Makroökonomische Vertiefung

WS22

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Contents

1 Introduction

Macroeconomists...

- empirically **describe** the aggregate economy.
- theoretically **explain** the determination of production, prices, employment, exchange rates, etc.
- quantitatively **evaluate** economic policies.

1.1 Trend versus Cycle

Macroeconomic time series are often decomposed into two components:

- **trend component**: long-term growth
- **cyclical component**: fluctuations around trend (business cycles)

1.2 HP-Filter

$$y_t = g_t + c_t$$

$$\min_{(g_t)_{t=1}^T} \quad \underbrace{\sum_{t=1}^{T} (y_t - g_t)^2}_{\text{cyclical fluctuation}} + \lambda \cdot \underbrace{\sum_{t=2}^{T-1} \big((g_{t+1} - g_t) - (g_t - g_{t-1})\big)^2}_{\text{change in growth trend}}$$

The parameter λ smoothes the trend:

- $\lambda = 0 \Rightarrow g_t = y_t$ (no cyclical fluctuations)
- $\lambda \to \infty \Rightarrow g_{t+1} - g_t = g_t - g_{t-1}$ (linear trend)

2 One-Period Real Macroeconomic Model

2.1 Assumptions

representative rousehold

- Life-time utility function $u(C, l)$ describes the preferences over consumption C and leisure l:

$$\begin{aligned} u_C(\cdot) > 0 \quad u_{CC}(\cdot) < 0 \\ u_l(\cdot) > 0 \quad u_{ll}(\cdot) < 0 \end{aligned}$$

- Time constraint: $l + N^S = h$, where N^S is labour supply and h is the available time.
- Flow budget constraint: $C \leq wN^S + rK^S$, where N^S and K^S are labour and capital supplied by the household. The household takes the wage w and the interest rate r as given.

representative firm

- The final good Y at given *total factor productivity* A is produced with $Y = AF\big(K^d, N^d\big)$, where $F(\cdot)$ is a *Neoclassical production function* combining capital K^d and N^d.

2.2 Representative Household: utility maximation

$$\begin{aligned} \max_{C,l,K^S} \quad & u(C,l) \\ \text{s.t.} \quad & C \leq w(h-l) + rK^S \\ & C \geq 0,\ l \in [0;h],\ K^S \in [0;K_0] \end{aligned}$$

$$\Rightarrow \quad \mathcal{L}\Big(\lambda, C, l, K^S\Big) = u(C,l) + \lambda \cdot \Big(w(h-l) + rK^S - C\Big)$$

The relevant FOCs for an optimum are:

I. $\frac{\partial \mathcal{L}(\cdot)}{\partial C} \overset{!}{=} 0 \quad \Rightarrow \quad u_C(\cdot) = \lambda$

II. $\frac{\partial \mathcal{L}(\cdot)}{\partial l} \overset{!}{=} 0 \quad \Rightarrow \quad u_l(\cdot) = \lambda w$

III. $\frac{\partial \mathcal{L}(\cdot)}{\partial \lambda} \overset{!}{=} 0 \quad \Rightarrow \quad C = w(h-l) + rK^S$

Solutions can be characterised by:

$$\begin{aligned} \text{MRS}_{l,C} \equiv \frac{u_l(\cdot)}{u_C(\cdot)} = w \\ C = w(h-l) + rK^S \end{aligned}$$

For given (w, r) and endowment K_0, **2** equations can be solved for **2** unknowns, (C, l).

2.3 Representative Firm: profit maximisation

$$\max_{N^d, K^d} \Pi^F\Big(N^d, K^d\Big) = AF\Big(N^d, K^d\Big) - wN^d - rK^d$$

The FOCs for an optimum are:

I. $\frac{\partial \Pi^F(\cdot)}{\partial K^d} \stackrel{!}{=} 0 \quad \Rightarrow \quad AF_K(\cdot) = r$

II. $\frac{\partial \Pi^F(\cdot)}{\partial N^d} \stackrel{!}{=} 0 \quad \Rightarrow \quad AF_N(\cdot) = w$

2.4 Competitive Equilibrium

- Every market participant is a price-taker.
- Households **maximise utility** and firms **maximise profit**. $\Rightarrow$ The decision of households and firms are consistent with each other, therefore all markets are clear:
  - Good market clearing: $C = Y$
  - Labour market clearing: $N^d = N^S$
  - Capital market clearing: $K^d = K^S$

The system can be reduced to:

$$\begin{aligned}
\frac{u_l(C,l)}{u_C(C,l)} &= AF_N(K, h-l) \\
C &= AF_N(K, h-l)(h-l) + AF_K(K, h-l)K \\
C &= AF(K, h-l)
\end{aligned}$$

3 T-Period Real Macroeconomic Model

3.1 Assumptions

representative household

- Life-time utility function $U\Big((C_t, l_t)_{t=0}^T\Big)$ describes the preferences over consumption C_t and leisure l_t and is time-separable:

$$U(\cdot) = \sum_{t=0}^{T} \beta^t u(C_t, l_t)$$

- $\beta \in (0, 1)$ is the household's discount factor.
- Time constraint: $l_t + N_t = h, \quad \forall t$
- Household's initial capital endowment is $K_0 > 0$.
- Net investment in the capital stock via savings, $S_t = I_t$:
  $K_{t+1} - K_t = I_t - \delta K_t, \quad \forall t$, where $\delta \in [0; 1]$ is the *rate of depreciation* and I_t is *gross investment.*
- Flow budget constraint: $C_t + I_t \leq w_t N_t + r_t K_t, \quad \forall t$

representative firm

- The final good Y at given *total factor productivity* A is produced with $Y = AF\big(K^d, N^d\big)$, where $F(\cdot)$ is a *Neoclassical production function* combining capital K^d and N^d.

3.2 Representative Household: utility maximisation

$$\begin{aligned}
\max_{(C_t, l_t, I_t)_{t=0}^T} \quad & \sum_{t=0}^{T} \beta^t u(C_t, l_t) \\
\text{s.t.} \quad & C_t + I_t \leq w_t(h - l_t) + r_t K_t \\
& K_{t+1} = (1 - \delta) K_t + I_t \\
& C_t \geq 0,\ l_t \in [0; h],\ K_{t+1} \geq 0
\end{aligned}$$

$$\Rightarrow \mathcal{L} = \sum_{t+1}^{T} \beta^t u(C_t, l_t) + \lambda_t \big(w_t(h - l_t) + (1 - \delta + r_t) K_t - C_t - K_{t+1}\big)$$

The relevant FCOs are:

I. $\frac{\partial \mathcal{L}(\cdot)}{\partial C_t} \stackrel{!}{=} 0 \quad \Rightarrow \quad \beta^t u_C(C_t, l_t) = \lambda_t, \quad \forall t$

II. $\frac{\partial \mathcal{L}(\cdot)}{\partial l_t} \overset{!}{=} 0 \quad \Rightarrow \quad \beta^t u_l(C_t, l_t) = \lambda_t w, \quad \forall t$

III. $\frac{\partial \mathcal{L}(\cdot)}{\partial K_{t+1}} \overset{!}{=} 0 \quad \Rightarrow \quad (1 - \delta + r_{t+1})\lambda_{t+1} = \lambda_t, \quad \forall t \in [0; T-1]$

IV. $\frac{\partial \mathcal{L}(\cdot)}{\partial \lambda_t} \overset{!}{=} 0 \quad \Rightarrow \quad K_{t+1} = w_t(h - l_t) - C_t + (1 - \delta + r_t)K_t, \quad \forall t$

Solutions can be characterised by:

$$
\begin{aligned}
\text{MRS}_{l_t, C_t} &\equiv \frac{u_l(C_t, l_t)}{u_C(C_t, l_t)} = w_t && \forall t \\
\text{MRS}_{C_t, C_{t+1}} &\equiv \frac{u_C(C_t, l_t)}{\beta u_C(C_{t+1}, l_{t+1})} = 1 - \delta + r_{t+1} && \forall t \in [0; T-1] \\
K_{t+1} &= w_t(h - l_t) + (1 - \delta + r_t)K_t - C_t && \forall t
\end{aligned}
$$

3.3 Representative Firm: profit maximisation

$$\max_{N_t, K_t} \Pi^F(N_t, K_t) = AF(N_t, K_t) - w_t N_t - r_t K_t$$

The FOCs for an optimum are:

I. $\frac{\partial \Pi^F(\cdot)}{\partial K_t} \overset{!}{=} 0 \quad \Rightarrow \quad AF_K(\cdot) = r_t$

II. $\frac{\partial \Pi^F(\cdot)}{\partial N_t} \overset{!}{=} 0 \quad \Rightarrow \quad AF_N(\cdot) = w_t$

3.4 Competitive Equilibrium

- Every market participant is a price-taker.
- Households **maximise utility** and firms **maximise profit**. $\Rightarrow$ The decision of households and firms are consistent with each other, therefore all markets are clear.

The system can be reduced to a system of $\mathbf{3T+2}$ nonlinear equations and $\mathbf{3T+2}$ unknowns (endogenous variables):

$$
\begin{aligned}
\frac{u_l(C_t, l_t)}{u_C(C_t, l_t)} &= A_t F_N(K_t, h - l_t) \\
\frac{u_C(C_t, l_t)}{\beta u_C(C_{t+1}, l_{t+1})} &= 1 - \delta + AF_K(K_{t+1}, h - l_{t+1}) \\
K_{t+1} &= AF(K_t, h - l_t) - C_t + (1 - \delta)K_t
\end{aligned}
$$

4 Ramsey-Cass-Koopmans Model

4.1 Assumptions

representative household

- The representative household grows at rate $n \geq 0$:
  $N_{t+1} = (1+n)N_t, \quad \forall t, \quad N_0 = 1$
- Life-time utility function $U\big((c_t)_{t=0}^{\infty}\big)$ describes preferences over consumption c_t:

$$U(\cdot) = N_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t u(c_t) \qquad c_t \equiv \frac{C_t}{N_t}$$

- $T \to \infty$ can be justified since the representative household is a family, where altruistic parents care about their offspring (**dynasty**).
- The *instantenious utility function* is isoelastic:

$$u(c_t) = \begin{cases} \frac{c_t^{1-\sigma}-1}{1-\sigma} & \sigma \neq 1 \\ \ln(c_t) & \text{otherwise} \end{cases}$$

- σ measures *relative risk aversion*: $\sigma(c) \equiv -\frac{u_{cc} \times c}{u_c(c)}$
- Household's initial capital endowment is $K_0 > 0$.
- Net investment in the capital stock via savings, $S_t = I_t$:
  $K_{t+1} - K_t = I_t - \delta K_t, \quad \forall t$
- Flow budget constraint: $C_t + I_t \leq w_t N_t + r_t K_t, \quad \forall t$

representative firm

- A_t is interpreted as an exogenous labour-augmenting technology, $g \geq 0$ being the *rate of technological progress*: $A_{t+1} = (1+g)A_t, \quad \forall t, \quad A_0 = 1$

$$\begin{aligned} y_t &= F(k_t, A_t) & y_t &\equiv Y_t/N_t \\ \tilde{y}_t &= F\left(\tilde{k}_t, 1\right) = f\left(\tilde{k}_t\right) & \tilde{y}_t &\equiv Y_t/(A_t N_t) \end{aligned}$$

$$\Rightarrow F_K(K_t, A_t N_t) = F_k(k_t, A_t) = f_{\tilde{k}}\left(\tilde{k}_t\right)$$

4.2 Representative Household: utility maximisation

$$\begin{aligned} \max_{(c_t,i_t)_{t=0}^{\infty}} \quad & N_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t u(c_t) \\ \text{s.t.} \quad & c_t + i_t \leq w_t + r_t k_t \\ & (1+n)k_{t+1} = (1-\delta)k_t + i_t \end{aligned}$$

$$\Rightarrow \mathcal{L} = \sum_{t=0}^{\infty} \beta^t (1+n)^t u(c_t) + \lambda_t \big(w_t + (1-\delta+r_t)k_t - c_t - (1+n)k_{t+1} \big)$$

The relevant FCOs are:

I. $\frac{\partial \mathcal{L}(\cdot)}{\partial c_t} \stackrel{!}{=} 0 \quad \Rightarrow \quad \beta^t (1+n)^t u_C(C_t, l_t) = \lambda_t$

II. $\frac{\partial \mathcal{L}(\cdot)}{\partial k_{t+1}} \stackrel{!}{=} 0 \quad \Rightarrow \quad (1-\delta+r_{t+1})\lambda_{t+1} = (1+n)\lambda_t$

III. $\frac{\partial \mathcal{L}(\cdot)}{\partial \lambda_t} \stackrel{!}{=} 0 \quad \Rightarrow \quad (1+n)k_{t+1} = w_t - c_t + (1-\delta+r_t)k_t$

Optimal behaviour requires the *transversality condition* to hold:

$$\lim_{T\to\infty} \beta^T (1+n)^{T+1} \lambda_T k_{T+1} = 0$$

Solutions can be characterised by:

$$\begin{aligned} \text{MRS}_{c_t,c_{t+1}} \equiv \frac{u_c(c_t)}{\beta u_c(c_{t+1})} &= 1-\delta+r_{t+1} \\ k_{t+1}(1+n) &= w_t + (1-\delta+r_t)k_t - c_t \end{aligned}$$

4.3 Representative Firm: profit maximisation

$$\max_{N_t,K_t} \Pi^F(N_t, K_t) = F(K_t, A_t N_t) - w_t N_t - r_t K_t$$

The FOCs for an optimum are:

I. $\frac{\partial \Pi^F(\cdot)}{\partial K_t} \stackrel{!}{=} 0 \quad \Rightarrow \quad F_K(\cdot) = r_t$

II. $\frac{\partial \Pi^F(\cdot)}{\partial N_t} \stackrel{!}{=} 0 \quad \Rightarrow \quad F_N(\cdot) = w_t$

4.4 Competitive Equilibrium

- Every market participant is a price-taker.
- Households **maximise utility** and firms **maximise profit**. $\Rightarrow$ The decision of households and firms are consistent with each other, therefore all markets are clear.

The system can be reduced to:

$$
\begin{aligned}
\frac{u_c(c_t)}{\beta u_c(c_{t+1})} &= 1-\delta+F_k(k_{t+1},A_{t+1}) \\
k_{t+1}(1+n) &= F(k_t,A_t)+(1-\delta+r_t)k_t-c_t
\end{aligned}
$$

It is helpful, to define the composite parameter $(1+z)\equiv(1+g)(1+n)$. With the isolelastic utility function and by normalising by A_t:

$$
\begin{aligned}
\frac{\tilde{c}_{t+1}}{\tilde{c}_t} &= \frac{\beta^{1/\sigma}\left((1-\delta)+f_{\tilde{k}}\left(\tilde{k}_{t+1}\right)\right)^{1/\sigma}}{1+g} \\
\tilde{k}_{t+1}-\tilde{k}_t &= \frac{f\left(\tilde{k}_t\right)-\tilde{c}_t}{1+z}-\frac{(z+\delta)\tilde{k}_t}{1+z}
\end{aligned}
$$

4.5 Steady State

Steady-state quilibrium with population growth and technological progress is an equilibrium path with $\tilde{k}_t=\tilde{k}^{ss}$, therefore $\Delta\tilde{k}_t=0$, $\Delta\tilde{c}_t=0$ and $\Delta\tilde{y}_t=0$.

- A steady state equilibrium is a fixed point of a dynamic system.
- No growth in per effective labour variables implies sustained growth in per capita and aggregate variable if $g>0$.

$\Delta\tilde{c}_t=0$ determines $\tilde{k}^{ss}$:

$$
f_{\tilde{k}}\left(\tilde{k}^{ss}\right)-\delta=\beta^{-1}(1+g)^{\sigma}-1
$$

$\Delta\tilde{k}_t=0$ and $\tilde{k}^{ss}$ determine $\tilde{c}^{ss}$:

$$
\tilde{c}^{ss}=f\left(\tilde{k}^{ss}\right)-(z+\delta)\tilde{k}^{ss}
$$

4.6 Log-linearised Equilibrium Conditions

assumptions: $u(c_t)=\ln(c_t), f\left(\tilde{k}_t\right)=\tilde{k}_t^{\alpha}, \sigma=1$

$$
\begin{aligned}
\Rightarrow \tilde{c}_{t+1} &= \frac{\beta\left(1-\delta+\alpha\tilde{k}_{t+1}^{\alpha-1}\right)}{1+g}\tilde{c}_t \\
\Rightarrow \tilde{k}_{t+1} &= \frac{1-\delta}{1+z}\tilde{k}_t+\frac{\tilde{k}_t^{\alpha}-\tilde{c}_t}{1+z}
\end{aligned}
$$

Euler equation

$$
\tilde{c}_{t+1}\approx\tilde{c}^{ss}+\left.\frac{\partial\tilde{c}_{t+1}}{\partial\tilde{c}^{ss}}\right|_{\substack{\tilde{c}_t=\tilde{c}^{ss}\\ \tilde{k}_{t+1}=\tilde{k}^{ss}}}(\tilde{c}_t-\tilde{c}^{ss})+\left.\frac{\partial\tilde{c}_{t+1}}{\partial\tilde{k}^{ss}}\right|_{\substack{\tilde{c}_t=\tilde{c}^{ss}\\ \tilde{k}_{t+1}=\tilde{k}^{ss}}}\left(\tilde{k}_{t+1}-\tilde{k}^{ss}\right)
$$

$$
\frac{\tilde{c}_{t+1}-\tilde{c}_t}{\tilde{c}_t}=\hat{\tilde{c}}_{t+1}=b_{ck}\hat{\tilde{k}}_{t+1}+b_{cc}\hat{\tilde{c}}_t
$$

capital accumulation

$$\tilde{k}_{t+1} \approx \tilde{k}^{ss} + \left.\frac{\partial \tilde{k}_{t+1}}{\partial \tilde{k}^{ss}}\right|_{\substack{\tilde{c}_t=\tilde{c}^{ss} \\ \tilde{k}_t=\tilde{k}^{ss}}} \left(\tilde{k}_t - \tilde{k}^{ss}\right) + \left.\frac{\partial \tilde{k}_{t+1}}{\partial \tilde{c}^{ss}}\right|_{\substack{\tilde{c}_t=\tilde{c}^{ss} \\ \tilde{k}_t=\tilde{k}^{ss}}} (\tilde{c}_t - \tilde{c}^{ss})$$

$$\frac{\tilde{k}_{t+1} - \tilde{k}_t}{\tilde{k}_t} = \hat{\tilde{k}}_{t+1} = b_{kk}\hat{\tilde{k}}_t + b_{kc}\hat{\tilde{c}}_t$$

4.7 Equilibrium Dynamics

$$\begin{pmatrix} \hat{\tilde{k}}_{t+1} \\ \hat{\tilde{c}}_{t+1} \end{pmatrix} = \begin{pmatrix} b_{kk} & b_{kc} \\ b_{ck}b_{kk} & b_{cc} + b_{ck}b_{kc} \end{pmatrix} \begin{pmatrix} \hat{\tilde{k}}_t \\ \hat{\tilde{c}}_t \end{pmatrix}$$

repeated substitution

RLOM guess

verification

5 Real Business Cycle Model

5.1 Assumptions

rational expectations

- E_t denotes the **rational expectations operator** - agents form expectations in time t about realisations of future random variables (utility, consumption, total factor productivity, etc.):
  - using all available information in time t including the structure of the underlying model, summarised in their **information set** in time t, $\mathcal{I}_t$
  - considering this as their **dominant strategy**
- The expectation operator for random variable X_{t+1} is defined as:

$$E_t X_{t+1} \equiv E\big(X_{t+1} \mid \mathcal{I}_t\big)$$

- In t the realisation of the variable in t is known:

$$E_t X_t = X_t$$

representative household

- Life-time utility function $U\big((C_t, N_t)_{t=0}^{\infty}\big)$ describes preferences over consumption C_t and labour N_t:

$$U(\cdot) = \sum_{t=0}^{\infty} \beta^t \big(\ln(C_t) - \zeta N_t\big)$$

- Household's initial capital endowment is $K_{-1} > 0$.
- Net investment in the capital stock via savings, $S_t = I_t$:
  $K_t = (1-\delta)K_{t-1} + I_t$
- Feasibility constraint: $C_t + I_t \leq Y_t$
- Flow budget constraint: $C_t + I_t \leq w_t N_t + r_t K_t$

representative firm

- Final good Y_t is produced with the Cobb-Douglas production function:
  $Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}$
- Through $\varepsilon_t^a \sim \mathcal{N}\big(0, \sigma_a^1\big)$ exogenous shocks in total factor productivity can be modelled:

$$A_t = A_{ss}^{1-\rho_a} A_{t-1}^{\rho_a} \exp(\varepsilon_t^a)$$

5.2 Social planner: utility maximation

$$
\begin{aligned}
\max_{(C_t,N_t,K_t,I_t)_{t=0}^{\infty}} \quad & E_0 \sum_{t=0}^{\infty} \beta^t \big(\ln(C_t) - \zeta N_t\big) \\
\text{s.t.} \quad & C_t + I_t \leq Y_t \\
& K_t = (1-\delta)K_{t-1} + I_t
\end{aligned}
$$

$$
\Rightarrow \mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \big(\ln(C_t) - \zeta N_t\big) + \lambda_t \big(A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta)K_{t-1} - K_t - C_t\big)
$$

The FOCs for an optimum are:

I. $\frac{\partial \mathcal{L}(\cdot)}{\partial C_t} \overset{!}{=} 0 \quad \Rightarrow \beta^t C_t^{-1} = \lambda_t$

II. $\frac{\partial \mathcal{L}(\cdot)}{\partial N_t} \overset{!}{=} 0 \quad \Rightarrow \beta^t \zeta = \lambda_t (1-\alpha) A_t K_{t-1}^{\alpha} N_t^{-\alpha}$

III. $\frac{\partial \mathcal{L}(\cdot)}{\partial K_t} \overset{!}{=} 0 \quad \Rightarrow \lambda_{t+1} E_t \big[\alpha A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha} + (1-\delta)\big] = \lambda_t$

5.3 Competitive Equilibrium

The FOCs can be simplified to a system of six equations and six unknowns (the real rate is was defined ex post):

$$
\begin{aligned}
\text{Euler equation:} \quad & 1 = \beta E_t \left[\frac{C_t}{C_{t+1}} R_{t+1}\right] \\
\text{labour/leisure:} \quad & \zeta C_t = (1-\alpha)\frac{Y_t}{N_t} \\
\text{production:} \quad & Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \\
\text{TFP:} \quad & A_t = A_{ss}^{1-\rho_a} A_{t-1}^{\rho_a} e^{\varepsilon_t^a} \\
\text{real return:} \quad & R_t \equiv \alpha \frac{Y_t}{K_{t-1}} + (1-\delta) \\
\text{feasibility:} \quad & C_t + K_t = Y_t + (1-\delta)K_{t-1}
\end{aligned}
$$

5.4 Steady State

Supposing that all variables are constant: $\forall t : X_t = X^{ss}$. K^{ss}/Y^{ss}, Y^{ss}/N^{ss}, C^{ss}, N^{ss} and R^{ss} can be expressed as functions of structural parameters scaled by A^{ss}.

- The Euler equation becomes: $R^{ss} = \beta^{-1}$

- The real return equation yields:

$$\frac{Y^{ss}}{K^{ss}} = \frac{R^{ss} - 1 + \delta}{\alpha} \Leftrightarrow \frac{K^{ss}}{Y^{ss}} = \frac{\alpha}{\beta^{-1} - 1 + \delta}$$

- The TFP equation trivially implies: $A^{ss} = A^{ss}$
- The production function yields:

$$\frac{Y^{ss}}{N^{ss}} = A_{ss}^{1/(1-\alpha)} \left(\frac{Y^{ss}}{K^{ss}}\right)^{\alpha/(1-\alpha)} = A_{ss}^{1/(1-\alpha)} \left(\frac{\alpha}{\beta^{-1} - 1 + \delta}\right)^{\alpha/(1-\alpha)}$$

- The labour/leisure trade-off can be solved for:

$$C^{ss} = \frac{1-\alpha}{\zeta} \cdot \frac{Y^{ss}}{N^{ss}} = \frac{1-\alpha}{\zeta} \cdot \left(A_{ss}^{1/(1-\alpha)} \left(\frac{\alpha}{\beta^{-1} - 1 + \delta}\right)^{\alpha/(1-\alpha)}\right)$$

- Feasibility results in:

$$N^{ss} = C^{ss} \left(\left(1 - \delta\frac{K^{ss}}{Y^{ss}}\right)\frac{Y^{ss}}{N^{ss}}\right)^{-1}$$

5.5 Log-linearised Equilibrium Conditions

The model dynamics in the neighbourhood of the steady state can be analysed with a **linear approximation**. The variable x_t denotes the logarithmic deviation of random variable X_t from its steady state X^{ss}:

$$x_t \equiv \ln(X_t) - \ln(X^{ss})$$

$\Rightarrow$ Using $e^X \approx (1 + X)$, $100 \times x_t$ ist the percentage deviation of X_t from its steady state:

$$\frac{X_t - X^{ss}}{X^{ss}} \approx \frac{X^{ss}(1 + x_t) - X^{ss}}{X^{ss}} = x_t$$

The log-linearised system becomes:

$$\begin{aligned}
\text{Euler equation:} \quad & 0 = E_t[c_t - c_{t+1} + r_{t+1}] \\
\text{labour/leisure:} \quad & c_t = y_t - n_t \\
\text{production:} \quad & y_t = a_t + \alpha k_{t-1} + (1-\alpha) n_t \\
\text{TFP:} \quad & a_t = \rho_a a_{t-1} + \varepsilon_t^a \\
\text{real return:} \quad & R^{ss} r_t = \alpha \frac{Y^{ss}}{K^{ss}} (y_t - k_{t-1}) \\
\text{feasibility:} \quad & C^{ss} c_t = Y^{ss} y_t + (1-\delta) K^{ss} k_{t-1} - K^{ss} k_t
\end{aligned}$$

5.6 Equilibrium Dynamics

repeated substitution

- Labour/leisure trade off, production function, real return and Euler equation yield

$$E_t[c_t - b_1 c_{t+1} + b_2 a_{t+1}] = 0$$

- Feasability constraint, production function and labor/leisure trade off yield

$$E_t[b_3 c_t + b_4 k_t - b_5 a_t - b_6 k_{t-1}] = 0$$

RLOM guess

$$\begin{aligned} k_t &= \mu_{kk} k_{t-1} + \mu_{ka} a_t \\ c_t &= \mu_{ck} k_{t-1} + \mu_{ca} a_t \end{aligned}$$

verification

6 Basic New Keynesian Model

6.1 Assumptions

monopolistic competition

- Firms set prices and charge a markup over marginal costs.
- Markup depends on the elasticity of substitution between a continuum of differentiated goods.

⇒ This **marketpower** is one source of inefficiency.

price rigidity

- Firms adjust prices in any given period only with probability θ.
- If shocks hit the economy, they can adjust quantities immediately.

⇒ This creates **price dispersion**, another source of inefficiency.

representative household

- Each household owns a **non-tradable** asset - an equal share in the portfolio of producers, which pays nominal dividend: $\int_0^1 D_t(i)\, di$
- C_t is an index which aggregates all goods produced and consumed in the economy. The goods are indexed by $i \in [0;1]$, $\varepsilon_p > 1$ being the price elasticity of demand.

$$C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\varepsilon_p - 1}{\varepsilon_p}}\, di \right)^{\frac{\varepsilon_p}{\varepsilon_p - 1}}$$

- **Instantaneous utility** is given by:

$$U(C_t, N_t, Z_t) = \begin{cases} \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} \right) Z_t & \sigma \neq 1 \\ \\ \left(\ln(C_t) - \frac{N_t^{1+\phi}}{1+\phi} \right) Z_t & \text{otherwise} \end{cases}$$

  Z_t being an exogenous preference shifter, which only affects inter-temporal decision.

- N_t are hours worked and W_t is the nominal wage.
- B_t is a riskless nominal one-period discount bond purchased in period t at price Q_t. At maturity in $t+1$ it pays one unit of money. The gross yield on the bond is $1/Q_t$ - therefore the nominal interest rate is defined as $i_t \equiv \ln\big(1/Q_t\big)$.

- Inter-temporal household solvency can be ensured by imposing the no-Ponzi condition:
$$\lim_{T\to\infty} E_t\left[\Lambda_{t,T}\frac{B_T}{P_T}\right] \geq 0$$

representative firm

- **Continuum of firms** indexed by $i \in [0;1]$.
- Each firm i operates with identical technology:
$$Y_t(i) = A_t N_t(i)^{1-\alpha}, \quad \alpha \in]0;1[$$
- Firms hire $N_t(i)$ on a **perfectly competitive** labour market.
- Each firm i chooses $Y_t(i), N_t(i), P_t^*(i)$ to maximise profits, which equal dividens: $D_t(i) \equiv P_t^*(i)Y_t(i) - W_t N_t(i)$

6.2 Representative Household

expenditure minimisation

$$\min_{C_t(i)} \int_0^1 P_t(i)C_t(i)\,di \quad s.t. \quad C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\varepsilon_p-1}{\varepsilon_p}}\,di\right)^{\frac{\varepsilon_p}{\varepsilon_p-1}}$$

$$\Rightarrow \mathcal{L} = \int_0^1 P_t(i)C_t(i)\,di + \lambda_t\left(C_t - \left(\int_0^1 C_t(i)^{\frac{\varepsilon_p-1}{\varepsilon_p}}\,di\right)^{\frac{\varepsilon_p}{\varepsilon_p-1}}\right)$$

The implied FOC $\forall i$ is:

$$\frac{\partial\mathcal{L}(\cdot)}{\partial C_t(i)} \stackrel{!}{=} 0 \quad \Rightarrow \quad P_t(i) = \lambda_t\frac{\varepsilon_p}{\varepsilon_p-1}\left(\int_0^1 C_t(i)^{\frac{\varepsilon_p-1}{\varepsilon_p}}\,di\right)^{\frac{\varepsilon_p}{\varepsilon_p-1}-1}\frac{\varepsilon_p-1}{\varepsilon_p}C_t(i)^{-\frac{1}{\varepsilon_p}}$$
$$= \lambda_t C_t^{\frac{1}{\varepsilon_p}} C_t(i)^{-\frac{1}{\varepsilon_p}}$$

For any two goods $i \neq j$ it follows that:

$$C_t(i) = C_t(j)\left(\frac{P_t(i)}{P_t(j)}\right)^{-\varepsilon_p}$$

$\Rightarrow$ The optimal demand for godd i relative to any good $j \neq i$ is a declining function of its relative price.

λ_t can be interpreted as the aggregate price index P_t - the marginal cost of an additional unit of the basket C_t:

$$
\begin{aligned}
C_t &= \left(\int_0^1 C_t(i)^{\frac{\varepsilon_p - 1}{\varepsilon_p}}\, di\right)^{\frac{\varepsilon_p}{\varepsilon_p - 1}} = \left(\int_0^1 \left(C_t(j)\left(\frac{P_t(i)}{P_t(j)}\right)^{-\varepsilon_p}\right)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} di\right)^{\frac{\varepsilon_p}{\varepsilon_p - 1}} \\
&= C_t(j) P_t(j)^{\varepsilon_p} \left(\int_0^1 P_t(i)^{1-\varepsilon_p}\, di\right)^{\frac{\varepsilon_p}{\varepsilon_p - 1}} \\
\lambda_t &= \left(\frac{C_t(j)}{C_t}\right)^{\frac{1}{\varepsilon_p}} P_t(j) = \left(\int_0^1 P_t(i)^{1-\varepsilon_p}\, di\right)^{\frac{1}{1-\varepsilon_p}} \equiv P_t
\end{aligned}
$$

The household's total **consumption expenditure** results in:

$$
\begin{aligned}
\int_0^1 P_t(i) C_t(i)\, di &= \int_0^1 P_t(i) \left(\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p} C_t\right) di = P_t^{\varepsilon_p} C_t \int_0^1 P_t(i)^{1-\varepsilon_p}\, di \\
&= P_t^{\varepsilon_p} C_t \int_0^1 P_t^{1-\varepsilon_p}\, di = C_t P_t
\end{aligned}
$$

Optimal demand for good i relative to the aggregate price level is:

$$
C_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p} C_t
$$

utility maximisation

$$
\begin{aligned}
\max_{\left(C_t, N_t, B_t/P_t\right)_{t=0}^{\infty}} \quad & E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t, Z_t) \\
\text{s.t.} \quad & P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t + D_t
\end{aligned}
$$

The household's optimality conditions can be rearranged to yield

$$
\begin{aligned}
\beta Q_t &= E_t \left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \frac{Z_{t+1}}{Z_t} \frac{P_t}{P_{t+1}}\right] \\
C_t^{\sigma} N_t^{\phi} &= \frac{W_t}{P_t}
\end{aligned}
$$

6.3 Representative Firm: profit maximisation

flexible prices

$$\begin{aligned} \max_{Y_t(i), N_t(i), P_t^*(i)} \quad & P_t^*(i) Y_t(i) - W_t N_t(i) \\ \text{s.t.} \quad & Y_t(i) = A_t N_t(i)^{1-\alpha} \\ & Y_t(i) = \left(\frac{P_t^*(i)}{P_t} \right)^{-\varepsilon_p} C_t \\ & 0 \leq Y_t(i), N_t(i), P_t^*(i) \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathcal{L} = & P_t^*(i) A_t N_t(i)^{1-\alpha} - W_t N_t(i) \\ & + \Psi_t(i) \Big(A_t N_t(i)^{1-\alpha} - Y_t(i) \Big) \\ & + \xi_t(i) \left(\left(\frac{P_t^*(i)}{P_t} \right)^{-\varepsilon_p} C_t - Y_t(i) \right) \end{aligned}$$

The FOCs are

I. $\frac{\partial \mathcal{L}(\cdot)}{\partial Y_t(i)} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \xi_t(i) = P_t^*(i) - \Psi_t(i)$

II. $\frac{\partial \mathcal{L}(\cdot)}{\partial N_t(i)} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \Psi_t(i) = \frac{W_t}{(1-\alpha) A_t N_t(i)^{-\alpha}} = \frac{W_t}{\mathrm{MPN}_t(i)}$

III. $\frac{\partial \mathcal{L}(\cdot)}{\partial P_t^*(i)} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \xi_t(i) = P_t^*(i) \varepsilon_p$

Combining (I.) & (II.) yields

$$\frac{P_t^*(i)}{\Psi_t(i)} = \frac{\varepsilon_p}{\varepsilon_p - 1} \equiv \mathcal{M}_p$$

sticky prices

$$\sum_{k=0}^{\infty} \theta_p^k E_t \left[\frac{\Lambda_{t,t+k} Y_{t+k|t}(i)}{P_{t+k}} \big(P_t^*(i) - \mathcal{M}_p \Psi_{t+k|t}(i) \big) \right] = 0$$

6.4 Equilibrium

6.5 Steady State

Linearisation

6.6 Monetary Policy

6.7 Equilibrium Dynamics

6.8 Model Analysis

monetary policy shock

discount rate shock

technology shock