varnothing \ \Rightarrow\ span(O_I) \cap span(O_J) = \{0\}$$

Basis in Boolean Semimodule

- An ordered set of vectors Y is a basis for X if:

  (a) Y is independent, (b) span(Y) = X

Basis in Boolean Semimodule

- An ordered set of vectors Y is a basis for X if:

  (a) Y is independent, (b) span(Y) = X

- **Theorem (Basis)** If $X \subseteq B^n$ has a basis Y then Y is unique.

Basis in Boolean Semimodule

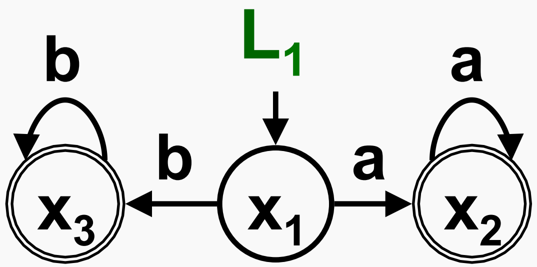
- An ordered set of vectors Y is a basis for X if:

  (a) Y is independent, (b) span(Y) = X

- Theorem (Basis) If $X \subseteq B^n$ has a basis Y then Y is unique.

O =

A^nC	ε	a	b	aa	ab	ba	bb
x_1	0	1	1	1	0	0	1
x_2	1	1	0	1	0	0	0
x_3	1	0	1	0	0	0	1



$[x_1\ x_2\ x_3]$: row basis

Basis in Boolean Semimodule

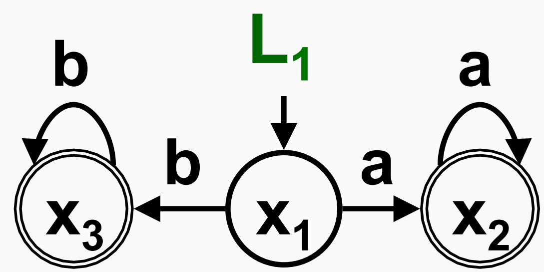
- An ordered set of vectors Y is a basis for X if:

  (a) Y is independent, (b) span(Y) = X

- Theorem (Basis) If $X \subseteq B^n$ has a basis Y then Y is unique.

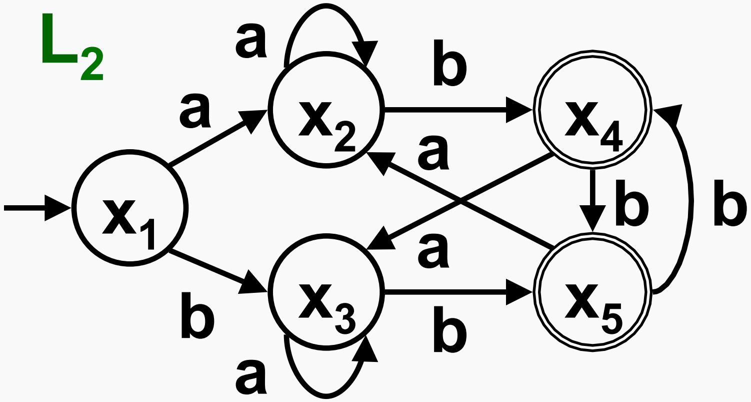
O =

A^nC	ε	a	b	aa	ab	ba	bb
x_1	0	1	1	1	0	0	1
x_2	1	1	0	1	0	0	0
x_3	1	0	1	0	0	0	1



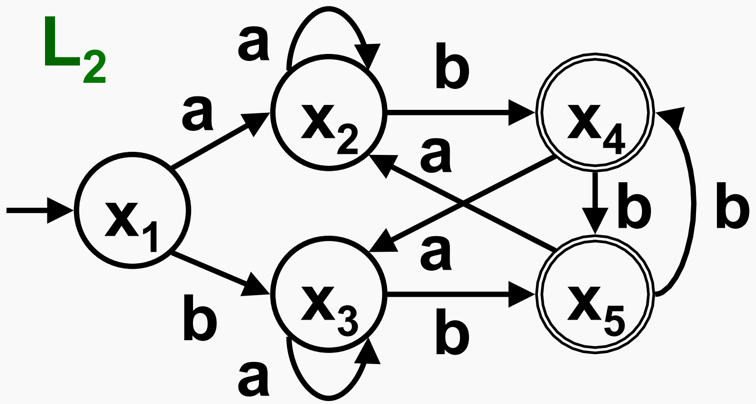
$[x_1\ x_2\ x_3]$: row basis, $[C(\varepsilon)\ AC(a)\ AC(b)]$: column basis.

Observability Reduction by Rows



$$A(a) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad A(b) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad x_0(\varepsilon) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C(\varepsilon) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Observability Reduction by Rows

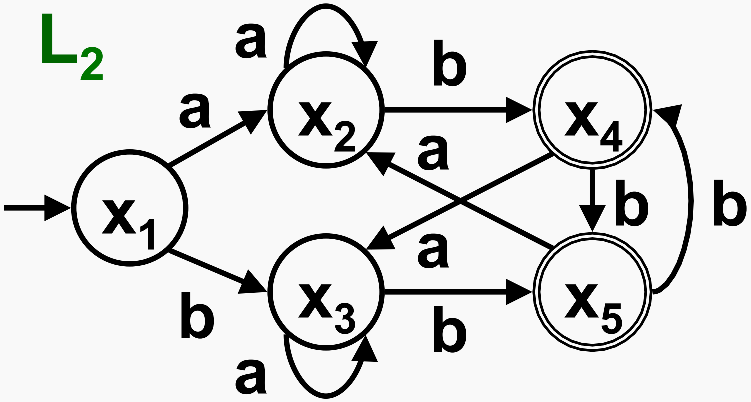


$$A(a) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad A(b) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad x_0(\varepsilon) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C(\varepsilon) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

O	ε	b	ab	bb	aab	abb	bab	bbb
x_1	0	0	1	1	1	1	1	1
x_2	0	1	1	1	1	1	1	1
x_3	0	1	1	1	1	1	1	1
x_4	1	1	1	1	1	1	1	1
x_5	1	1	1	1	1	1	1	1

Observability Reduction by Rows

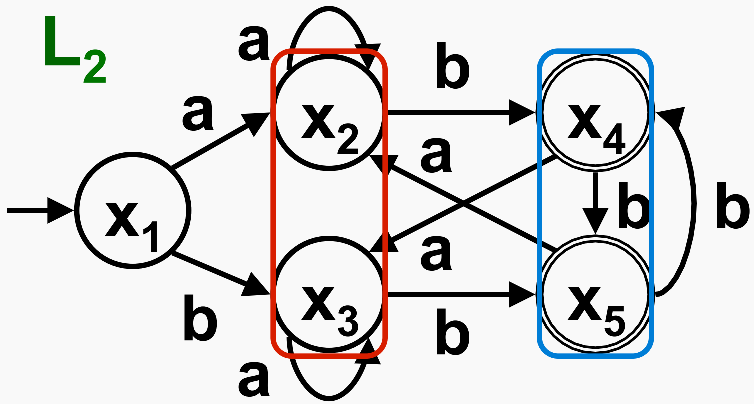
L_2



$$A(a) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad A(b) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad x_0(\varepsilon) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C(\varepsilon) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

O	ε	b	ab	bb	aab	abb	bab	bbb
x_1	0	0	1	1	1	1	1	1
x_2	0	1	1	1	1	1	1	1
x_3	0	1	1	1	1	1	1	1
x_4	1	1	1	1	1	1	1	1
x_5	1	1	1	1	1	1	1	1

Observability Reduction by Rows



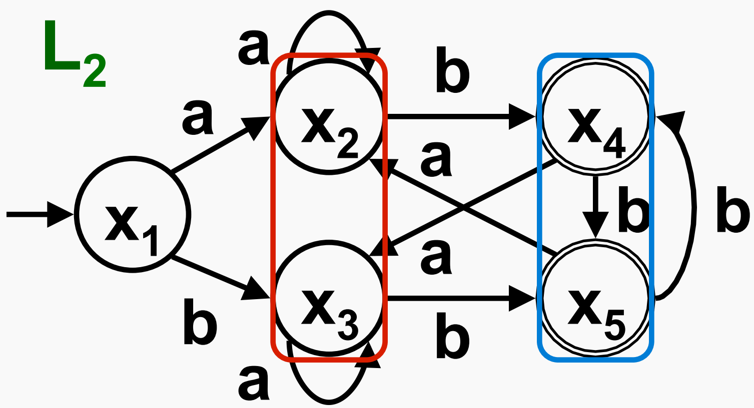
$$A(a) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad A(b) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad x_0(\varepsilon) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C(\varepsilon) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

O	ε	b	ab	bb	aab	abb	bab	bbb
x_1	0	0	1	1	1	1	1	1
x_2	0	1	1	1	1	1	1	1
x_3	0	1	1	1	1	1	1	1
x_4	1	1	1	1	1	1	1	1
x_5	1	1	1	1	1	1	1	1

Define linear transf $\bar{x} = xT$:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Observability Reduction by Rows



$$A(a)=\begin{bmatrix}0&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&1&0&0\\0&1&0&0&0\end{bmatrix}\quad A(b)=\begin{bmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&1&0\end{bmatrix}\quad x_0(\varepsilon)=\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\quad C(\varepsilon)=\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}$$

O	ε	b	ab	bb	aab	abb	bab	bbb
x_1	0	0	1	1	1	1	1	1
x_2	0	1	1	1	1	1	1	1
x_3	0	1	1	1	1	1	1	1
x_4	1	1	1	1	1	1	1	1
x_5	1	1	1	1	1	1	1	1

Define linear transf $\bar{x} = x\,T$:

$$T=\begin{bmatrix}1&0&0\\0&1&0\\0&1&0\\0&0&1\\0&0&1\end{bmatrix}$$

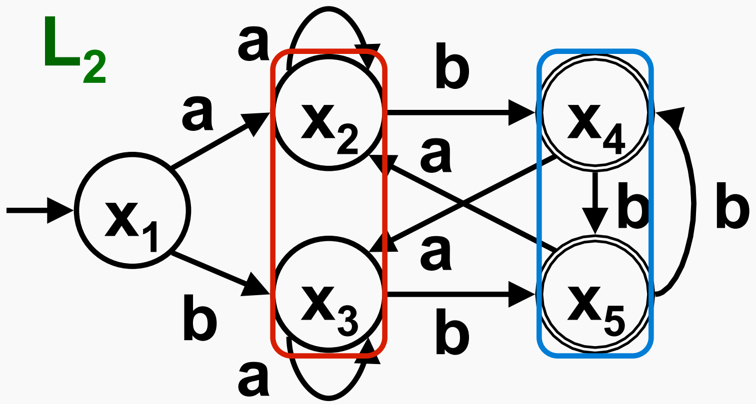
$$\bar{x}(n+1) = x(n+1)T = x(n)AT$$

$$= \bar{x}(n)T^{-1}AT = \bar{x}(n)\bar{A}$$

$$\bar{x}_0(\varepsilon) = x_0(\varepsilon)T$$

$$\bar{C}(\varepsilon) = T^{-1}C(\varepsilon)$$

Observability Reduction by Rows



$$A(a) = \begin{bmatrix} 0&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&1&0&0\\0&1&0&0&0 \end{bmatrix} \quad A(b) = \begin{bmatrix} 0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&1&0 \end{bmatrix} \quad x_0(\varepsilon) = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} \quad C(\varepsilon) = \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}$$

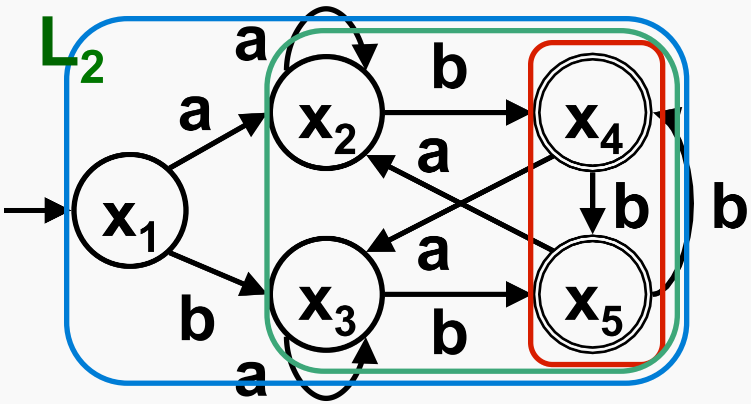
O	ε	b	ab	bb	aab	abb	bab	bbb
x_1	0	0	1	1	1	1	1	1
x_2	0	1	1	1	1	1	1	1
x_3	0	1	1	1	1	1	1	1
x_4	1	1	1	1	1	1	1	1
x_5	1	1	1	1	1	1	1	1

Define linear transf $\bar{x} = x\,T$:

$$T = \begin{bmatrix} 1&0&0\\0&1&0\\0&1&0\\0&0&1\\0&0&1 \end{bmatrix} \quad \bar{A}(a) = \begin{bmatrix} 0&1&0\\0&1&0\\0&1&0 \end{bmatrix} \quad \bar{A}(b) = \begin{bmatrix} 0&1&0\\0&0&1\\0&0&1 \end{bmatrix} \quad \bar{x}_0(\varepsilon) = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \bar{C}(\varepsilon) = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$\bar{A}(x) = [A(x)T]_T$ $\quad \bar{x}_0(\varepsilon) = x_0(\varepsilon)T$ $\quad \bar{C}(\varepsilon) = [C(\varepsilon)]_T$

Observability Reduction by Columns



$$A(a) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad A(b) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad x_0(\varepsilon) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C(\varepsilon) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

O	ε	b	ab	bb	aab	abb	bab	bbb
x_1	0	0	1	1	1	1	1	1
x_2	0	1	1	1	1	1	1	1
x_3	0	1	1	1	1	1	1	1
x_4	1	1	1	1	1	1	1	1
x_5	1	1	1	1	1	1	1	1

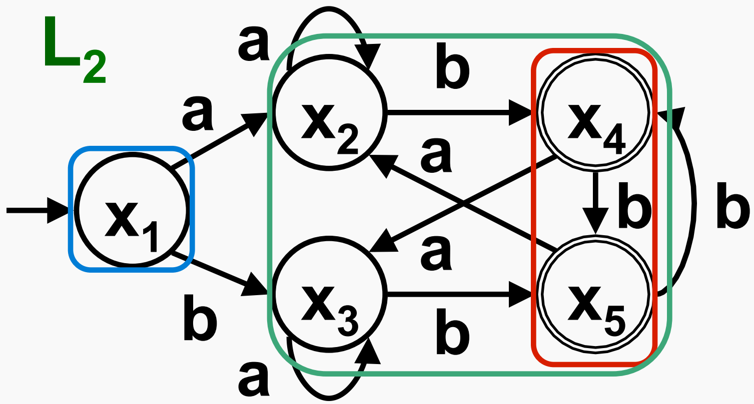
Define linear transf $\bar{x} = x\,T$:

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \bar{A}(a) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \bar{A}(b) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \bar{x}_0(\varepsilon) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \bar{C}(\varepsilon) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\bar{A}(x) = [A(x)T]_T$ $\quad \bar{x}_0(\varepsilon) = x_0(\varepsilon)T$ $\quad \bar{C}(\varepsilon) = [C(\varepsilon)]_T$

Mixed Observability Reduction

L_2



$$A(a) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad A(b) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad x_0(\varepsilon) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C(\varepsilon) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

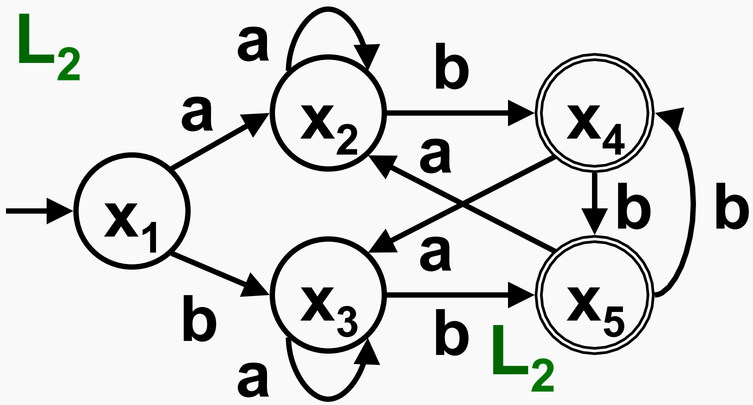
O	ε	b	ab	bb	aab	abb	bab	bbb
x_1	0	0	1	1	1	1	1	1
x_2	0	1	1	1	1	1	1	1
x_3	0	1	1	1	1	1	1	1
x_4	1	1	1	1	1	1	1	1
x_5	1	1	1	1	1	1	1	1

Define linear transf $\bar{x} = x\,T$:

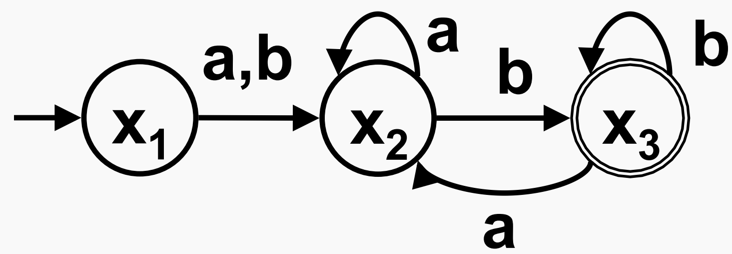
$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad \bar{A}(a) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \bar{A}(b) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \bar{x}_0(\varepsilon) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \bar{C}(\varepsilon) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\bar{A}(x) = [A(x)T]_T \quad \bar{x}_0(\varepsilon) = x_0(\varepsilon)T \quad \bar{C}(\varepsilon) = [C(\varepsilon)]_T$

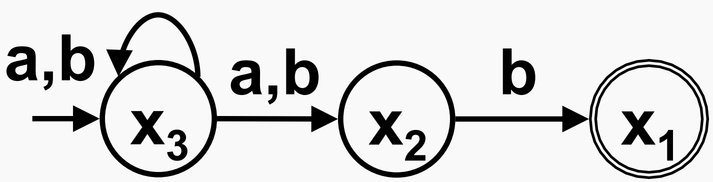
Original and Reduced Automata



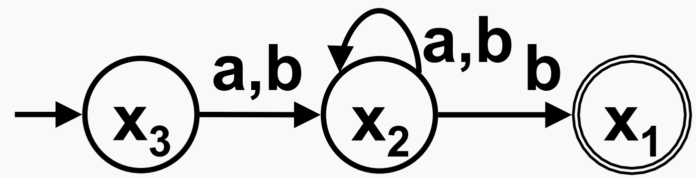
DFA L_{21} by rows



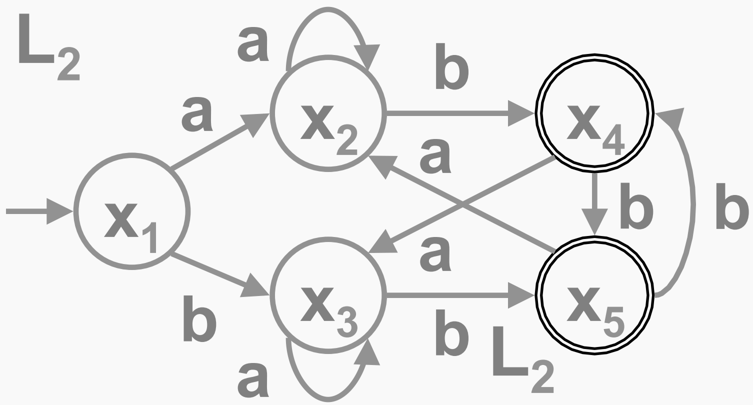
NFA L_{22} by columns



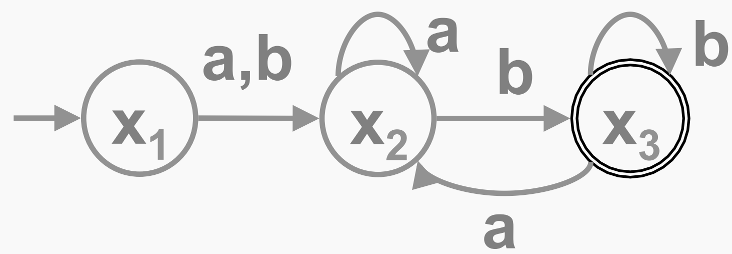
NFA L_{23} mixed



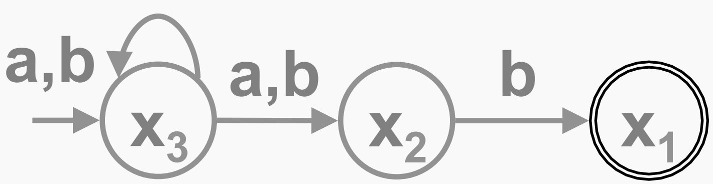
Original and Reduced Automata



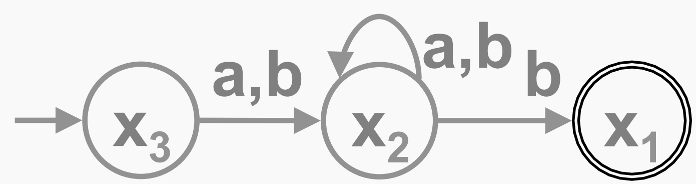
DFA L_{21} by rows



NFA L_{22} by columns



NFA L_{23} mixed



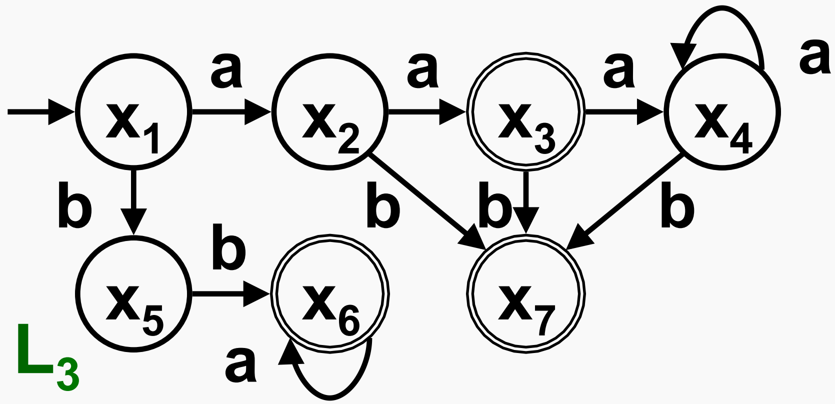
Let $\bar{x} = x\,T$ in L_{21} where

Then $L_{22} = [\bar{A}_{21}, \bar{x}^{t}_{21,0}, \bar{C}_{21}]$

$L_{23} = [\bar{A}'_{21}, \bar{x}'^{t}_{21,0}, \bar{C}'_{21}]$

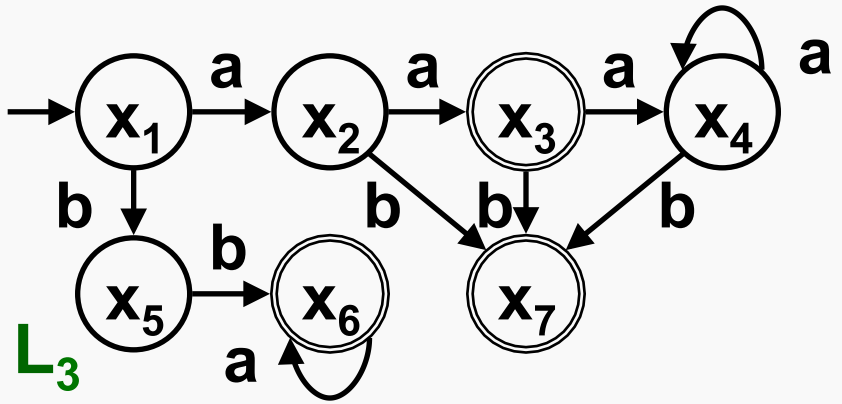
$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad T' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Row Basis but No Column Basis



O	ε	a	b	ab	ba	bb	aaa
x_1	0	0	0	1	0	1	0
x_2	0	1	1	1	0	0	0
x_3	1	0	1	1	0	0	0
x_4	0	0	1	1	0	0	0
x_5	0	0	1	0	0	0	0
x_6	1	1	0	0	1	0	1
x_7	1	0	0	0	0	0	0

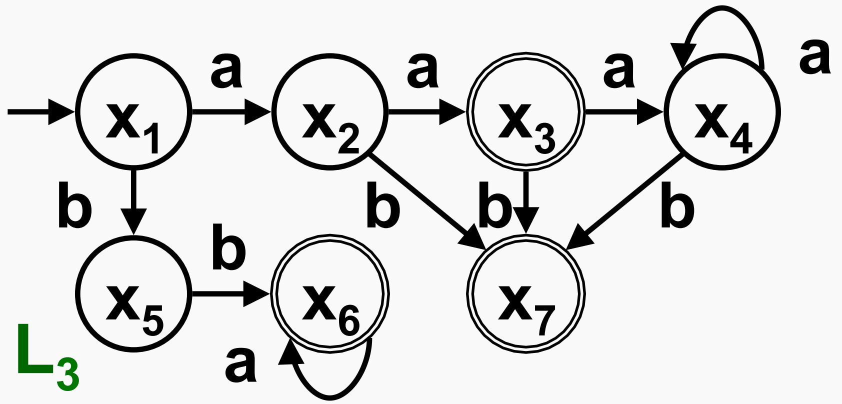
Row Basis but No Column Basis



L_3

O	ε	a	b	ab	ba	bb	aaa
x_1	0	0	0	1	0	1	0
x_2	0	1	1	1	0	0	0
x_3	1	0	1	1	0	0	0
x_4	0	0	1	1	0	0	0
x_5	0	0	1	0	1	0	0
x_6	1	1	0	0	0	0	1
x_7	1	0	0	0	0	0	0

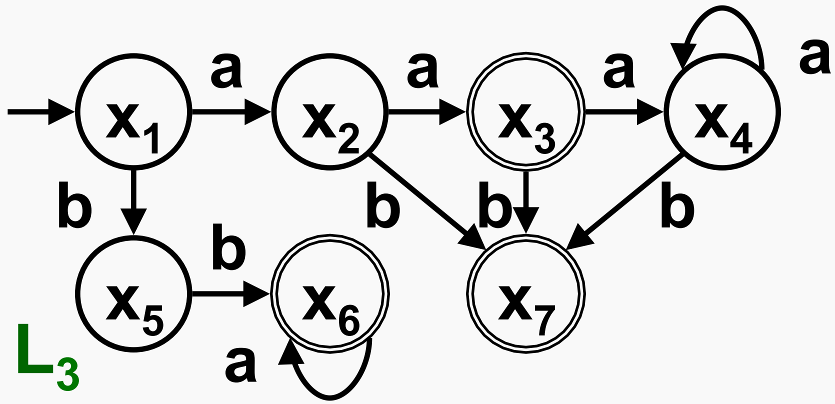
Row Basis but No Column Basis



O	ε	a	b	ab	ba	bb	aaa
x_1	0	0	0	1	0	1	0
x_2	0	1	1	1	0	0	0
x_3	1	0	1	1	0	0	0
x_4	0	0	1	1	0	0	0
x_5	0	0	1	0	1	0	0
x_6	1	1	0	0	0	0	1
x_7	1	0	0	0	0	0	0

O	ε	a	b	ab	ba	bb	aaa
x_1	0	0	0	1	0	1	0
x_2	0	1	1	1	0	0	0
x_3	1	0	1	1	0	0	0
x_4	0	0	1	1	0	0	0
x_5	0	0	1	0	1	0	0
x_6	1	1	0	0	0	0	1
x_7	1	0	0	0	0	0	0

Row Basis but No Column Basis



O	ε	a	b	ab	ba	bb	aaa
x_1	0	0	0	1	0	1	0
x_2	0	1	1	1	0	0	0
x_3	1	0	1	1	0	0	0
x_4	0	0	1	1	0	0	0
x_5	0	0	1	0	1	0	0
x_6	1	1	0	0	0	0	1
x_7	1	0	0	0	0	0	0

O	ε	a	b	ab	ba	bb	aaa
x_1	0	0	0	1	0	1	0
x_2	0	1	1	1	0	0	0
x_3	1	0	1	1	0	0	0
x_4	0	0	1	1	0	0	0
x_5	0	0	1	0	1	0	0
x_6	1	1	0	0	0	0	1
x_7	1	0	0	0	0	0	0

Observabilty Reduction

- **Theorem (Cover):** Finding a (possibly mixed) basis T for O_L is equivalent to finding a minimal cover for O_L.
  - either as its set basis cover or as its Karnaugh cover.
- **Theorem (Complexity):** Determining a cover T for O_L is NP-complete (set basis problem complexity).
- **Theorem (Rank):** The row (= column) rank of O_L is the size of the set cover T (size of Karnaugh cover).

Reachability: Dual of Observability

- Let L = [S,A,C] be an n-state automaton. It's output:

$$[y(0)\ y(1)\ \ldots\ y(n-1)]^t = C^t[x_0\ \ A^t x_0\ \ \ldots\ (A^t)^{n-1} x_0] = C^t R^t \quad (3)$$

where x_0 is now a column vector.

L is reachable if C is uniquely determined by (3).

Reachability: Dual of Observability

- Let L = [S,A,C] be an n-state automaton. It's output:

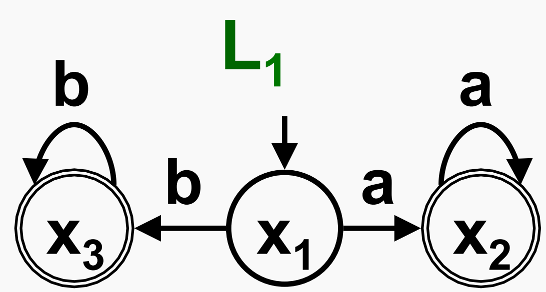
$$[y(0)\ y(1)\ \ldots\ y(n\text{-}1)]^t = C^t[x_0\ \ A^t x_0\ \ \ldots\ (A^t)^{n-1}x_0] = C^t R^t \qquad (3)$$

L is reachable if C is uniquely determined by (3).

- **Example:** the **reachability matrix** of L_1 is:

R^t =

$(A^t)^n x_0$	ε	a	b	a a	a b	b a	b b
x_1	1	0	0	0	0	0	0
x_2	0	1	0	1	0	0	0
x_3	0	0	1	0	0	0	1

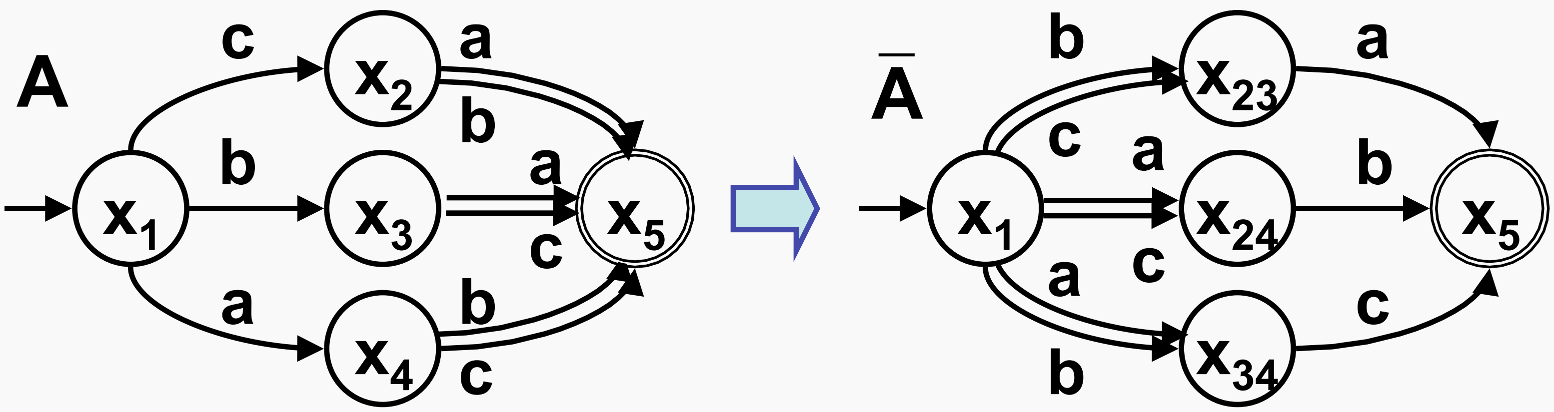


- **Row basis** $[x_1\ \ x_2\ \ x_3] = [x_0(\varepsilon)\ A^t x_0(a)\ A^t x_0(b)]$ **col basis.**

Observabilty, Reachability and More

- **DFA Minimization:** Is a particular case of observability reduction (single initial state requires distinctness only)
- **NFA Determinization:** Is a particular case of reachability transformation (take all distinct columns as “basis”)
- **Minimal automata:** Are related by linear maps (but not by graph isomorphisms!). Better definition of minimality
- **Other techniques:** Are easily formalized in this setting: Pumping lemma, NFA to RE, Z-transforms, etc.

Arnold, Dicky & Nivat's Example Revisited
(Observability Reduction)



Define linear transf $\bar{x}^t = x^t T$:

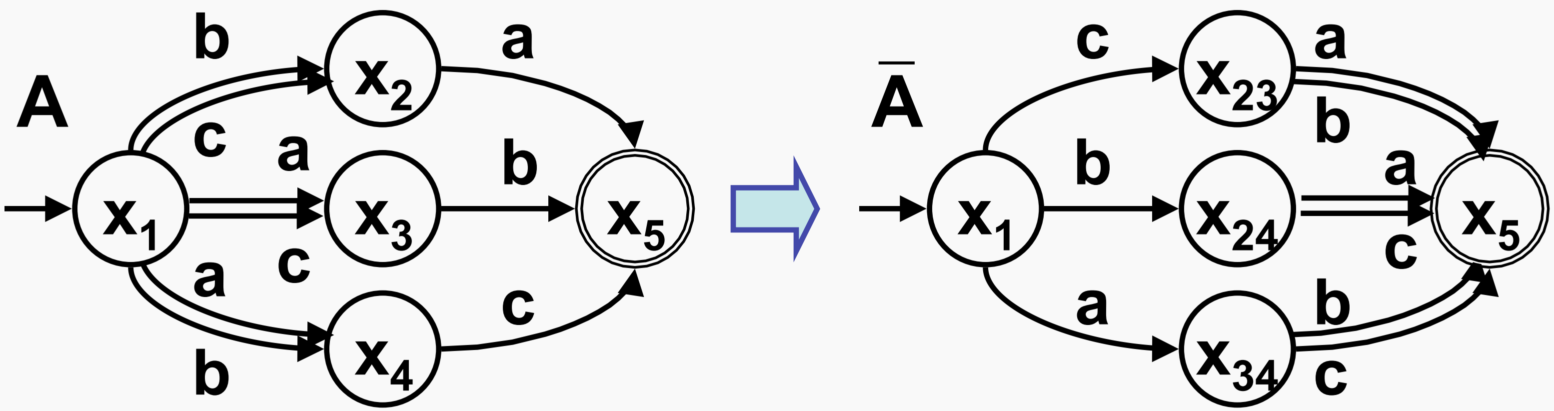
$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{A}(x) = [A(x)T]_T$$

$$\bar{x}_0^t(\varepsilon) = x_0^t(\varepsilon)T$$

$$\bar{C}(\varepsilon) = [C(\varepsilon)]_T$$

Arnold, Dicky & Nivat's Example Revisited
(Reachability Reduction)



Define linear transf $\bar{x} = x\,T$:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\bar{A}(x) = [A(x)T]_T$

$\bar{x}_0(\varepsilon) = x_0(\varepsilon)T$

$\bar{C}(\varepsilon) = [C(\varepsilon)]_T$