Name:

> You are allowed to use your own copy of the lecture notes, a formulary and a calculator. Any other documents, especially pre-calculated examples, are forbidden.

Problem 1 (13 points) Let us consider the following system model

$$\underline{Y} = \underline{\underline{W}}\,\underline{X} + \underline{V},$$

where $\underline{Y}$ is the observed random vector and $\underline{X} = (X_1 \quad X_2)^T$ is the random vector we would like to estimate, with $\sigma_{X_1}^2 = \sigma_{X_2}^2 = 4$, $R_{X_1,X_2} = 2$, and $\underline{\mu}_X = (1.5 \quad 1.5)^T$. Further assume that $\underline{V} = (V_1 \quad V_2)^T$ is a zero-mean noise vector with $\sigma_{V_1}^2 = \sigma_{V_2}^2 = 1.5$, $R_{V_1,V_2} = 1$ and $\underline{\underline{R}}_{\underline{X},\underline{V}} = 0$. The constant matrix $\underline{\underline{W}}$ is given by

$$\underline{\underline{W}} = \begin{pmatrix} 2 & 1.5 \\ 1.5 & 2 \end{pmatrix}.$$

a) Are $\underline{X}$ and $\underline{V}$ uncorrelated? Justify your answer.

b) Find the homogeneous LMMSE estimator for estimating $\underline{X}$ from the observed random vector $\underline{Y}$.

c) Find the corresponding MMSE.

Problem 2 (13 points) Let $X[n]$ and $Y[n]$ be two jointly wide-sense stationary (WSS), complex-valued, correlated processes. Consider two new random processes $W[n] = X[n] + Y[n]$ and $Z[n] = X[n] - Y[n]$.

a) Are $W[n]$ and $Z[n]$ individually WSS? Are $W[n]$ and $Z[n]$ jointly WSS? Justify your answers.

b) Assume that $X[n]$ and $Y[n]$ are zero-mean. Calculate the cross-correlation coefficients $\rho_{W,Z}[m] = c_{W,Z}[m]/(\sigma_W \sigma_Z)$ and $\rho_{W,X}[m] = c_{W,X}[m]/(\sigma_W \sigma_X)$.

c) Assume that $X[n]$ and $Y[n]$ are statistically independent. Is $U[n] = X[n]Y[n]^*$ WSS? Justify your answer.

Problem 3 (23 points) Consider the joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} K, & x^2 + y^2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

The random variables X and Y are mapped onto two new random variables Z and R as shown in Fig. 1.

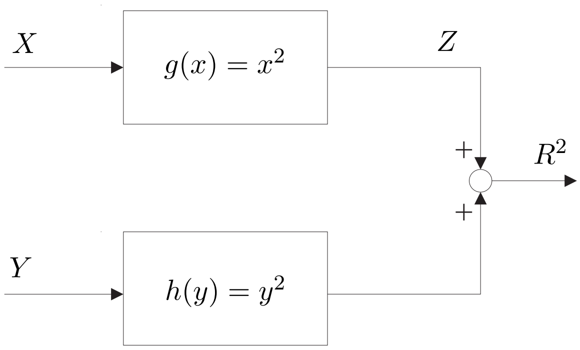

Figure 1: Relation between random variables X, Y, R, and Z.

a) Sketch the joint pdf $f_{X,Y}(x,y)$ and calculate the constant K.

b) Calculate the expectations $E\{R\}$ and $E\{R^2\}$.

c) Calculate the probability $\mathrm{P}\big\{X^2 + Y^2 < 1/2\big\}$.

d) Calculate the marginal pdfs $f_X(x)$ and $f_Z(z)$.

e) Calculate the mixed moments $m_{XY}^{(1,1)}$ and $m_{XZ}^{(1,1)}$.

f) Are X and Y uncorrelated and/or statistically independent? Justify your answer.

Hints:

- $\int_0^1 \sqrt{1-u^2}du = \frac{\pi}{4}$
- Conversion between cartesian coordinates and polar coordinates: $dxdy = rdrd\phi$

Problem 4 (21 points) Consider a Gaussian distributed random variable $A \sim \mathcal{N}(0, \sigma^2)$ and a discrete random variable $B \in \{-1, 1\}$ with $\mathrm{P}\{B = 1\} = \mathrm{P}\{B = -1\} = 1/2$, i.e., $p_B(b) = 1/2[\delta(b-1) + \delta(b+1)]$. The random variables A and B are statistically independent.

a) Calculate the joint probability density function (pdf) $f_{A,W}(a, w)$ for $W = AB$.

b) Calculate the marginal pdf $f_W(w)$.

c) Find out whether W and A are statistically independent and/or uncorrelated and/or orthogonal. Justify your answer.

d) Repeat subtasks a) – c) for the case:
A is a discrete random variable with probability $\mathrm{P}\{A = 1\} = \mathrm{P}\{A = -1\} = 1/2$ and B is a Gaussian distributed random variable with $\mathcal{N}(0, \sigma^2)$.