Processing of Stochastic Signals

(Execerises 8)

December 22, 2011

8.2.a: $\underline{z} = \underline{x} + j\underline{y}$, $\underline{\underline{C}}\underline{z}, \underline{x}, \underline{y}$ real-valued

according to (3.177)

$$\underline{\underline{C}}_{\underline{z}} = \underline{\underline{C}}_{\underline{x}} + \underline{\underline{C}}_{\underline{y}} + \left(\underline{\underline{C}}_{\underline{y},\underline{x}} - \underline{\underline{C}}^{T}_{\underline{y},\underline{x}}\right)$$

in case of circular symmetry about the mean we have according to (3.185)

$$\underline{\underline{C}}_{\underline{x}} = \underline{\underline{C}}_{\underline{y}}, \underline{\underline{C}}_{\underline{y},\underline{x}} = -\underline{\underline{C}}_{\underline{x},\underline{y}}$$

and therefore

$$\underline{\underline{C}}_{\underline{z}} = 2\underline{\underline{C}}_{\underline{x}} + 2j\underline{\underline{C}}_{\underline{y},\underline{x}}$$

8.2.b:

1. diagonal elements of $\underline{\underline{C}}_{\underline{x},\underline{y}}$ zero

   (3.185) and (3.186) $\quad \underline{\underline{C}}_{\underline{y},\underline{x}} = \underline{\underline{C}}^{T}_{\underline{x},\underline{y}}$, $\underline{\underline{C}}_{\underline{y},\underline{x}} = -\underline{\underline{C}}_{\underline{x},\underline{y}}$

   $\Rightarrow$ Diagonal Elements: $\left(\underline{\underline{C}}_{\underline{x},\underline{y}}\right)_{i,i} = -\left(\underline{\underline{C}}_{\underline{x},\underline{y}}\right)_{i,i}$

   $\Rightarrow$ and this is only possible if $\left(\underline{\underline{C}}_{\underline{x},\underline{y}}\right)_{i,i} = 0$

2. $\underline{x}, \underline{y}$ uncorrelated iff $\underline{\underline{C}}_{\underline{z}} \in R^{NxN}$ $\Rightarrow$ imaginary part has to become zero:

   $\underline{\underline{C}}_{\underline{z}} = 2\underline{\underline{C}}_{\underline{x}} + 2j\underbrace{\underline{\underline{C}}_{\underline{y},\underline{x}}}_{=0}$

   $\underline{\underline{C}}_{\underline{y},\underline{x}} = 0$ gives that $\underline{x}, \underline{y}$ are uncorrelated.

3. $\underline{\underline{C}}_{\underline{z}}\underline{\underline{C}}^{H}_{\underline{z}}$ real-valued iff $\underline{\underline{C}}_{\underline{x}}\underline{\underline{C}}^{T}_{\underline{x},\underline{y}} = \underline{\underline{C}}_{\underline{x},\underline{y}}\underline{\underline{C}}_{\underline{x}}$

   $$\begin{pmatrix} \underline{\underline{C}}_{\underline{x}}\underline{\underline{C}}^{T}_{\underline{x},\underline{y}} = \underline{\underline{C}}_{\underline{x},\underline{y}}\underline{\underline{C}}^{T}_{\underline{x}} = \underline{\underline{C}}_{\underline{x},\underline{y}}\underline{\underline{C}}_{\underline{x}} \Rightarrow \underline{\underline{C}}_{\underline{x}} = \underline{\underline{C}}^{T}_{\underline{x}} \\ \underline{\underline{C}}^{T}_{\underline{x},\underline{y}} = \underline{\underline{C}}_{\underline{y},\underline{x}} = -\underline{\underline{C}}_{\underline{x},\underline{y}} \end{pmatrix}$$

$$
\begin{aligned}
\underline{\underline{C}}_{\underline{z}}\underline{\underline{C}}_{\underline{z}}^{H} &= \left(2\underline{\underline{C}}_{\underline{x}} + 2j\underline{\underline{C}}_{\underline{x,y}}^{T}\right)\left(2\underline{\underline{C}}_{\underline{x}} + 2j\underline{\underline{C}}_{\underline{x,y}}^{T}\right)^{H} \\
&= \left(2\underline{\underline{C}}_{\underline{x}} + 2j\underline{\underline{C}}_{\underline{x,y}}^{T}\right)\left(2\underline{\underline{C}}_{\underline{x}} - 2j\underline{\underline{C}}_{\underline{x,y}}^{T}\right) \\
&= 4\underline{\underline{C}}_{\underline{x}} + 4j\underline{\underline{C}}_{\underline{x,y}}^{T}\underline{\underline{C}}_{\underline{x}} - 4j\underline{\underline{C}}_{\underline{x}}\underline{\underline{C}}_{\underline{x,y}} + 4\underline{\underline{C}}_{\underline{x,y}}^{T}\underline{\underline{C}}_{\underline{x,y}} \\
&= 4\left(\underline{\underline{C}}_{\underline{x}}\underline{\underline{C}}_{\underline{x}} + \underline{\underline{C}}_{\underline{x,y}}^{T}\underline{\underline{C}}_{\underline{x,y}}\right) + 4j\left(\underline{\underline{C}}_{\underline{x,y}}^{T}\underline{\underline{C}}_{\underline{x}} + \underline{\underline{C}}_{\underline{x}}\underline{\underline{C}}_{\underline{x,y}}^{T}\right)
\end{aligned}
$$

For this to be real valued the following must hold

$$
\begin{aligned}
\underline{\underline{C}}_{\underline{x}}\underline{\underline{C}}_{\underline{x,y}}^{T} - \underline{\underline{C}}_{\underline{x,y}}\underline{\underline{C}}_{\underline{x}} &= 0 \\
\underline{\underline{C}}_{\underline{x}}\underline{\underline{C}}_{\underline{x,y}}^{T} &= \underline{\underline{C}}_{\underline{x,y}}\underline{\underline{C}}_{\underline{x}}
\end{aligned}
$$

therefore this holds under our assumption.

8.3.a:

$\underline{s} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \mu_s = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \sigma_{s_1}^2 = \sigma_{s_2}^2 = 7/4, R_{s_1,s_2} = 1/2$

$\underline{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \underline{\mu_n} = \underline{0}, \sigma_{n_1}^2 = \sigma_{n_2}^2 = 1, R_{n_1,n_2} = -1/4$

$\underline{y} = \underline{n} + \underline{s}$

according to (3.272)

$A_{LMMSE} = \underline{\underline{R}}_{s}\left(\underline{\underline{R}}_{s} + \underline{\underline{R}}_{n}\right)^{-1} = \underline{\underline{R}}_{s}\underline{\underline{R}}_{y}^{-1}$

$\underline{\underline{R}}_{\underline{s}} = \begin{pmatrix} \sigma_{s_1}^2 + \mu_{s_1}^2 & 1/2 \\ 1/2 & \sigma_{s_2}^2 + \mu_{s_2}^2 \end{pmatrix} = \begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}$

$\underline{\underline{R}}_{\underline{n}} = \begin{pmatrix} 1 & -1/4 \\ -1/4 & 1 \end{pmatrix} \quad \underline{\underline{R}}_{\underline{y}} = \begin{pmatrix} 3 & 1/4 \\ 1/4 & 3 \end{pmatrix}$

$\underline{\underline{R}}_{\underline{y}}^{-1}: \quad \left(\begin{array}{cc|cc} 3 & 1/4 & 1 & 0 \\ 1/4 & 3 & 0 & 1 \end{array}\right) =^{Z_1 - \frac{Z_2}{12}} \left(\begin{array}{cc|cc} \frac{143}{48} & 0 & 1 & -\frac{1}{12} \\ 1/4 & 3 & 0 & 1 \end{array}\right)$

$=^{\frac{143}{12}Z_2 - Z_1} \left(\begin{array}{cc|cc} 143/48 & 0 & 1 & -1/12 \\ 0 & 143/4 & -1 & 12 \end{array}\right) =^{12 Z_1} \left(\begin{array}{cc|cc} 143/4 & 0 & 12 & -1 \\ 0 & 143/4 & -1 & 12 \end{array}\right)$

$\underline{\underline{R}}_{\underline{y}}^{-1} = \frac{4}{143}\begin{pmatrix} 12 & -1 \\ -1 & 12 \end{pmatrix} = \begin{pmatrix} 48/143 & -4/143 \\ -4/143 & 48/143 \end{pmatrix}$

$A_{LMMSE} = \frac{4}{143}\begin{pmatrix} 47/2 & 4 \\ 4 & 47/2 \end{pmatrix} = \begin{pmatrix} 94/143 & 16/143 \\ 16/143 & 94/143 \end{pmatrix}$

$\hat{s}_{LMMSE} = \begin{pmatrix} 94/143 & 16/143 \\ 16/143 & 94/143 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

8.3.b:

(3.265)

$$
\begin{aligned}
\epsilon_{LMMSE} &= tr\left\{\underline{\underline{R}}_{\underline{s}} - \underline{\underline{R}}^T_{\underline{y,s}}\underline{\underline{R}}^{-1}_{\underline{y}}\underline{\underline{R}}_{\underline{y,s}}\right\} \\
\underline{\underline{R}}_{\underline{y,s}} &= E\left\{(\underline{n}+\underline{s}\underline{s}^T\right\} = E\left\{\underline{n}\underline{s}^T + \underline{s}\underline{s}^T\right\} = \\
&= E\left\{\underline{n}\underline{s}^T\right\} + E\left\{\underline{s}\underline{s}^T\right\} = \underline{\underline{R}}_{\underline{s}} \\
\epsilon_{LMMSE} &= tr\left\{\begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}^T \begin{pmatrix} 48/143 & -4/143 \\ -4/143 & 48/143 \end{pmatrix} \begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}\right\} \\
&= tr\left\{\begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix} - \begin{pmatrix} 94/143 & 16/143 \\ 16/143 & 94/143 \end{pmatrix} \begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}\right\} \\
&= tr\left\{\begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix} - \begin{pmatrix} 196/143 & 79/143 \\ 79/143 & 196/143 \end{pmatrix}\right\} \\
&= tr\left\{\begin{pmatrix} 90/143 & -15/286 \\ -15/286 & 90/143 \end{pmatrix}\right\} \\
&= \tfrac{180}{143} = \epsilon_{LMMSE}
\end{aligned}
$$

8.3.c:

Because the signal has non-zero mean it would be better to use the inhomogeneous LMMSE estimator instead of the homogeneous LMMSE estimator.

8.4.a:

from $\underline{\underline{W}} \in R^M xN$ and definition of $\underline{y}$

$\Rightarrow \underline{x} \in R^N, \underline{y} \in R^M, \underline{v} \in R^M$

8.4.b:

$$
\begin{aligned}
&\text{(page 101 ff.)} \quad \hat{\underline{\underline{x}}}_{LMMSE} &&= \underline{\underline{A}}_{LMMSE}\underline{y} = \underline{\underline{R}}^T_{\underline{x,y}}\ \underline{\underline{R}}^{-1}_{\underline{y}}\ \underline{y} \\
&\text{(3.270)} && \left(\underline{\underline{W}}\underline{\underline{R}}_{\underline{x}}\right)^T \left(\underline{\underline{W}}\underline{\underline{R}}_{\underline{x}}\underline{\underline{W}}^T + \underline{\underline{R}}_{\underline{y}}\right)^{-1}\ \underline{y} \\
&\text{(3.262)} && \underline{\underline{R}}_{\underline{x}}\underline{\underline{W}}^T \left(\underline{\underline{W}}\underline{\underline{R}}_{\underline{x}}\underline{\underline{W}}^T + \underline{\underline{R}}_{\underline{y}}\right)^{-1}\ \underline{y}
\end{aligned}
$$

8.4.c:

$$\epsilon_{LMMSE} = tr\left\{\underline{\underline{R}}_x - \underline{\underline{R}}^T_{\underline{y,x}}\underline{\underline{R}}^{-1}_{\underline{y}}\ \underline{\underline{R}}_{\underline{y,x}}\right\}$$

$$\underline{\underline{R}}_{\underline{y,x}} = \underline{\underline{W}}\underline{\underline{R}}_x$$

$$\underline{\underline{R}}^{-1}_{\underline{y}} = \left(\underline{\underline{W}}\underline{\underline{R}}_{\underline{x}}\underline{\underline{W}}^T + \underline{\underline{R}}_{\underline{y}}\right)^{-1}$$

8.4.d:

straight forward, just apply the hint given on the sheet