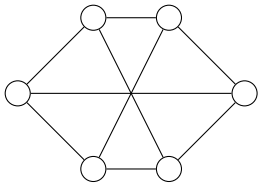
Exercise 1

Discrete Mathematics

October 7, 2021

1 Cubic graphs

a) The following is a cubic graph with 6 vertices.



b) Simple undirected cubic graphs have $\forall v \in V : \deg v = 3$.

The degree sum formula [1]

$$\sum_{v \in V} \deg v = 2 \mid E \mid$$

holds for simple undirected cubic graphs.

Proof by contradiction:
Assume there is a simple undirected cubic graph with an odd number n of vertices. Then it has

$$\mid E \mid = \frac{\sum_{v \in V} \deg v}{2} = \frac{3n}{2}$$

edges. $3n/2$ is not a natural number. However, all graphs have a natural number $E \in \mathbb{N}$ of edges. Contradiction.

c) *Exercise presentation: This was done using induction in the exercise solution presentation*

K_4 is a cubic graph for $n = 2$. $K_{3,3}$ is a cubic graph for $n = 3$. For $n \geq 2$ there are exactly 2 cases:

[1] `https://en.wikipedia.org/wiki/Degree_(graph_theory)#Handshaking_lemma`

Case 1: $2n \mod 4 = 0$
The graph disjoint union [2] of $\frac{n}{4}$ times K_4 is a cubic graph with $2n$ vertices.

Case 2: $2n \mod 4 = 2$
The graph disjoint union of $\frac{n-2}{4} - 1$ times K_4 and a single $K_{3,3}$ is a cubic graph with $2n$ vertices.

Consequently, for all $n \geq 2$ there is a cubic graph with $2n$ vertices.

2 Graph theoretical models

a) The vertices $v \in V$ are inhabitants. Two vertices are adjacent if the respective inhabitants are neighbors.

Proof by induction.
Basis Step: In the graph K_2 both inhabitants have exactly one neighbor.
Inductive Step: Assume there is a graph $G = (V, E)$ with $\mid V \mid = n$ that fulfills the property. Then there is a set of vertices Q with $Q \subseteq V$ of which all vertices have the same degree.

Case 1 The new vertex v_0 is not connected to G.

Case 2 v_0 is only connected to vertices that do not form the property, that is to vertices in $G \setminus Q$. The number of neighbors of all vertices $q \in Q$ remains equal.

Case 3 v_0 is connected to at least one vertex $q \in Q$.

Case 3.1 $Q = K_2$.

Case 3.1.1 $\deg(v_0) = 1$. v_0 has one neighbor q_0 and q_1 keeps a single neighbor.

Case 3.1.2 $\deg(v_0) \neq 1$. This forms K_3 in which all vertices have degree 2.

Case 3.2 $Q \neq K_2$. Then $\mid Q \mid > 2$ by definition.

Case 3.2.1 v_0 has exactly one neighbor $q_0 \in Q$ (and possibly other neighbors in $G \setminus Q$). At least two vertices $q_1, q_2 \in Q$ with $\deg(q_1) = \deg(q_2)$ remain.

Case 3.2.2 v_0 is connected to at least two vertices $q_1, q_2, \ldots, q_n \in Q$. All those vertices had a certain equal degree k before and now, with the edge to v_0, they all have degree $k + 1$.

In each case, the property remains fulfilled for $n + 1$ and the proof is done.

b) Each of the n friends is modeled as vertex $v_i \in V$ with $1 \leq i \leq n$ of a graph $G = (V, E)$. Sending a postcard from v_i to v_j with $1 \leq j \leq n, i \neq j$ creates a

[2] `https://mathworld.wolfram.com/GraphUnion.html`

directed edge (v_i, v_j). Every member of the group receives postcards from precisely those friends to whom he/she sent postcards if each edge occurs only in a pair $(v_i, v_j), (v_j, v_i)$ and replacing each pair with an undirected edge $\{v_i, v_j\}$ leads to a simple undirected cubic graph.

3 Isomorphic

There is a mapping

$$A \mapsto a, B \mapsto 2, C \mapsto c, D \mapsto 4, \alpha \mapsto 3, \beta \mapsto b, \gamma \mapsto 1, \delta \mapsto d,$$

that is an edge-preserving bijection between the two graphs. They are are isomorphic as a consequence.

4 Walks and Triangles

a) *Exercise Presentation: There should be a nice pattern in the adjacency matrices.*

In his first lecture, Prof. Drmota didn't mention the difference between walk and path. It seems he meant paths where vertices may appear multiple times, that means walks. [3]:

The lth power of an adjacency matrix A has as its i, j entry the total number of l-step paths (redundant and non-redundant, that means walks) from i to j. This also includes the adjacency matrix

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

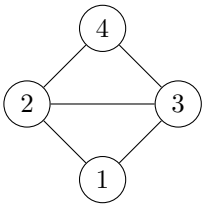
of the given graph.

b) Entries in A^3 symbolise paths of length 3. The number of triangles is $tr(A^3)/6$ where the trace tr is the sum of elements on the main diagonal (upper left to lower right) of A. Each distinct trinagle will be counted twice (clockwise and counter-clockwise) for each of the three nodes in the graph.

Computations on two graphs follow.

1.) The graph

[3] This is supported by Festinger 1949 after Ross and Harary 1952 (easier to access in TU VPN)

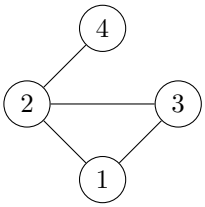


has the matrix

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad A_1^3 = \begin{pmatrix} 2 & 5 & 5 & 2 \\ 5 & 4 & 5 & 5 \\ 5 & 5 & 4 & 5 \\ 2 & 5 & 5 & 2 \end{pmatrix}$$

and the number of triangles is $(2+4+4+2)/6 = 2$.

2.) The graph



has the matrix

$$A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad A_2^3 = \begin{pmatrix} 2 & 4 & 3 & 1 \\ 4 & 2 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 2 & 4 & 3 \\ 1 & 3 & 1 & 0 \end{pmatrix}$$

and the number of triangles is $(2+2+2+0)/6 = 1$.

5 Complement

Let $\bar{G} = (V, \bar{E})$. Then the graph $G' = (V, E \cup \bar{E})$ with $|E \cup \bar{E}| = n$ is the complete graph K_n.

As a consequence, the problem can be restated as:

> Suppose in the complete graph K_n we color some of the edges red and the rest blue. Show that there is either a an all-red cycle or an all-blue cycle.

If there are more than $n-1$ red edges, then the subgraph consisting of red edges has a cycle (because then it has more edges than a tree can have). On the other hand if there are more than $n-1$ blue edges, then the blue subgraph has a cycle.

If neither of these are true, then there are at most $2(n-1)$ edges. We know that complete graphs have $\frac{n(n-1)}{2}$ edges. That means if $\frac{n(n-1)}{2} > 2n-2$ then this case is not possible and one of the initial two cases (where there exists either a red or a blue cycle) is the case. With at least five vertices ($n \geq 5$), this inequality $\frac{n(n-1)}{2} > 2n-2$ is never fulfilled.

Consequently, we are always in one of the initial two cases and there is either an all-red or an all-blue cycle. Translated back, this means that either G or $\bar{G}$ contains a cycle. [4]

6 Equivalent statements

Exercise presentation: It seems my proofs are a bit long here. One cycle $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow e \Rightarrow a$ is sufficient.

a) $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$

b) Connectivity is fulfilled by definition: A graph G is connected if any two vertices of it are connected. What remains is the cycles-part.

   $(a) \Rightarrow (b)$ Proof by contradiction. Assume (a). Assume two arbitrary vertices v_0, v_1 are connected by two different paths p_0, p_1. Using one edge multiple times is not possible on a single path (in contrast to walks). Therefore, there must be at least one intermediate node v_x that is wlog part of p_0 but not of p_1. As both paths p_0, p_1 have the same start v_0 and the same end v_1 (or vice versa), they form a cycle. This contradicts our initial assumption. Hence, every two vertices of G are connected by a unique path.

   $(a) \Leftarrow (b)$ Proof by contradiction. Assume (b). Assume G has cycles. Then there exist at least two vertices v_0, v_1 that have at least two different paths from the same start v_0 to the same end v_1. This contradicts our initial assumption. Hence, G is connected and has no cycles.

   Consequently, $(a) \Leftrightarrow (b)$.

c) Connectivity is given in (a) as well as (c).

   $(a) \Rightarrow (c)$ Proof by induction on $n = |V|$. Induction basis: $n = 1$. Then the graph consists of a single vertex. There are no edges. Therefore $1 = 0 + 1$. Induction step: $n \to n+1$ with $n \geq 1$. Then $n+1 \geq 2$. We apply the following lemma:

   > If T is a tree and $|V(T)| \geq 2$, then T has at least 2 leaves.

[4] https://math.stackexchange.com/questions/2961120/given-a-simple-graph-and-its-complement-prove-that-either-of-them-has-a-cycle

So there must be a leaf v. Consider $T' := T \setminus \{v\}$ (we remove the leaf and also the single edge that connects it to the rest of the tree). T' is also a tree. Also, we know that for the vertices $|V(T')| = |V(T)| - 1 = n + 1 - 1 = n$ and that for the edges $|E(T')| = |E| - 1$ hold. We now use the induction hypothesis and know that $|V(T')| = n = |E(T')| + 1 = |E| - 1 + 1$. Therefore, $|E| = n$. As $|V| = n + 1$ and $|E| = n$ the relation $|V| = |E| + 1$ is fulfilled. This concludes the proof.

$(a) \Leftarrow (c)$ Proof by induction. To make the statement precise, for any $n \in \mathbb{N}$

> $P(n)$: If $G = (V, E)$ is connected and $n = |V| = |E| + 1$ then G is a tree (= connected and no cycles).

Basis step $P(1)$: Let G have a single $n = 1$ vertex and be connected. Then G has no edges $|E| = n - 1 = 0$. Therefore it is connected and cycle-free, and thus a tree. Inductive Step: Assume $P(n)$ (Induction hypothesis). Show $P(n + 1)$. Let G be an arbitrary connected graph with $n = |V| = |E| + 1$. Then, by the induction hypothesis, G has no cycles. Let $v \in V$ be an arbitrary vertex of G. We create a new graph G' with a single new vertex x and a single (undirected) edge between v and x, that is $G' = (V \cup \{x\}, E \cup \{(v, x)\})$. As G was connected and the only new vertex x is connected to the rest of the graph, G' is connected, too. Also, for G' we have $n + 1 = |V| + 1 = |E| + 2$ vertices. Creating G' in such a way cannot add a new cycle. As G was cycle-free, G' is cycle-free, too. That means G' is a tree. Consequently, $P(n + 1)$ holds. Therefore, $P(n) \Rightarrow P(n + 1)$. This concludes the proof.

Consequently, $(a) \Leftrightarrow (c)$.

d) A bridge of a connected graph is a graph edge whose removal disconnects the graph.

$(d) \Rightarrow (b)$ Proof by contrapositive $\neg b \implies \neg d \iff d \implies b$. First, assume $(\neg b)$ that not every two vertices of G are connected by a simple [5] unique path. Second, assume (d) that G is a minimally connected graph. Then by definition, every edge of G is a bridge. Following from the first assumption, there exist two vertices $v \in V(G), w \in V(G)$ that fall into one of two cases:

Case 1 not connected at all. This contradicts the part of the second assumption that G is connected.

Case 2 connected by at least two paths. Then there is at least one edge that can be deleted without disconnecting v and w. Then G is not disconnected. This contradicts our second assumption.

Both cases contradict the second assumption. For any statement P, $P \vee \neg P$ is true. Therefore, the negation $(\neg d)$ of the second assumption must hold. As the contrapositive is now complete, the initial implication is also completely shown.

$(b) \Rightarrow (d)$ Proof by contrapositive $\neg d \implies \neg b \iff b \implies d$. First, assume $(\neg d)$ that G is not a minimally connected graph. Second, assume (b) that every two

[5] paths with distinct vertices and edges, and therefore different to walks

vertices of G are connected by a unique path. From the first assumption follows, that there is an edge (v, w) that is not a bridge. This means, there are at least two paths that connect v and w: The path p consisting of the single edge (v, w) and some other path q. This contradicts the second assumption (b). For any statement P, $P \vee \neg P$ is true. Therefore, the negation $(\neg b)$ of the second assumption must hold. As the contrapositive is now complete, the initial implication is also completely shown.

Consequently, $(b) \Leftrightarrow (d)$.

e) $(b) \Rightarrow (e)$ Direct proof. First, assume (b) that every two vertices of G are connected by a unique path. Unless $G = K_2$, there are two vertices $v \in V(G), w \in V(G)$ that are not adjacent. Except this condition, let those two vertices be arbitrary. This set of vertices defines the set of edges that can still be added to the graph. From the first assumption follows, that there is a path p with start vertex v and end vertex w. Adding an edge (v, w) adds a second connection between start vertex v and end vertex w of p. This means adding any edge yields a cycle and completes the proof.

$(b) \Leftarrow (e)$ Proof by contrapositive $\neg b \implies \neg e \iff e \implies b$. First, assume $(\neg b)$ that not every two vertices of G are connected by a unique path. Then there exist two vertices $v \in V(G), w \in V(G)$ that fall into one of two cases:

Case 1 one of the two is not adjacent to any other vertex and thus not connected to the graph at all. Then we can add the edge v, w without creating a cycle. Hence $(\neg e)$, G is not a maximally acyclic graph.

Case 2 connected by at least two paths. Then there are two paths with v and w as end points. This means there is a cycle. Hence $(\neg e)$, G is not a maximally acyclic graph.

In any case $(\neg e)$, G is not a maximally acyclic graph. As the contrapositive is now complete, the initial implication is also completely shown.

Consequently, $(b) \Leftrightarrow (e)$.

7 Edge count, vertex count

Exercise presentation: Could have also been a direct proof

Proof by contradiction. Let G be an arbitrary graph with n vertices and at least n edges. Assume G does not contain any cycle. Then every connected component $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_k = (V_k, E_k)$ of G is a tree by definition. Component G_i has $|E_i| = |V_i| - 1$ edges for $1 \leq 1 \leq k$. However, we get

$$\sum_{i=1}^{k} E_i = \sum_{i=1}^{k} (|V_i| - 1) = n - k < n$$

meaning that the number of edges in our graph is smaller than n. Contradiction. StackExchange

StackExchange

8 Internal nodes count, leaf node count

Partition the set of vertices V into two sets L (for leaves) and I (for internal nodes), such that $|L| + |I| = |V|$. We know that for trees $|V| = |E| + 1$.

The absence of vertices of degree 2 means each vertex is

- an internal node and has degree at least 3 or
- a leaf.

Therefore, we can see that

$$|L| + 3|I| \leq \sum_{v \in V} \deg v$$

With the handshaking lemma

$$\sum_{v \in V} \deg v = 2|E|$$

and the initial formulas, we can performe some substitutions

$$|L| + 3|I| \leq 2|E| = 2(|V| - 1) = 2(|L| + |I| - 1) = 2|L| + 2|I| - 2$$

which, with a few calculations, finally gives us

$$\begin{aligned} |L| + 3|I| &\leq 2|L| + 2|I| - 2 \\ |I| &= |L| - 2 < |L| \end{aligned}$$

This concludes the proof.

9 Average degree

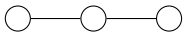
a) We show that deleting a vertex of maximal degree Δ cannot increase the average vertex degree. Let G have n vertices and e edges, so its average vertex degree is $\frac{2e}{n}$. Clearly, $\frac{2e}{n} < \Delta$. If $x \in V(G)$ has maximum degree Δ, then $G - x$ has average degree $\frac{2(e-\Delta)}{n-1}$. Thus we need to show $\frac{2(e-\Delta)}{n-1} \leq \frac{2e}{n}$. Using the fact that $\frac{2e}{n} \leq \Delta$ in

the first step, this is verified as follows:

$$\begin{aligned}\frac{2(e-\Delta)}{n-1} &\leq \frac{2(e-\frac{2e}{n})}{n-1} \\ &= \frac{2e(n-2)}{n(n-1)} \\ &= \frac{2e}{n} \cdot \frac{n-2}{n-1} \\ &\leq \frac{2e}{n}\end{aligned}$$

Note that $\leq$ and not $<$ is required in the last step because $e = 0$ is possible. This completes the proof.

b) We provide a counterexample to the task description to show that deleting a vertex of minimal degree δ can indeed decrease the average degree. Consider the following graph



It has average degree $\frac{4}{3}$. If we remove a vertex of minimum degree, we get

which has an average degree of 1 and $1 < \frac{4}{3}$. This completes the counterexample.

10 Permutations

Any permutation can be expressed as the composition of transpositions. When the set permuted is $\{1, 2, \ldots, n\}$, then any permutation can be expressed as a composition of adjacent transpositions $(1, 2), (2, 3), (3, 4)$, and so on. In other words, adjacent transpositions form a set of generators that generate the symmetric group of $n!$ permutations

More visually:
One way to show a graph is connected is to show that from any vertex of the graph, there is a path to a (special) vertex of the graph. For the present problem, we show that from any permutation $(a_1, a_2, \ldots, a_n)$ we can reach the identity permutation $(1, 2, \ldots, n)$ using adjacent transpositions.

To do so, we identify i so that $a_i = 1$, and perform adjacent transpositions $(a_{i-1}, a_i), \ldots,$ (a_1, a_2) in sequence to get element 1 to the first position. This results in the permutation $(1, a_2, a_3, \ldots a_n)$.

This can be repeated to get element 2 in the correct place and so on. This shows that any permutation is connected to the identity permutation by a path using adjacent transpositions. Hence the graph is connected.